AN EFFICIENT METHOD FOR SOLVING TWO-ASSET TIME FRACTIONAL BLACK-SCHOLES OPTION PRICING MODEL

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ABSTRACT. In this paper, we investigate an efficient hybrid method for solving two-asset time fractional Black-Scholes partial differential equations. The proposed method is based on the Crank-Nicolson radial basis functions methods. We show that, this method is convergent and we obtain good approximations for solution of our problems. The numerical results show high accuracy of the proposed method without needing high computational cost.

1. INTRODUCTION

The financial markets are becoming more complex with trading many types of financial derivatives. A financial derivative is a contract with a value dependant on one or several underlying assets.

Options are some of the most common derivatives. There are two main types of options:

A call option gives its owner the right, but not the obligation, to buy some asset for a price referred to as the exercise price or strike price until a specified time called the expiration time of the option.

A put option gives its owner the right, but not the obligation, to sell some asset for an exercise price until the expiration time of the option.

Black-Scholes [1] and Merton [2] introduced a parabolic partial differential equation (PDE) that the price of the European option satisfies under certain assumption. During the last decades, researchers have been presenting some numerical methods in order to solve Black-Scholes equation such as finite difference method [3, 4] and radial basis functions (RBFs) method [5, 6, 7, 8].

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Fractional differential equations (FDEs) arise in many areas of science, economics and engineering, such as biophysics, finance, bioengineering, electrodynamics of complex media, signal processing and viscoelastic materials. (for instance see [9, 10, 11, 12])

Discovering of the fractional structure in financial market the fractional Black-Scholes equations are introduced. Wyss [13] derived time fractional Black-Scholes model to price European options. Jumaire used the fractional Taylor formula to derive the time and space fractional Black-Scholes equations [14]. In recent years, various numerical methods have been introducing to solve fractional Black-Scholes equations, for instance, finite difference method [15, 16, 17, 18]. Hagh et al. [19] used residual power series method and collocation based meshfree method to solve time fractional Black-Scholes models. The implicit numerical scheme for one-asset time fractional Black-Scholes model was presented in [20]. In order to see more results see [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].

RBFs method is known as a powerful tool for interpolation of scattered data. The main advantage of radial basis functions method, is its meshless characteristic. The simplicity of the method and its ability to interpolate the scattered data as well as its direct extension to higher dimension has made this method an important subject of numerical solution of PDEs. For a set of \( N \) distinct data points \( \{x_j\}_{j=1}^N \) and corresponding data values \( \{f_j\}_{j=1}^N \), the RBF interpolation is given by

\[
S(x) = \sum_{j=1}^{N} \lambda_j \varphi(r),
\]

where \( \varphi(r) = \varphi(\|x - x_j\|) \) and \( \lambda_j \) are real coefficients satisfying the interpolation conditions \( S(x) = f(x_j) \) for \( j = 1, \ldots, N \) which lead to the following symmetric linear system

\[
A \lambda = f
\]

where \( A_{j,k} = \varphi(\|x_j - x_k\|) \), \( \lambda = [\lambda_1, \ldots, \lambda_N]^T \) and \( f = [f_1, \ldots, f_N]^T \).

There are two kinds of RBFs, the piecewise smooth and the infinitely smooth RBFs. Infinitely smooth RBFs have a shape parameter \( c \), as the shape parameter has significant effect on the accuracy of the method, and the infinitely smooth RBFs can be spectrally accurate [34, 35]. Some well-known RBFs are listed in table 1.

In this paper, we applied RBFs in order to solve the two-asset time fractional Black-Scholes PDE for Exchange and Call on Maximum models. In order to this purpose, Multiquadric radial basis functions and Caputo derivative are considered.

The rest of this paper is organized as follows: In section 2 we discuss fractional calculus. The two-asset Black-Scholes PDE, and in special case the two-asset time fractional Black-Scholes PDE are studied in section 3. In section 4, a proposed method based on \( \theta \) method and RBFs method for solving two-assets time fractional Black-Scholes PDE is presented. In addition, convergency of the proposed method is proved in section 5. In section 6, Exchange option and Call on Maximum Rainbow option are introduced. The proposed method is applied to solve these problems and their obtained numerical results are presented.
### Table 1. Some well-known RBFs

<table>
<thead>
<tr>
<th>Name of function</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Infinitely smooth RBFs</strong></td>
<td></td>
</tr>
<tr>
<td>Gaussian (GA)</td>
<td>$\varphi(r) = e^{-c^2r^2}$</td>
</tr>
<tr>
<td>Inverse quadratic (IQ)</td>
<td>$\varphi(r) = (c^2 + r^2)^{-1}$</td>
</tr>
<tr>
<td>Multiquadric (MQ)</td>
<td>$\varphi(r) = \sqrt{c^2 + r^2}$</td>
</tr>
<tr>
<td>Inverse Multiquadric (IMQ)</td>
<td>$\varphi(r) = \sqrt{c^2 + r^2} - 1$</td>
</tr>
<tr>
<td><strong>Piecewise smooth RBFs</strong></td>
<td></td>
</tr>
<tr>
<td>Linear</td>
<td>$\varphi(r) = r$</td>
</tr>
<tr>
<td>Cubic</td>
<td>$\varphi(r) = r^3$</td>
</tr>
<tr>
<td>Thin plate spline (TSP)</td>
<td>$\varphi(r) = r^2 \log(r)$</td>
</tr>
</tbody>
</table>

### 2. Fractional Calculus

In this section, we recall some essential facts of fractional calculus[11]. There are various definitions for fractional derivatives. However, three definitions of fractional derivatives are more applicable than others and are used in modelling the problems in different fields of sciences. These definitions are Grunwald-Letnikov, Riemann-Liouville and Caputo. Among these fractional derivatives, Riemann-Liouville and Caputo derivatives are usually considered. Riemann-Liouville derivative has a lot of problems in modelling real-world phenomena, for example the derivative of a constant function is not zero in Riemann-Liouville approach. But Caputo definition resolves problems of Riemann-Liouville definitions in modelling real-world phenomena, and so is more practical in science and engineering. The main advantage of the Caputo approach is that the initial conditions for FDEs is sufficient to prove the uniqueness of the solution, and so we use Caputo derivative in this paper.

**Definition 2.1.** The Caputo definition for the fractional-order derivatives is defined as

$$
\begin{align*}
D_t^\alpha u(x, t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{k-\alpha-1} \varphi^k u(x, \tau) \, d\tau, \\
\frac{\partial^k u(x, t)}{\partial t^k} &= \frac{\partial^k u(x, t)}{\partial t^k},
\end{align*}
$$

(2.1)

\[\alpha = k \in \mathbb{N} \cup 0\]

### 3. Two-Asset Time Fractional Black-Scholes PDE

Multi-asset options are based on more than one underlying asset. A two-asset option is a special case of multi-asset options, where the number of underlying asset is two. The two-asset Black-Scholes equation is a two-dimensional parabolic PDE:

$$
\frac{\partial U}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 U}{\partial S_1^2} + \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 U}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 U}{\partial S_2^2} + r S_1 \frac{\partial U}{\partial S_1} + r S_2 \frac{\partial U}{\partial S_2} - r U = 0,
$$
where \( \sigma_1 \) and \( \sigma_2 \) are the volatility of assets \( S_1 \) and \( S_2 \), respectively, \( \rho \) is the correlation coefficient between \( S_1 \) and \( S_2 \) and \( r \) is the risk-free rate.

The solution domain is \( \{ S_1 \in [0, \infty), S_2 \in [0, \infty), t \in [0, T] \} \), where \( T \) is the expiration time.

The final condition is:

\[
U(S_1, S_2, T) = \text{payoff}(S_1, S_2). \quad (3.1)
\]

From the change of variable \( \tau = T - t \), we obtain

\[
\frac{\partial U}{\partial \tau} - \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 U}{\partial S_1^2} - \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 U}{\partial S_1 \partial S_2} - \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 U}{\partial S_2^2} - r S_1 \frac{\partial U}{\partial S_1} - r S_2 \frac{\partial U}{\partial S_2} + r U = 0. \quad (3.2)
\]

Now, the final condition (3.1) becomes initial condition:

\[
U(S_1, S_2, 0) = \text{payoff}(S_1, S_2). \quad (3.3)
\]

Two-asset options are usually classified into:

- Exchange option
- Spread option
- Basket option
- Rainbow option

In this paper we focus on Exchange and Rainbow options which will be described in section 6.

Two-asset time fractional Black-Scholes PDE of fractional order \( 0 < \alpha \leq 1 \) becomes as

\[
\frac{D_\alpha^\alpha}{t} U - \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 U}{\partial S_1^2} - \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 U}{\partial S_1 \partial S_2} - \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 U}{\partial S_2^2} - r S_1 \frac{\partial U}{\partial S_1} - r S_2 \frac{\partial U}{\partial S_2} + r U = 0.
\]

(3.4)

for \( \alpha = 1 \), the Eq. (3.4) reduces to (3.2), that is the general two-asset Black-Scholes equation.

4. Proposed method for two-asset time fractional option pricing

In this paper, we consider two-asset time fractional Black-Scholes PDE

\[
\frac{D_\alpha^\alpha}{t} U(x, y, t) - \frac{1}{2} \sigma_1^2 x^2 \frac{\partial^2 U}{\partial x^2}(x, y, t) - \sigma_1 \sigma_2 \rho xy \frac{\partial^2 U}{\partial x \partial y}(x, y, t) - \frac{1}{2} \sigma_2^2 y^2 \frac{\partial^2 U}{\partial y^2}(x, y, t)
\]

\[
- r x \frac{\partial U}{\partial x}(x, y, t) - r y \frac{\partial U}{\partial y}(x, y, t) + r U(x, y, t) = 0,
\]

(4.1)

where \( 0 < \alpha < 1 \) and \( D_\alpha^\alpha \) denotes the Caputo derivatives on \( t \).

The domain for each asset price is \([0, +\infty)\), but in a numerical method, we usually truncate the domain to \([0, L_1]\) and \([0, L_2]\) respectively. The choice of \( L_1 \) and \( L_2 \) usually depends on the evaluation area we are interested in.

We consider (4.1) with initial condition:

\[
U(x, y, 0) = \text{payoff}(x, y),
\]

(4.2)
We discretize the domain with \( N \) \( \kappa \) \( t \) \( \kappa \) \( t \) that and boundary conditions:

\[
U(0, y, t) = \kappa(y, t), \quad U(L_1, y, t) = \beta(y, t), \quad U(x, 0, t) = g(x, t), \quad U(x, L_2, t) = \delta(x, t),
\]

where \( \kappa(y, t), \beta(y, t), g(x, t) \) and \( \delta(x, t) \) functions are consistent to the exact solution of (4.1).

We discretize the domain with \( N \) division in x-axis and y-axis, not necessarily uniform as \( \{x_i\}_{i=1}^N \) and \( \{y_j\}_{j=1}^N \) and \( M \) time steps, so interval \([0, T]\) is discretized with \( \Delta t = \frac{T}{M} \), such that \( t^n = n\Delta t, n = 0, 1, ..., M \) and \( T \) denotes the expiration time.

By substituting \( t^{n+1} \) into (2.1), we obtain

\[
D_t^\alpha U(x, y, t^{n+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t^{n+1}} \frac{\partial U(x, y, \tau)}{\partial \tau} (t^{n+1} - \tau)^{-\alpha} d\tau
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \int_{k\Delta t}^{(k+1)\Delta t} \frac{\partial U(x, y, \tau)}{\partial \tau} (t^{n+1} - \tau)^{-\alpha} d\tau.
\] (4.3)

Using the forward difference relation:

\[
\frac{\partial U(x, y, t)}{\partial t} = \frac{U(x, y, t + \Delta t) - U(x, y, t)}{\Delta t} + O(\Delta t),
\] (4.4)

and substituting (4.4) into (4.3) we obtain

\[
D_t^\alpha U(x, y, t^{n+1}) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \left( \frac{U^{k+1} - U^k}{\Delta t} + O(\Delta t) \right) \int_{k\Delta t}^{(k+1)\Delta t} (t^{n+1} - \tau)^{-\alpha} d\tau,
\] (4.5)

where \( U^n = U(x, y, t^n) \). Here we have

\[
\int_{k\Delta t}^{(k+1)\Delta t} (t^{n+1} - \tau)^{-\alpha} d\tau = \int_{t^k}^{t^{k+1}} (t^{n+1} - \tau)^{-\alpha} d\tau = \int_{t^{n+1} - t^k}^{t^{n+1} - t^{k+1}} -r^{-\alpha} dr
\]

\[
= \frac{1}{1-\alpha} \left( (t^{n+1} - t^k)^{1-\alpha} - (t^{n+1} - t^{k+1})^{1-\alpha} \right)
\]

\[
= \frac{1}{1-\alpha} \left( ((n+1)\Delta t - (k+1)\Delta t)^{1-\alpha} - ((n+1)\Delta t - k\Delta t)^{1-\alpha} \right)
\]

\[
= \frac{1}{1-\alpha} \left( (\Delta t)^{1-\alpha}(n-k)^{1-\alpha} - (\Delta t)^{1-\alpha}(n-k+1)^{1-\alpha} \right)
\]

\[
= \frac{(\Delta t)^{1-\alpha}}{1-\alpha} \left( (n-k+1)^{1-\alpha} - (n-k)^{1-\alpha} \right).
\] (4.6)
Now by substituting (4.6) into (4.5) we find

\[
D_t^\alpha U(x, y, t^{n+1}) = \frac{1}{\Gamma(1 - \alpha)} \sum_{k=0}^n \left( \frac{U^{k+1}_t - U^k_t}{\Delta t} + O(\Delta t) \right) \frac{(\Delta t)^{1-\alpha}}{1-\alpha} \left( (n - k + 1)^{1-\alpha} - (n - k)^{1-\alpha} \right)
\]

\[
= \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=0}^n (U^{k+1} - U^k) \left( (n - k + 1)^{1-\alpha} - (n - k)^{1-\alpha} \right) + O((\Delta t)^{2-\alpha}) \quad (4.7)
\]

Now, we approximate function \( U \) with RBF method according to:

\[
U(x, y, t) = \sum_{i=1}^{N^2} \lambda_i(t) \varphi_i(x, y) = \varphi \lambda,
\]

where \( \varphi(x, y) = [\varphi(r_{1,1}), ..., \varphi(r_{1,N}), \varphi(r_{2,1}), ..., \varphi(r_{2,N}), ..., \varphi(r_{N,1}), ..., \varphi(r_{N,N})] \), \( r_{i,j} = \sqrt{(x-x_i)^2 + (y-y_j)^2} \) and \( \varphi \) is a radial basis function. By defining the operator

\[
D = -\frac{1}{2} \sigma_1^2 \frac{\partial^2}{\partial x^2} - \sigma_1 \sigma_2 \rho xy \frac{\partial^2}{\partial x \partial y} - \frac{1}{2} \sigma_2 \rho y^2 \frac{\partial^2}{\partial y^2} - rx \frac{\partial}{\partial x} - ry \frac{\partial}{\partial y} + r,
\]

we can rewrite (4.1) to:

\[
D_t^\alpha U(x, y, t) + Du(x, y, t) = 0.
\]

Using the \( \theta \) method and (4.7) we have

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=0}^n (U^{k+1} - U^k) \left( (n - k + 1)^{1-\alpha} - (n - k)^{1-\alpha} \right)
\]

\[
+ O((\Delta t)^{2-\alpha}) + (1 - \theta)DU^{n+1} + \theta DU^{n} = 0,
\]

(4.9)

where the parameter \( \theta \) is chosen in interval \([0, 1]\). By rearranging (4.9) we have

\[
c_\alpha U^{n+1} + (1 - \theta)DU^{n+1} = c_\alpha U^{n} - c_\alpha \sum_{k=0}^{n-1} (U^{k+1} - U^k)d_\alpha(k) - \theta DU^{n}.
\]

So,

\[
(c_\alpha + (1 - \theta)D)U^{n+1} = (c_\alpha - \theta D)U^{n} - c_\alpha \sum_{k=0}^{n-1} (U^{k+1} - U^k)d_\alpha(k),
\]

where \( c_\alpha = \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \) and \( d_\alpha(k) = \left( (n - k + 1)^{1-\alpha} - (n - k)^{1-\alpha} \right) \).

Defining \( A_1 = (c_\alpha + (1 - \theta)D) \) and \( B_1 = (c_\alpha - \theta D) \) we have

\[
A_1U^{n+1} = B_1U^{n} - c_\alpha \sum_{k=0}^{n-1} (U^{k+1} - U^k)d_\alpha(k).
\]

(4.10)
By using RBF approximation, we find
\[ U_{n+1} = \sum_{i=1}^{N^2} \lambda_n^{i+1} \varphi_i(x, y), \tag{4.11} \]
\[ U^n = \sum_{i=1}^{N^2} \lambda_n^i \varphi_i(x, y). \tag{4.12} \]
Substituting values from (4.11) and (4.12) into (4.10) for all internal and boundary points, we get the scheme in matrix form:
\[ A_1 \Phi^{\lambda_n+1} = B_1 \Phi^{\lambda_n} + f^{n+1} + g^{n+1}, \tag{4.13} \]
where \( \Phi = [\varphi(r_{i,j})]_{i,j=1}^{N} \), \( f = [-c_\alpha \sum_{k=0}^{n-1} (U_i^{k+1} - U_i^k) d\alpha(k)]_{i=1}^{N^2} \) and \( g^{n+1} \) is a \( N^2 \times 1 \) vector, such that according to internal points its components are equal to zero and its other components are obtained by substituting boundary points into their boundary conditions. Subsequently (4.13) can be written as
\[ \lambda_n^{i+1} = (A_1 \Phi)^{-1} (B_1 \Phi) \lambda_n^i + (A_1 \Phi)^{-1} f^{n+1} + (A_1 \Phi)^{-1} g^{n+1}. \tag{4.14} \]
So,
\[ \lambda_n^{i+1} = H \lambda_n^i + G, \tag{4.15} \]
In above relation \( \Phi^0 \) vector is obtained using initial condition (4.2).

5. **The convergence of the proposed method**

In this section, we prove the convergency of the scheme (4.15).

We define matrix \( E = \Phi H \Phi^{-1} \). The components of the matrix \( E \) depends on the constant \( \gamma = \frac{\Delta t}{h^s} \), where \( h \) is the distance between any two nodes, and \( s \) is the highest order of partial differential operator, where \( s \) is equal to 2 for mention problem (4.1).

We know that \( |D_t^\alpha u(x, y) - D_t^\alpha u(x, y)| \leq \beta_l h^{l-\alpha} |u|_{n+\alpha} \), where \( l \in \mathbb{N}, \alpha \leq 1 \), \( N_{\alpha}(\Omega) \) is a native space of radial basis function \( \varphi \) and \( u^n(x, y) \) is the exact solution of (4.1) at time \( n\Delta t \).

We assume that (4.15) is accurate of order \( p \), it yields that
\[ u^{n+1} = \Phi H \Phi^{-1} u^n + \Phi G^{n+1} + O((\Delta t)^{2-\alpha} + h^p), \quad \Delta t \to 0, h \to 0, \forall n \tag{5.1} \]
Now we define \( e^n(x, y) = u^n(x, y) - U^n(x, y) \). By subtracting (4.15) from (5.1) we get:
\[ e^{n+1} = E e^n + O((\Delta t)^{2-\alpha} + h^p), \quad \Delta t \to 0, h \to 0 \]
By Lax-Richtmyer definition of convergency the scheme in (4.15) is convergent if
\[ \| E \| \leq 1, \]
(5.2)
hence, there exist a constant \( \eta \) such that
\[ \| e^{n+1} \| \leq \| E \| \| e^n \| + \eta ((\Delta t)^{2-\alpha} + h^p). \]
It is seen that \( e^0 = 0 \), using the initial condition. So we have
\[ \| e^{n+1} \| \leq (1 + \| E \| + \| E \|^2 + \ldots + \| E \|^n) \eta ((\Delta t)^{2-\alpha} + h^p). \]
By considering (5.2), we obtain
\[ \| e^{n+1} \| \leq (n + 1) \eta ((\Delta t)^{2-\alpha} + h^p). \]
So convergence of the scheme is proved.

6. Implementation of the Proposed Method

In this section we introduce Exchange and Rainbow options. The numerical solutions for them are further considered using the proposed method.

6.1. Exchange option. Exchange option is usually used in energy market. The payoff of this option is:

\[ \text{payoff} = \max(S_1 - S_2, 0). \]
Since there is no strike price term, the classification of call and put option is not used for this option.
There are various types of boundary conditions, in this paper we consider the following:

\[ U(0, S_2, t) = 0, \quad 0 \leq S_2 \leq L_2, 0 < t \leq T, \]
(6.1)
\[ U(L_1, S_2, t) = \max(L_1 - S_2 e^{-r(T-t)}, 0), \quad 0 \leq S_2 \leq L_2, 0 < t \leq T, \]
(6.2)
\[ U(S_1, 0, t) = S_1, \quad 0 < S_1 < L_1, 0 < t \leq T, \]
(6.3)
\[ U(S_1, L_2, t) = \max(S_1 - L_2 e^{-r(T-t)}, 0), \quad 0 < S_1 < L_1, 0 < t \leq T. \]
(6.4)
An analytical solution formula to two-asset Black-Scholes equation for Exchange option was introduced by Margrabe[37]. The exact solution is as bellow:
\[ C(S_1, S_2, t) = S_1 N(d_1) - S_2 N(d_2), \]
where
\[d_1 = \frac{\ln(S_1/S_2) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = \frac{\ln(S_2/S_1) - (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = d_1 - \sigma\sqrt{T - t},\]
\[\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \quad N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz.\]

The numerical solutions of Exchange option using proposed method is presented in next section.

6.1.1. An example of Exchange option. In this section, we consider PDE (3.4) with initial condition (3.3) and boundary conditions (6.1)-(6.4), when
\[\sigma_1 = \sigma_2 = 0.2, T = 0.5, \rho = 0.1, r = 0.1, L_1 = L_2 = 40\]

In order to use the proposed method, we suppose \(M = 30\) and \(\theta = 0.5\). This problem is solved by the proposed method by Multiquadratic (MQ) RBF and appropriate shape parameter. All numerical computations have been done by using MATLAB. The results of the proposed method the exact solution of the problem for \(\alpha = 1\) are shown in table 2. It can be seen that the proposed method has high accuracy. Also, the computational time of the proposed method for \(N = 10\) and \(M = 30\) is 0.3 second.

<table>
<thead>
<tr>
<th>(S_1)</th>
<th>(S_2)</th>
<th>Approx by the proposed method (N = 10, \theta = 0.5, \alpha = 1)</th>
<th>Exact solution for (\alpha = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>8</td>
<td>(9.404853879854 \times 10^{-5})</td>
<td>(9.404893887087 \times 10^{-5})</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
<td>(1.880364912671 \times 10^{-4})</td>
<td>(1.8809787777417 \times 10^{-4})</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>(6.195082337376)</td>
<td>(6.195082339727)</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>(1.610562395720)</td>
<td>(1.610562395729)</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>(4.891152587957)</td>
<td>(4.891152587955)</td>
</tr>
</tbody>
</table>

The approximated solutions of the example of Exchange option for \(N = 10\) and various \(\alpha = 0.5, 0.7, 0.9, 1\), with \(t = 0.5\), \(S_2 = 16\) are shown in Fig. 1. It can be seen from Fig. 1 that the numerical solutions for different \(0 < \alpha < 1\) are tending to the solution related to \(\alpha = 1\).

For a fixed \(N = 10\), the numerical solutions at times \(5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t\), with \(\alpha = 1\), \(S_1 = 20\) are depicted in Fig. 2. Moreover, the absolute errors are shown in Fig. 3.

The absolute errors between the exact and the obtained numerical solutions are done for proposed method in the domain \([0, 40] \times 16\), \(t = 0.5\) and \(\alpha = 1\) for various \(N = 4, 6, 8\) and 10. The results are depicted in Fig. 4. The graphs indicates the thoroughness of the proposed technique.
6.2. **Rainbow option.** Rainbow option is based on a combination of various assets like a rainbow is a combination of various colors. There are different forms of Rainbow option. Some of the typical models of Rainbow options and their payoffs are listed in table 3. For more details refer to\cite{38, 39, 40}.

In this paper we consider Call on Maximum option. The boundary conditions are:
Figure 3. The absolute errors of the example of Exchange option at times $t = 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t$, with $N = 10$, $\alpha = 1$ and $S_1 = 20$.

Figure 4. The absolute errors of the example of Exchange option for $\alpha = 1$ and various $N = 4, 6, 8, 10$, with $S_2 = 16$.

$\Box C(0, 0, t) = 0$.

$\Box$ If $S_1 = 0$ and $S_2 \neq 0$, the option value $C$ depends only on $S_2$ and $t$:

$$\frac{\partial C}{\partial t}(S_2, t) - \frac{1}{2} \sigma^2 S_2^2 \frac{\partial^2 C}{\partial S_2^2}(S_2, t) - r S_2 \frac{\partial C}{\partial S_2}(S_2, t) + r C(S_2, t) = 0.$$
### Table 3. Examples of Rainbow option

<table>
<thead>
<tr>
<th>Name</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multi-asset Rainbow option</td>
<td>$\max(S_1 - E_1, S_2 - E_2, 0)$</td>
</tr>
<tr>
<td>Pyramid Rainbow option</td>
<td>$\max(</td>
</tr>
<tr>
<td>Max option</td>
<td>$\max(S_1, S_2)$</td>
</tr>
<tr>
<td>Call on Maximum option</td>
<td>$\max(S_1, S_2)$</td>
</tr>
<tr>
<td>Call on Minimum option</td>
<td>$\min(S_1, S_2)$</td>
</tr>
<tr>
<td>Put on Maximum option</td>
<td>$\max(E - \max(S_1, S_2), 0)$</td>
</tr>
<tr>
<td>Put on Minimum option</td>
<td>$\max(E - \min(S_1, S_2), 0)$</td>
</tr>
</tbody>
</table>

□ If $S_1 \neq 0$ and $S_2 = 0$, the option value $C$ depends only on $S_1$ and $t$:

$$
\frac{\partial C}{\partial t}(S_1, t) - \frac{1}{2} \sigma_1^2 S_1 \frac{\partial^2 C}{\partial S_1^2}(S_1, t) - r S_1 \frac{\partial C}{\partial S_1}(S_1, t) + r C(S_1, t) = 0.
$$

□ If $S_1 \to \infty$ and $S_2 \to \infty$, the option value is approximately equal to $S_1$ or $S_2$.

□ If $S_1 \to \infty$ and $S_2$ is finite, the option value is approximately equal to $S_1$.

□ If $S_1$ is finite and $S_2 \to \infty$, the option value is approximately equal to $S_2$.

The exact solution is:

$$
C(S_1, S_2, t) = S_1[N(\delta_1) - N'(d_1, \delta_1, \rho_1)] + S_2[N(\delta_2) - N'(d_2, \delta_2, \rho_2)] + E e^{-r(T-t)} N'(d_1 + \sigma_1 \sqrt{T-t}, -d_2 + \sigma_2 \sqrt{T-t}, \rho) - E e^{-r(T-t)},
$$

where

$$
\begin{align*}
\delta_1 &= \ln(S_1/E) + (r + \frac{1}{2} \sigma_1^2)(T-t), \\
\delta_2 &= \ln(S_2/E) + (r + \frac{1}{2} \sigma_2^2)(T-t), \\
d_1 &= \frac{\ln(S_1/E) + (r + \frac{1}{2} \sigma_1^2)(T-t)}{\sigma_1 \sqrt{T-t}}, \\
d_2 &= \frac{\ln(S_2/E) + (r + \frac{1}{2} \sigma_2^2)(T-t)}{\sigma_2 \sqrt{T-t}}, \\
\rho_1 &= \frac{\rho \sigma_2 - \sigma_1}{\sigma}, \\
\rho_2 &= \frac{\rho \sigma_1 - \sigma_2}{\sigma}, \\
\sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2},
\end{align*}
$$

$$
N(\delta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\delta} e^{-\frac{z^2}{2}} \, dz.
$$

$$
N'(d, \delta, \rho) = \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{d} \int_{-\infty}^{\delta} e^{-\frac{x^2 - 2 \rho xy + y^2}{2(1-\rho^2)}} \, dxdy,
$$
and $E$ is the strike price. The numerical solutions for Call on Maximum option are discussed in rest of this paper.

6.2.1. An example of Call on Maximum option. Here we consider PDE (3.4) with initial and boundary conditions consistent with Call on Maximum Rainbow option when

$E = 10, \sigma_1 = \sigma_2 = 0.2, T = 0.5, \rho = 0.1, r = 0.1, L_1 = L_2 = 40$

In order to use the proposed method we suppose $M = 30$ and $\theta = 0.5$. This problem is solved by the proposed method by MQ RBF and appropriate shape parameter. All numerical computations have been done by using MATLAB. The results of the proposed method and the exact solution of the problem for $\alpha = 1$, are shown in table 4. The computational time of the proposed method for $N = 10$ and $M = 30$ is 1.5 second, and it can be seen that the proposed method has high accuracy.

**Table 4. Results for a Call on Maximum option example**

<table>
<thead>
<tr>
<th>$S_1$</th>
<th>$S_2$</th>
<th>Approx by the proposed method</th>
<th>Exact solution for $\alpha = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>6.487819515821</td>
<td>6.487819515264</td>
</tr>
<tr>
<td>10</td>
<td>8</td>
<td>0.858850135128</td>
<td>0.858850135433</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>7.696995177509</td>
<td>7.696995177078</td>
</tr>
<tr>
<td>20</td>
<td>8</td>
<td>10.487706094292</td>
<td>10.487706094291</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>10.687059187041</td>
<td>10.687059187049</td>
</tr>
</tbody>
</table>

The option prices for different values of $\alpha$ are shown in table 5. The results show the convergence property of the proposed scheme.

**Table 5. Call on Maximum option prices for different values of $\alpha$, $N = 10$**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Approx solution for $S_1 = 8, S_2 = 16$</th>
<th>Approx solution for $S_1 = 10, S_2 = 8$</th>
<th>Approx solution for $S_1 = 16, S_2 = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.487819515821</td>
<td>0.858850135128</td>
<td>7.696995177509</td>
</tr>
<tr>
<td>0.8</td>
<td>6.767074281504</td>
<td>0.979842444227</td>
<td>7.869315238913</td>
</tr>
<tr>
<td>0.6</td>
<td>6.919186398316</td>
<td>1.114209984923</td>
<td>8.049413265431</td>
</tr>
<tr>
<td>0.4</td>
<td>7.017000160938</td>
<td>1.239646310190</td>
<td>8.246213299863</td>
</tr>
</tbody>
</table>

The approximated solutions of the example of Call on Maximum option for $N = 10$ at times $t = 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t$, with $S_2 = 16$ and $\alpha = 1$ and $\alpha = 1$ are shown respectively in Figs. 5 and 6.
Figure 5. The approximated solutions of the example of Call on Max option for $N = 10$ at times $t = 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t$, with $S_2 = 16, \alpha = 1$.

Figure 6. The approximated solutions of the example of Call on max option for $N = 10$ at times $t = 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t$, with $S_2 = 16, \alpha = 0.5$.

Also, the absolute errors of the example of Call on Maximum option at times $t = 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t$, with $N = 10, \alpha = 1$ and $S_2 = 16$ are shown in Fig. 7.
The absolute errors of the example of Call on Max option at times $t = 5\Delta t, 10\Delta t, 15\Delta t, 20\Delta t, 25\Delta t, 30\Delta t$, with $N = 10$, $\alpha = 1$ and $S_2 = 16$ are shown in Fig. 8. The graphs indicates the thoroughness of the proposed technique.

The absolute errors of the example of Call on Max option for $\alpha = 1$ and various $N = 4, 6, 8, 10$, with $S_2 = 16$ are shown in Fig. 8. The graphs indicates the thoroughness of the proposed technique.
7. Conclusion

The two-asset time fractional Black-Scholes PDE depends on two underlying assets, and also has time-dependent fractional derivative of order $0 < \alpha \leq 1$, for $\alpha = 1$ it leads to general two-asset Black-Scholes equation. In this paper, an efficient hybrid numerical method for solving time fractional PDE (3.4) based on the radial basis function and the Crank-Nicolson methods was introduced. Furthermore, the convergency of the mentioned method was proved. Two-asset options are categorized into: Exchange, Spread, Basket and Rainbow options. Here, we focus on Exchange and Call on Maximum Rainbow options. The proposed scheme can be easily used for other two-asset options, and also can be extended to higher dimensions. The numerical results show high accuracy of the proposed method without needing high computational cost.

References


[40] R. Stulz, Options in the minimum or the maximum of two risky assets, J. Finac. Econ, 10 (1982), 161–185.