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GREEN’S FUNCTION APPROACH TO THERMAL DEFLECTION OF A THIN HOLLOW CIRCULAR DISK UNDER AXISYMMETRIC HEAT SOURCE

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ABSTRACT. A Green’s function approach is adopted to solve the two-dimensional thermoeelastic problem of a thin hollow circular disk. Initially, the disk is kept at temperature $T_0(r, z)$. For times $t > 0$, the inner and outer circular edges are thermally insulated and the upper and lower surfaces of the disk are subjected to convection heat transfer with convection coefficient $h_c$ and fluid temperature $T_\infty$, while the disk is also subjected to the axisymmetric heat source. As a special case, different metallic disks have been considered. The results for temperature and thermal deflection has been computed numerically and illustrated graphically.

1. INTRODUCTION

Roy Choudhury [1] discussed the normal deflection of a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper face and the lower face of the plate kept at zero temperature while the circular edge is thermally insulated. Grysa and Kozlowski [2] investigated an inverse one-dimensional transient thermoelastic problem and obtained the temperature and heat flux on the surface of an isotropic infinite slab. Ootao et al. [3, 4] studied the theoretical analysis of a three-dimensional transient thermal stress problem for a nonhomogeneous/functionally graded hollow circular cylinder due to a moving heat source in the axial direction from the inner and outer surfaces. Tanigawa et al. [5] discussed the theoretical analysis of thermoelastoplastic deformation of a circular plate due to a partially distributed heat supply. Noda et al. [6] discussed the transient thermoelastoplastic bending problems, making use of the strain increment theorem, and determined the temperature field and the thermoelastic deformation for the heating and cooling processes in a thin circular plate subjected to a partially distributed and axisymmetric heat supply on the upper surface.
Chakraborty et al. [7] solved the deflection of a circular plate due to the heating of a concentric circular region. Recently, some cases of thermal deflection [7, 8, 9, 10, 11, 12], thermal stresses [13, 14, 15, 16, 17], or both of them [18, 19, 20] have been investigated on thin circular plates of solid [7, 8, 9, 12, 13, 15, 16, 18, 20] and annular disk [10, 11, 14, 19] under different initial and boundary conditions/input heat source. Moreover, most of these studies on the thermoelastic problem of thin-wall plates analysis involving integral transform [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20] methods were used to obtain the analytical solution.

In this article, we extended the work of Gaikwad [14] for a two-dimensional Green’s function approach to the transient thermoelastic problem of a thin hollow circular disk under the axisymmetric heat source. The convection boundary condition is assumed in this study for both the upper and lower surfaces of the disk with its inner and outer edges are thermally insulated and subjected to an axisymmetric heat source. The analytical method of Green’s function is employed with the help of integral transform technique to determine the temperature distribution function. The thermal deflection is also obtained based on the calculated temperature distribution considering the state of plane stresses.

The remainder of this study offers the following:

- The governing transient heat conduction equation with the thermoelastic equation of the thin hollow circular disk under study is formulated as a boundary value problem.
- The Green’s function method is used to solve the transient heat conduction equation.
- The finite Hankel and Fourier integral transform technique is used to derive Green’s function.
- Based on the temperature distribution, the thermal deflection in a thin hollow circular disk is also obtained.
- The mathematical model is prepared for different metallic disks and the results for temperature, and thermal deflection has been computed numerically and illustrated graphically with the help of Mathcad software.

It is believed that this particular problem has not been considered by anyone. This is a new and novel contribution to the field of thermoelasticity. The results presented here will be more useful in engineering problems particularly, in the determination of the state of strain in a thin hollow circular disk constituting foundations of containers for hot gases or liquids, in the foundations for furnaces, etc.

2. Analysis

2.1. Transient Heat Conduction Problem.

We consider a thin hollow circular disk as shown in Fig. 1, of radius $a$ and thickness by $h$, the occupying space $D$ is $a \leq r \leq b$, $-h/2 \leq z \leq h/2$ and is initially at temperature $T_0(r, z)$. For $t > 0$, the fixed circular edges ($r = a, r = b$) are thermally insulated and the upper and lower surfaces ($z = \pm h/2$) of the disk are subjected to convection heat transfer with convection coefficient $h_c$ and fluid temperature $T_\infty$, while the disk is subjected to the
axisymmetric heat source $g_0 \ (W.m^{-3})$. Under these realistic prescribed conditions temperature, and thermal deflection/stresses in a thin hollow disk due to the axisymmetric heat source are required to be determined.

The temperature of the hollow circular disk $T(r, z, t)$ at time $t$ satisfies the differential equation

$$
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{k_t} \frac{\partial^2 T}{\partial z^2} + \frac{g_0}{k_t} = \frac{1}{\alpha} \frac{\partial T}{\partial t}, \quad \text{in} \quad a \leq r \leq b, \quad -h/2 \leq z \leq h/2, \quad t > 0 \quad (2.1)
$$

with the boundary conditions,

$$
T \bigg|_{r=0} = \infty \quad (2.2)
$$
$$
\frac{\partial T}{\partial r} \bigg|_{r=a} = 0 \quad (2.3)
$$
$$
\frac{\partial T}{\partial r} \bigg|_{r=b} = 0 \quad (2.4)
$$

$$
-k_t \frac{\partial T}{\partial z} \bigg|_{z=h/2} = h_c \left( T \bigg|_{z=h/2} - T_\infty \right) \quad (2.5)
$$

$$
-k_t \frac{\partial T}{\partial z} \bigg|_{z=-h/2} = h_c \left( T \bigg|_{z=-h/2} - T_\infty \right) \quad (2.6)
$$

and the initial condition,

$$
T \bigg|_{t=0} = T_0(r, z) \quad (2.7)
$$

where $k_t$ is the thermal conductivity, the thermal diffusivity is defined as $\alpha = k_t/\rho c_p$ with $\rho$ and $c_p$ denoting the density and specific heat of the material of the hollow disk respectively.

$g(r, z, t) = g_i(t) \delta(r - r_c) \delta(z - z_c)$ represents an axisymmetric heat source, where $g_i$ is an instantaneous line heat source, and $\delta$ is a Dirac delta function that characterizes the location of the line heat source at $r_c$ and $z_c$.

Here the hollow disk is assumed sufficiently thin.
2.2. Determination of the Temperature.

First, we modified the formulated boundary value problem with homogeneous boundary conditions. The temperature field $T(r, z, t)$ is divided into two components, $T(r, z, t) = \psi(r, z, t) + T_\infty$, where the constant ambient component $T_\infty$ satisfies Eq. (2.1) and the dynamic component $\psi$ satisfies the following equation:

$$
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + \frac{g_0}{k_t} = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}
$$

with the boundary conditions,

$$
\psi \bigg|_{r=0} < \infty
$$

$$
\frac{\partial \psi}{\partial r} \bigg|_{r=a} = 0
$$

$$
\frac{\partial \psi}{\partial r} \bigg|_{r=b} = 0
$$

$$
\left( \frac{\partial \psi}{\partial z} + h_{s1} \psi \right) \bigg|_{h/2} = 0
$$

$$
\left( \frac{\partial \psi}{\partial z} - h_{s2} \psi \right) \bigg|_{-h/2} = 0
$$

and the initial condition,

$$
\psi \big|_{t=0} = T_0 - T_\infty = A(r, z)
$$

where $h_{s1} = h_c/k_t$ and $h_{s2} = h_c/k_t$ be the relative heat transfer coefficients on the upper and lower surface of the thin hollow circular disk.

To determine the Green’s function, we consider the homogeneous form of Eq. (2.8) with $g_0(r, z, t) = 0$:

$$
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\alpha} \frac{\partial \psi}{\partial t}
$$

with the boundary conditions,

$$
\psi \bigg|_{r=0} < \infty
$$

$$
\frac{\partial \psi}{\partial r} \bigg|_{r=a} = 0
$$

$$
\frac{\partial \psi}{\partial r} \bigg|_{r=b} = 0
$$

$$
\left( \frac{\partial \psi}{\partial z} + h_{s1} \psi \right) \bigg|_{h/2} = 0
$$

$$
\left( \frac{\partial \psi}{\partial z} - h_{s2} \psi \right) \bigg|_{-h/2} = 0
$$

and the initial condition,

$$
\psi \big|_{t=0} = T_0 - T_\infty = A(r, z)
$$
Secondly, for the temperature function $\psi(r, z, t)$, we introduce the finite Hankel transform $H$ over the variable $r$ and its inverse transform defined in [21] as:

$$T(r, z, t) = \sum_{m=1}^{\infty} \frac{R_0(\beta_m, r)}{N(\beta_m)} T(\beta_m, z, t)$$  \hspace{1cm} (2.17)

where

$$R_0(\beta_m, r) = \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m b)}{Y_0(\beta_m b)} \right]$$

and $\beta_1, \beta_2, \beta_3, \ldots$ are the positive roots of the transcendental equation

$$\frac{J_0'(\beta_m a)}{J_0'(\beta_m b)} - \frac{Y_0'(\beta_m a)}{Y_0'(\beta_m b)} = 0.$$

This transform satisfies the relation

$$H \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = -\beta_m^2 T(\beta_m, z, t)$$  \hspace{1cm} (2.18)

Thirdly, for the Hankel-transformed function $\mathcal{T}(\beta_m, z, t)$, we introduce the finite Fourier transform over the variable $z$ and its inverse transform defined in [21] as

$$\mathcal{T}(\beta_m, \eta_p, t) = \int_{z=-h/2}^{h/2} Z(\eta_p, z') \mathcal{T}(\beta_m, z', t) dz'$$  \hspace{1cm} (2.19)

$$\mathcal{T}(\beta_m, z, t) = \sum_{p=1}^{\infty} \frac{Z(\eta_p, z)}{N(\eta_p)} \mathcal{T}(\beta_m, \eta_p, t)$$  \hspace{1cm} (2.20)

where

$$Z(\eta_p, z) = \eta_p \cos(\eta_p z) + h_{s1} \sin(\eta_p z)$$

$$\frac{1}{N(\eta_p)} = \sqrt{2} \left[ \left( \eta_p^2 + h_{s1}^2 \right) \left( \frac{h}{2} + \frac{h_{s2}}{\eta_p^2 + h_{s2}^2} \right) + h_{s1} \right]^{-1}$$

and $\eta_1, \eta_2, \ldots$ are the positive roots of the transcendental equation

$$\tan \left( \frac{\eta_p h}{2} \right) = \frac{\eta_p (h_{s1} + h_{s2})}{\eta_p^2 - h_{s1} h_{s2}}, \quad p = 1, 2, 3, \ldots$$
Applying the finite Hankel transform and finite Fourier transform defined in Eqs. (2.16) and (2.19) and their respective inverses defined in Eqs. (2.17) and (2.20) and operate them on Eqs. (2.9)—(2.15):

\[
\psi(r, z, t) = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{R_0(\beta_m, r)Z(\eta_p, z)}{N(\beta_m)N(\eta_p)} e^{-\alpha(\beta_m^2 + \eta_p^2)t} \tilde{A}(\beta_m, \eta_p)
\]

where

\[
\tilde{A}(\beta_m, \eta_p) = \int_{r' = a}^{b} \int_{z' = -h/2}^{h/2} r' R_0(\beta_m, r') Z(\eta_p, z') (T_0(r', z') - T_\infty) \, dr' \, dz'
\]

\[
\psi(r, z, t)
\]
can also be given from the Green's function approach [21]:

\[
\psi(r, z, t) = \int_{r' = a}^{b} \int_{z' = -h/2}^{h/2} G(r, z, t|r', z', \tau)|_{\tau = 0}(T_0(r', z') - T_\infty) \, r' \, dr' \, dz'
\]

From the above, the Green's function can be obtained as:

\[
G(r, z, t|r', z', \tau) = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{R_0(\beta_m, r)Z(\eta_p, z)R_0(\beta_m, r')Z(\eta_p, z')}{N(\beta_m)N(\eta_p)} e^{-\alpha(\beta_m^2 + \eta_p^2)(t-\tau)}
\]

Finally, the solution of the nonhomogeneous problem of Eqs. (2.1) and (2.2)—(2.7) in terms of the above Green's function is given as:

\[
T(r, z, t) = T_\infty + \int_{r' = a}^{b} \int_{z' = -h/2}^{h/2} G(r, z, t|r', z', \tau)|_{\tau = 0}(T_0(r', z') - T_\infty) \, dr' \, dz'
\]

\[
+ \frac{\alpha}{k_t} \int_{\tau = 0}^{t} \int_{r' = a}^{b} \int_{z' = -h/2}^{h/2} r' G(r, z, t|r', z', \tau) g_0(r', z', \tau) \, dr' \, dz' \, dt'
\]

(2.21)

2.3. Special Case. :

Setting, \(T_0(r, z) = T_\infty\) and \(g(r, z, t) = 1(t)\delta(r - r_c)(z - z_c)\) with \(1(t)\) denoting a unit step function in Eq. (2.21), the transient temperature field is given as follows:

\[
T(r, z, t) = T_0 + \frac{r_c}{k_t} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} R_0(\beta_m, r)R_0(\beta_m, r_c) \left[ \frac{Z(\eta_p, z)Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)} \right]
\]

(2.22)

3. Determination of Thermal Deflection

The thermal bending problem of a thin disk with a thickness \(h\), it will be assumed that the deflection, which means a deformation in the out-of-plane direction of the disk, is small. By Kirchhoff-Love hypothesis that the plane initially perpendicular to the neutral plane of the disk remains a plane after deformation and is perpendicular to the deformed neutral plane.
The differential equation satisfied the deflection function $\omega(r, t)$ as defined in [22] as

$$\nabla^2 \nabla^2 \omega = -\frac{1}{(1-\nu)}D \nabla^2 M_T$$  \hspace{1cm} (3.1)

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$$

and $M_T$ is the thermal moment of the disk, $\nu$ is the Poisson’s ratio of the plate material, $D$ is the flexural rigidity of the disk denoted by

$$D = \frac{E h^3}{12(1-\nu^2)}$$

The term $M_T$ is defined as

$$M_T = \alpha t E \int_{-h/2}^{h/2} (T(r,z,t) - T_0) z dz$$  \hspace{1cm} (3.2)

where $\alpha$ and $E$ are the coefficients of the linear thermal expansion and the Young’s modulus, respectively.

For out-of-plane deformation, the boundary conditions are given as

$$\frac{\partial \omega}{\partial r} \bigg|_{r=a} = \frac{\partial \omega}{\partial r} \bigg|_{r=b} = 0$$  \hspace{1cm} (3.3)

Initially

$$T \big|_{t=0} = \omega \big|_{t=0} = T_0(r,z)$$  \hspace{1cm} (3.4)

Assume the solution of Eq. (3.1) satisfying conditions (3.3) as

$$\omega(r,t) = \sum_{m=1}^{\infty} C_m(t) \left[ J_0(\beta_m r) - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)} \frac{J_0'(\beta_m b)}{Y_0'(\beta_m b)} \right]$$  \hspace{1cm} (3.5)

where $\beta'_m$ s are the positive roots of transcendental equation,

$$\frac{J_0(\beta_m a)}{J_0'(\beta_m b)} = \frac{Y_0(\beta_m a)}{Y_0'(\beta_m b)}$$

It can be easily shown that

$$\frac{\partial w}{\partial r} = \sum_{m=1}^{\infty} C_m(t) \left[ \frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \frac{J_0'(\beta_m b)}{Y_0'(\beta_m b)} \right]$$

$$\frac{\partial w}{\partial r} \bigg|_{r=a} = \frac{\partial w}{\partial r} \bigg|_{r=b} = 0$$

Hence, the solution of Eq. (3.5) satisfies the condition of Eq. (3.3).

Now

$$\nabla^2 \nabla^2 w = \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right)^2 \sum_{m=1}^{\infty} C_m(t) \left[ \frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right]$$  \hspace{1cm} (3.6)
Using the well-known result:
\[
\left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) J_0(\beta_m r) = -\beta_m^2 J_0(\beta_m r)
\]
\[
\left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) Y_0(\beta_m r) = -\beta_m^2 Y_0(\beta_m r)
\]
in Eq. (3.6), one obtains
\[
\nabla^2 \nabla^2 w = \sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right]
\]  
(3.7)

The thermal moment could be obtained by substituting Eq. (2.22) into Eq. (3.2):
\[
M_T = D_m \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} R_0(\beta_m, r) R_0(\beta_m, r_c) \frac{\sin(\eta_p h/2) Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)(\beta_m^2 + \eta_p^2)}
\]

where
\[
D_m = \alpha \nu \frac{E}{k \eta_p^2}
\]

Now
\[
\nabla^2 M_T = \nabla^2 D_m \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} R_0(\beta_m, r) R_0(\beta_m, r_c) \frac{\sin(\eta_p h/2) Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)(\beta_m^2 + \eta_p^2)}
\]

(3.8)

solving Eq. (3.8), one obtains
\[
\nabla^2 M_T = -D_m \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \beta_m^2 R_0(\beta_m, r) R_0(\beta_m, r_c) \frac{\sin(\eta_p h/2) Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)(\beta_m^2 + \eta_p^2)}
\]

(3.9)

Substituting Eqs. (3.7) and (3.9) into Eq. (3.1) yields
\[
\sum_{m=1}^{\infty} C_m(t) \beta_m^4 \left[ \frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)} \right] = \frac{D_m}{(1 - \nu) D} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \beta_m^2 R_0(\beta_m, r) R_0(\beta_m, r_c) \frac{\sin(\eta_p h/2) Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)(\beta_m^2 + \eta_p^2)}
\]

(3.10)

Solving Eq. (3.10), one obtains
\[
C_m(t) = \frac{D_m}{(1 - \nu) D} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{1}{\beta_m^2} R_0(\beta_m, r_c) \frac{\sin(\eta_p h/2) Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)(\beta_m^2 + \eta_p^2)}
\]

(3.11)
Finally, substituting Eq. (3.11) in Eq. (3.5), one obtains the expression for the quasi-static thermal deflection $\omega(r, t)$ as:

$$\omega(r, t) = \frac{D_m}{(1 - \nu)D} \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{1}{\beta_m^2} R_0(\beta_m, r) R_0(\beta_m, r_c) \left[ \frac{\sin(\eta_p h/2) Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2 + \eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)} \right]$$  

(3.12)

4. NUMERICAL RESULTS AND DISCUSSION

4.1. Dimension. :

The constants associated with the numerical calculation are taken as:

- Inner radius of a circular disk $a = 1$ m,
- Outer radius of a circular disk $b = 5$ m,
- Thickness of circular disk $h = 0.5$ m,
- Constant line heat source $g_i = 200$ W/m,
- Initial temperature $T_0(r, z) = 0$,
- Relative heat transfer coefficients $h_{s1} = 10$, $h_{s2} = 0$.

4.2. Material Properties. :

The four different materials was chosen for the purpose of numerical evaluation, for which we take the following values of the physical constants as[23]:

<table>
<thead>
<tr>
<th>Materials</th>
<th>$k_t$ (W/m-K)</th>
<th>$c_p$ (J/kg-K)</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$\alpha$ (10$^{-6}$ m$^2$/s)</th>
<th>$\alpha_t$ ($10^{-6}$ 1/K)</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum (Al)</td>
<td>204</td>
<td>896</td>
<td>2727</td>
<td>84.18</td>
<td>22.2</td>
<td>70</td>
<td>0.35</td>
</tr>
<tr>
<td>Copper (Cu)</td>
<td>386</td>
<td>383</td>
<td>8954</td>
<td>112.34</td>
<td>16.6</td>
<td>117</td>
<td>0.36</td>
</tr>
<tr>
<td>Iron (Fe)</td>
<td>72.7</td>
<td>452</td>
<td>7897</td>
<td>20.34</td>
<td>12</td>
<td>193</td>
<td>0.21</td>
</tr>
<tr>
<td>Steel</td>
<td>53.6</td>
<td>465</td>
<td>7833</td>
<td>14.74</td>
<td>13</td>
<td>200</td>
<td>0.26</td>
</tr>
<tr>
<td>0.5% carbon (St)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.3. Roots of the Transcendental Equation. :

Here $\beta_1 = 3.1965$, $\beta_2 = 6.3123$, $\beta_3 = 9.4445$, $\beta_4 = 12.5812$, $\beta_5 = 15.7199$ are the positive root of the transcendental equation

$$\left[ \frac{J'_0(\beta a)}{J'_0(\beta b)} - \frac{Y'_0(\beta a)}{Y'_0(\beta b)} \right] = 0$$
and η₁ = 1.4289, η₂ = 4.3058, η₃ = 7.2281, η₄ = 10.2003, η₅ = 13.2142 are positive roots of the transcendental equation

\[ \tan \left( \frac{\eta h}{2} \right) = \frac{\eta (h_{s1} + h_{s2})}{\eta^2 - h_{s1} h_{s2}} \]

which takes the form \( \eta \tan \left( \frac{\eta}{4} \right) = c \), for \( h = 0.5, h_{s1} = c = 10, h_{s2} = 0 \) in [21]. The numerical calculations have been presented by the PTC MATHCAD (Prime-3.1) and the results are depicted graphically.

The constants \( \lambda_0 \) and \( \mu_0 \) are given as:

\[ \lambda_0 = \frac{D_m 10^4}{(1 - \nu) D}, \quad \mu_0 = \frac{2\mu(1 + \nu) \alpha t}{10^7} \]

i.e. elastic material constants.

The obtained expressions for the temperature field and thermal deflection provide important intuition into the role of the thermomechanical material properties in elastic behaviors of the thin hollow circular disks under the axisymmetric heat source. The temperature distribution in the disk is only dependent on its thermal properties, on the other hand, the disk deflection is dependent on both thermal and mechanical properties. We have used the first 50 terms \( (p=1–50) \) for the inner series summation, as given by Eq. (2.22), and have used the first 10 terms \( (m=1–10) \) of the outer series summation. However, for very small times \( (t<1s) \), the convergence of the inner series, particularly at \( r \simeq 0 \), was closer to 2%; hence exploration of very small times would require additional terms for the outer series summation (i.e. \( m>10 \)) to achieve greater accuracy. The numerical calculations have been carried out for four different materials (Aluminum, Copper, Iron, and Steel), which have mechanical and thermal properties as shown in Table 1. Assume that the disk is subjected to a constant heat line source of \( g_i = 200 \text{ W/m} \) with initial temperature \( T_0(r, z) = 0 \).

Figure 2 shows the temperature distribution for the iron disk along with the disk radius at the lower surface \( (z = -h/2) \), when the heat source is located at \( (r_c=4 \text{ m} \) and \( z_c=0.5 \text{ m} \) at times different times ranged from 10s to 5000 s. It should be noted that the disk temperature gradually increases in the range \( 1 \leq r \leq 4 \) with increases time, attaining maximum value at the heat source location, and goes on decreasing towards the outer circular edge. It is clear that the disk temperature the rate slows down with respect to time as it approaches the steady-state.

Figure 3 shows the temperature distribution for the iron disk along with the disk radius at the lower surface \( (z = -h/2) \), when the heat source is specified with a constant intensity of \( g_i = 200 \text{ W/m} \), as the heating location with different radii’s on the upper surface \( (z = h/2) \) for steady-state temperature \( t=3000 \text{ s} \). It can be observed that the maximum temperature stays around \( 85^0 \) for \( r_c=1.5, 2.5, 3, \) and \( 3.5 \text{ m} \), and it becomes \( 100 \) and \( 102^0 \text{C} \) for \( r_c=4.5, 4.7 \text{ m} \) respectively. It should be noted that the heat source approaches toward the insulated boundary, the heat tends to be accumulated locally rather than being dissipated in all directions as it does at the median area of the disk.

Figure 4 shows the temperature distribution at the lower surface due to axisymmetric heat source \( (z_c=0.5 \text{ m}, \) and \( t=3000 \text{ s} \) at different radii with increasing heat source intensity \( r_c \), that
FIGURE 2. The temperature distribution at the lower surface due to axisymmetric heat source ($r_c=4$ m, $z_c=0.5$ m) at different times parameters.

FIGURE 3. The temperature distribution at the lower surface due to axisymmetric heat source ($z_c=0.5$ m, and $t=3000$ s) at different radii with constant heat source intensity ($g_i = 200$ W/m).

is $g_i=40$, 80, 120, 160, 200, and 240 W/m for $r_c=1.5$, 2.5, 3, 3.5, 4.5, and 4.7 m respectively. It should be noted that the temperature increases monotonically mainly due to the increasing heat source intensity with $r_c$, because less amount of heat is dissipated as the heat source approaches closer to the insulated boundary.
Figure 4. The temperature distribution at the lower surface due to axisymmetric heat source ($z_c=0.5$ m, and $t=3000$ s) at different radii with increasing heat source intensity $r_c$.

Figure 5. The thermal deflection at the mid-plane due to axisymmetric heat source ($r_c=4$ m, $z_c=0.5$ m, and $t=3000$ s).

Figure 5 shows the thermal deflection at the mid-plane ($z=0$) of the disk for four different materials under the same conditions used to obtain Fig. 2. It can be observed that for all materials, the deflection is maximum at the heat source location, and it decreases with increasing $r$ to reach to zeros. The obtained results show good agreement with boundary conditions (3.3). The steel (53.6 W/m-K) and iron (72.7 W/m-K) disks have smaller thermal conductivity compared
with the aluminum and copper disks, so the steel and iron disks have larger deflection than the aluminum (204 W/m-K) and silver (386 W/m-K) disks. It is due to the larger temperature gradients induced in the steel and iron disks which results from the lower thermal conductivity of these materials. However the steel and iron disks have larger Young’s moduli (200 and 193 GPa) compared with the aluminum (70 GPa) and copper (117 GPa) disks, so the steel and iron disks have larger deflection than the aluminum and copper disks. This conclusion gives us the knowledge that the material thermal properties have a dominant effect on the thermal deflection compared to the mechanical properties.

5. CONCLUSIONS

Green’s function approach to analyzing the two-dimensional transient thermoelastic problem of a thin hollow circular disk under the axisymmetric heat source proposed. The convection boundary condition is assumed in this study for both the upper and lower surfaces of the disk with its inner and outer edges are thermally insulated and subjected to an axisymmetric heat source. The analytical method of Green’s function is employed with the help of an integral transform technique to determine the temperature distribution function. The thermal deflection is also obtained based on the calculated temperature distribution considering the state of plane stresses. We introduce the two-dimensional treatment based on the so-called Kirchhoff-Love’s hypothesis; thereafter basic equations are derived from the problem. For the thermoelastic deformation, the analytical solution is obtained and some important conclusions have been drawn as follows:

- The temperature distribution per Eq. (2.22) shows the several interesting changes are rooted in the physics of the boundary condition. The temperature and thermal deflection occur near the heat source, due to the axisymmetric heat source.
- The convergence of the series summation is rapid for a large time.
- The numerical values of the temperature and thermal deflection for the disk of materials Steel, Iron, Aluminum, and Copper are in the proportion and follow the relation $\text{Steel} \leq \text{Iron} \leq \text{Aluminum} \leq \text{Copper}$. We conclude that the thermal conductivity of material decreases its deflection increases. Hence, these values are inversely proportional to their thermal conductivity.
- It is observed that the material thermal properties have a dominant effect on the thermal deflection compared to the mechanical properties.
- It should be noted that a high strength level is needed for a good thermal resistance material.

The rotating disk has applications in aerospace engineering, particularly in gas turbines and gears. The rotating disk represents work under thermo-mechanical loads. Also, any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (2.22)–(3.12).
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REFERENCES


EXISTENCE OF SOLUTION FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS VIA TOPOLOGICAL DEGREE METHOD

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ABSTRACT. This paper is studied the existence of a solution for the impulsive Cauchy problem involving the Caputo fractional derivative in Banach space by using topological structures. We based on using topological degree method and fixed point theorem with some suitable conditions. Further, some topological properties for the set of solutions are considered. Finally, an example is presented to demonstrate our results.

1. INTRODUCTION

Fractional differential equations have proved to be effective modeling of many physical phenomena and various fields for more details, see Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Deimling [4]. Topological degree method is one of the important tools that procedure needs weakly compact conditions instead of strongly compact conditions. In fact, topological methods become very closely to study the existence of solutions of fractional differential equations in the last decades, see Feckan [5], and Mawhin [6]. The fractional differential equations in Banach space have recently been receiving more attention by many researchers such as Agarwal et al. [7], Balachandran and Park [8], and Zhang [9]. In 2009, Benchohra and Seba [10], considered the existence of solutions for impulsive fractional differential equations in a Banach space by Monch’s fixed point theorem and the technique of measures of non compactness. In 2010, Ahmad and Sivasundaram [11], studied the existence of solutions for impulsive integral boundary value problems with fractional order by applying the contraction mapping principle and Krasnoselskii’s fixed point theorem. In 2012, Wang et.al [12, 13], studied existence, uniqueness and data dependence for the solutions for impulsive Cauchy problems with fractional order by degree method for condensing maps by a singular Gronwall inequality. In 2012, Feckan et. al [14], corrected a formula of solutions for impulsive fractional differential
equations which cited in the previous paper and they established some sufficient conditions for existence of the solutions by using fixed point methods. Motivated from some cited results, our aim in this paper is to confirm some new results on the following impulsive Cauchy problem (ICP) for fractional differential equations involving the Caputo fractional derivative by topological degree method and fixed point theorem.

\[
\begin{aligned}
&\mathcal{C}D^q x(t) = \xi(t, x(t)) \quad t \in \mathcal{J}/\{t_1, t_2, \ldots, t_m\}, \mathcal{J} := [0, T] \\
&x(0) = x_0, \\
&\Delta x(t_k) = I_k(x(t_k)) \\
&\quad k = 1, 2, \ldots, m,
\end{aligned}
\]

(1.1)

where \(\mathcal{C}D^q\) is the Caputo fractional derivative of order \(q \in (0, 1)\), \(x_0\) is an element of \(\mathcal{X}\), \(\xi : \mathcal{J} \times \mathcal{X} \rightarrow \mathcal{X}\) is a given jointly continuous linear map, and \(\mathcal{PC}(\mathcal{J}, \mathcal{X})\) is a Banach space with the norm \(\|x\|_{\mathcal{PC}} = \sup\{\|x(t)\| : t \in \mathcal{J}\}\), \(I_k : \mathcal{X} \rightarrow \mathcal{X}\) is a continuous map and \(t_k\) satisfies, \(0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = T\).

2. Preliminaries

In this section, we introduce some necessary definitions and theorems which are needed throughout this paper.

We define a Banach space \(\mathcal{PC}(\mathcal{J}, \mathcal{X}) = \{x : \mathcal{J} \rightarrow \mathcal{X} : x \in \mathcal{C}((t_k, t_{k-1}], \mathcal{X})\}\), for \(k = 0, \ldots, m\) and there exist \(x(t_k^+)\) and \(x(t_k^-)\) such that \(x(t_k^+) = x(t_k), x(t_k^-) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)\) and \(x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)\) represent the right and left limits of \(x(t)\) at \(t = t_k\).

**Definition 2.1.** ([2]) For a given function \(\xi\) on the closed interval \([a, b]\), the \(q\)th fractional order integral of \(\xi\) is defined by:

\[
\mathcal{I}^q_{a+}\xi(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \xi(s)ds,
\]

wherever \(\Gamma\) is the gamma function.

**Definition 2.2.** ([2]) For a given function \(\xi\) on the closed interval \([a, b]\), the \(q\)th Riemann-Liouville fractional-order derivative of \(\xi\), is defined by:

\[
(\mathcal{D}^q_{a+}\xi)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-q-1} \xi(s)ds.
\]

Here \(n = [q] + 1\) and \([q]\) denotes the integer part of \(q\).

**Definition 2.3.** ([2]) For a given function \(\xi\) on the closed interval \([a, b]\), the Caputo fractional order derivative of \(\xi\), is defined by:

\[
(\mathcal{C}D^q_{a+}\xi)(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} \xi^{(n)}(s)ds,
\]

where \(n = [q] + 1\).

**Theorem 2.1.** (Banach contraction mapping principle)([15])

Let \(\mathcal{X}\) be a Banach space, and \(\psi : \mathcal{X} \rightarrow \mathcal{X}\) is a contraction mapping with contraction constant \(K\), then \(\psi\) has a unique fixed point.
Theorem 2.2. (Schaefer’s fixed point theorem) ([15])
Let $K$ be a nonempty convex, closed and bounded subset of a Banach space $X$. If $\psi : K \to K$ is a complete continuous operator such that $\psi(K) \subset X$, then $\psi$ has at least one fixed point in $K$.

Lemma 2.1. ([14]) Let $q \in (0, 1)$ and $\xi : \mathcal{J} \times X \to X$ be continuous. A function $x \in C(\mathcal{J}, X)$ is said to be a solution of the fractional integral equation

$$x(t) = x_0 - \frac{1}{\Gamma(q)} \int_0^t (a-s)^{q-1} \xi(s, x(s))ds + \frac{1}{\Gamma(q)} \int_t^a (t-s)^{q-1} \xi(s, x(s))ds,$$

if and only if $x$ is a solution of the following fractional Cauchy problems

$$\left\{\begin{array}{ll}
cD^q x(t) &= \xi(t, x(t)), & t \in \mathcal{J}, \\
x(a) &= x_0, & a > 0
\end{array}\right.$$

3. Main Results

First of all, let us define the mean of a solution of the ICP (1.1).

Definition 3.1. If a function $x \in PC(\mathcal{J}, X)$ satisfies the equation $cD^q x(t) = \xi(t, x(t))$ almost everywhere on $\mathcal{J}$, and the condition $\Delta x(t_k) = I_k(x(t_k))$, $k = 1, 2, ..., m$ and $x(0) = x_0$ then, $x$ is said to be a solution of the fractional ICP (1.1).

In order to treat the problem of existence for a solution of ICP (1.1), we need the following assumptions:

H1: $\xi : \mathcal{J} \times X \to X$ is jointly continuous.
H2: For arbitrary $x, y \in X$, there exists a constant $\delta_\xi > 0$, such that

$$\|\xi(t, x) - \xi(t, y)\| \leq \delta_\xi \|x - y\|$$

H3: For arbitrary $(t, x) \in \mathcal{J} \times X$, there exist $\delta_1, \delta_2 > 0, q_1 \in [0, 1)$ such that

$$\|\xi(t, x)\| \leq \delta_1 \|x\|^{q_1} + \delta_2.$$

H4: $I_k : X \to X$ is continuous and there is a constant $\gamma_I \in (0, \frac{1}{m})$ such that

$$\|I_k(x) - I_k(y)\| \leq \gamma_I \|x - y\|, \text{ for all } x, y \in X, \ k = 1, 2, ..., m$$

H5: For arbitrary $x \in X$, there exist $\gamma_1, \gamma_2 > 0, q_2 \in [0, 1)$ such that

$$\|I_k(x)\| \leq \gamma_1 \|x\|^{q_2} + \gamma_2, \ k = 1, 2, ..., m$$

Lemma 3.1. The fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1} \xi(s, x(s))ds$$

$$+ \frac{1}{\Gamma(q)} \int_t^{t_k} (t-s)^{q-1} \xi(s, x(s))ds + \sum_{0 < t_k < t} I_k(x(t_k))$$

(3.1)
has a solution \( x \in \mathcal{PC}(\mathcal{J}, \mathcal{X}) \), for \( t \in (t_k, t_{k+1}), \ k = 1, 2, ..., m \) if and only if \( x \) is a solution of the fractional ICP(1.1).

**Proof.** First, assume \( x \in \mathcal{PC}(\mathcal{J}, \mathcal{X}) \) satisfies ICP(1.1), we have to show that the fractional integral Eq. (3.1) has at least one solution \( x \in \mathcal{PC}(\mathcal{J}, \mathcal{X}) \). Consider the operator \( \mathcal{F} : \mathcal{PC}(\mathcal{J}, \mathcal{X}) \to \mathcal{PC}(\mathcal{J}, \mathcal{X}) \) defined by:

\[
(\mathcal{F}x)(t) = x(t) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \xi(s, x(s))ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \xi(s, x(s))ds + \sum_{0<t_k<t} I_k(x(t_k)) \bigg|_{k=1, 2, ..., m}.
\]

It obvious that \( \mathcal{F} \) is well defined due to [H1] and [H4]. Let \( \{x_n\} \) be a sequence such that \( x_n \to x \) in \( \mathcal{PC}(\mathcal{J}, \mathcal{X}) \). Then, for each \( t \in \mathcal{J} \) we consider

\[
\| (\mathcal{F}x_n)(t) - (\mathcal{F}x_m)(t) \| \leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| \xi(s, x_n(s)) - \xi(s, x_m(s)) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| \xi(s, x_n(s)) - \xi(s, x_m(s)) \| ds + \sum_{0<t_k<t} \| I_k(x_n(t_k)) - I_k(x_m(t_k)) \|
\]

\[
\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| \xi(s, x_n(s)) - \xi(s, x(s)) + \xi(s, x(s)) - \xi(s, x_m(s)) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| \xi(s, x_n(s)) - \xi(s, x(s)) + \xi(s, x(s)) - \xi(s, x_m(s)) \| ds + \sum_{0<t_k<t} \| I_k(x_n(t_k)) - I_k(x(t_k)) + I_k(x(t_k)) - I_k(x_m(t_k)) \|
\]

\[
\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| \xi(s, x_n(s)) - \xi(s, x(s)) \| ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| \xi(s, x_m(s)) - \xi(s, x(s)) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| \xi(s, x_n(s)) - \xi(s, x(s)) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| \xi(s, x_m(s)) - \xi(s, x(s)) \| ds + \sum_{0<t_k<t} \| I_k(x_n(t_k)) - I_k(x(t_k)) \| + \sum_{0<t_k<t} \| I_k(x_m(t_k)) - I_k(x(t_k)) \|.
\]
Since $I_k$ is continuous and $\xi$ is also jointly continuous, then we have
$$\|\xi(s, x_n(s)) - \xi(s, x(s))\| \to 0 \text{ as } n \to \infty,$$
also, $\sum_{0 < t_k < t} \|I_k(x_n(t_k)) - I_k(x(t_k))\| \to 0 \text{ as } n \to \infty,$ therefore $\|(F x_n)(t) - (F x)(t)\| \to 0 \text{ as } n, m \to \infty.$ Consequently, by assumptions [H2] and [H4], it is not difficult to obtain that $\|(F x_n)(t) - (F x)(t)\| \to 0 \text{ as } n \to \infty,$ as follows:

$$\|(F x_n)(t) - (F x)(t)\| \leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds + \sum_{0 < t_k < t} \|I_k(x_n(t_k)) - I_k(x(t_k))\|$$

$$\leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \delta_1 \|x_n - x\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \delta_2 \|x_n - x\| ds + \sum_{0 < t_k < t} \gamma_k \|x_n - x\| \to 0 \text{ as } n \to \infty$$

Thus, $F$ is continuous and completely continuous. Consequently, by Schaefer’s fixed point theorem, one can deduce that $F$ has at least one fixed point on $PC(\mathcal{J}, \mathcal{X})$ which is a solution of the fractional ICP(1.1).

Conversely, assume that $x$ satisfies the fractional integral Eq. (3.1). If $t \in (0, t_1]$ then $x(0) = x_0$ and by using the fact that $^cD^q I^q_t$ is the left inverse of $I^q_t$ and by Lemma (2.1), one can obtain $^cD^q_t x(t) = \xi(t, x(t))$. If $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$ also by Lemma (2.1) and using that fact the Caputo derivative of a constant is equal to zero. It can deduced that $^cD^q_t x(t) = \xi(t, x(t))$ for $t \in (t_k, t_{k+1}]$ and $x(t_k^-) = x(t_k^+) = I_k(x(t_k))$ which completes the proof. \qed

**Lemma 3.2.** The operator $F : PC(\mathcal{J}, \mathcal{X}) \to PC(\mathcal{J}, \mathcal{X})$ is bounded.

**Proof.** It is sufficient to show that for any $\mu > 0$, there exists a constant $K > 0$ such that for each $x \in \beta_\mu = \{\|x\|_{PC} \leq \mu : x \in PC(\mathcal{J}, \mathcal{X})\}$, then we have $\|F x\|_{PC} \leq K$. Now, let $\{x_n\}$ be a sequence on a bounded subset $\mathcal{M} \subset \beta_\mu$, for every $x_n \in \mathcal{M}$ by assumptions [H3] and [H5], we have

$$\|(F x_n)(t)\|_{PC} \leq \|x_0\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x_n(s))\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|\xi(s, x_n(s))\| ds + \sum_{0 < t_k < t} \|J_k(x_n(t_k))\|, \quad k = 1, \ldots, m$$

$$\leq \|x_0\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} [\delta_1 \|x_n\|_{\mathcal{X}} + \delta_2] ds$$
\[ + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} [\delta_1 \|x_n\|^q_1 + \delta_2] ds + \sum_{0 < t_k < t} [\gamma_1 \|x_n\|^q_2 + \gamma_2], \quad k = 1, 2, \ldots, m \]

Which implies that

\[ \|(F_n x_n)(t)\|_{PC} \leq \|x_0\| + \frac{(m + 1)[\delta_1 \mu^n + \delta_2]T^q}{\Gamma(q + 1)} + m[\gamma_1 \mu^n_2 + \gamma_2] := K. \]

Therefore \((F_n x_n)\) is uniformly bounded on \(M\), which implies \(F(M)\) is bounded in \(\beta_\mu \subseteq PC(J, \mathcal{X})\).

**Lemma 3.3.** The operator \(F : PC(J, \mathcal{X}) \to PC(J, \mathcal{X})\) is equicontinuous.

**Proof.** Let \(\{x_n\}\) be a sequence on a bounded subset \(M \subset \beta_\mu\) as we defined in Lemma (3.2). For \(t_1, t_2 \in J\), and \(t_1 < t_2\), we consider

\[
\|(F_n x_n)(t_2) - (F_n x_n)(t_1)\| = \frac{1}{\Gamma(q)} \sum_{0 < t_k < t_2} \int_{t_k}^{t_1} (t - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
- \frac{1}{\Gamma(q)} \sum_{0 < t_k < t_1} \int_{t_k}^{t_1} (t - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
- \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_1 - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t_1} (t_1 - s)^{q-1} \xi(s, x_n(s)) ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t_2} (t_2 - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
- \frac{1}{\Gamma(q)} \int_{0}^{t_1} (t_1 - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t_1} (t_1 - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
- \frac{1}{\Gamma(q)} \int_{0}^{t_1} (t_1 - s)^{q-1} \xi(s, x_n(s)) ds
\]

\[
+ \sum_{0 < t_k < t_2-t_1} I_k(x_n(t_k))\]

As \(t_2 \to t_1\), then it is easy to deduce that the right hand side of the above inequality tends to zero. Therefore, \((F_n x_n)\) is equicontinuous. \(\square\)
Lemma 3.4. The operator $F : PC(J, \mathcal{X}) \to PC(J, \mathcal{X})$ is compact.

Proof. Consider a closed subset $H \subseteq PC(J, \mathcal{X})$. Since $F : PC(J, \mathcal{X}) \to PC(J, \mathcal{X})$ is bounded and equicontinuous then by the Arzela Ascoli theorem, we get $F : PC(J, \mathcal{X}) \to PC(J, \mathcal{X})$ is completely continuous which implies $F(H)$ is a relatively compact subset of $PC(J, \mathcal{X})$. Therefore $F : PC(J, \mathcal{X}) \to PC(J, \mathcal{X})$ is compact. □

Theorem 3.1. Assume that $[H1] - [H5]$ hold, then the fractional ICP(1.1) has at least one solution.

Proof. It is clear that the fixed points of the operator $F$ are solutions of the ICP(1.1). Obviously the operator $F : PC(J, \mathcal{X}) \to PC(J, \mathcal{X})$ is continuous and completely continuous, then we shall prove that $S(F) = \{x \in PC(J, \mathcal{X}) : x = \kappa Fx, \text{ for some } \kappa \in [0, 1]\}$ is bounded. Let $x \in S(F)$, then $x = \kappa Fx$ for some $\kappa \in [0, 1]$.

$$||x(t)||_{PC} \leq \kappa (||Fx(t)||) \leq ||x_0|| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}||\xi(s, x(s))||ds$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}||\xi(s, x(s))||ds + \sum_{0 < t_k < t} ||I_k(x(t_k))||, \quad k = 1, 2, ..., m$$

$$\leq ||x_0|| + \frac{(m + 1)[\delta_1 \mu^{q_1} + \delta_2 T^q]}{\Gamma(q + 1)} + m[\gamma_1 \mu^{q_2} + \gamma_2].$$

The above inequality at the same time with $q_1, q_2 \in [0, 1)$ and by result of Lemma (3.2) show that $S$ is bounded in $PC(J, \mathcal{X})$. As a consequence of Schaefer’s fixed point theorem, we can deduce that $F$ has a fixed point which is a solution of the fractional ICP(1.1). □

Theorem 3.2. Assume that $[H1] - [H5]$ hold, then the set of solutions for the fractional ICP(1.1) is convex.

Proof. By Theorem (3.1), it is obvious that the fractional ICP(1.1) has a solution in $PC(J, \mathcal{X})$. Set $\kappa = 1$, then the set solutions will be defined as $S(F) = \{x \in PC(J, \mathcal{X}) : x = Fx, \text{ for some } \kappa \in [0, 1]\}$. For each $x_1, x_2 \in S(F)$, $\lambda \in [0, 1]$ and $t \in J$, then by definition of $F$, we have

$$\lambda x_1(t) + (1 - \lambda) x_2(t) = \lambda (F x_1)(t) + (1 - \lambda) (F x_2)(t)$$

$$= \lambda[x_0 + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\xi(s, x_1(s))ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}\xi(s, x_1(s))ds]$$

$$+ \sum_{0 < t_k < t} I_k(x_1(t_k))] + (1 - \lambda)[x_0 + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\xi(s, x_2(s))ds$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1}\xi(s, x_2(s))ds + \sum_{0 < t_k < t} I_k(x_2(t_k)))]$$

Thus, \[ \lambda x_1 + (1 - \lambda)x_2 [t] = (\mathcal{F}[\lambda x_1 + (1 - \lambda)x_2]) (t) \]

Therefore, \( \lambda x_1 + (1 - \lambda)x_2 \in S(\mathcal{F}) \) which implies \( S(\mathcal{F}) \) is convex. Hence, the set solutions of ICP(1.1) is convex.

**Theorem 3.3.** Assume that \([H1] - [H5]\) hold, then the fractional ICP(1.1) has a unique solution on \( \mathcal{PC}(\mathcal{J}, \mathcal{X}) \).

**Proof.** It can be easily shown that \( \mathcal{F} \) is a contraction mapping on \( \mathcal{PC}(\mathcal{J}, \mathcal{X}) \) by [H2] and [H4] as follows, for arbitrary \( x, y \in \mathcal{PC}(\mathcal{J}, \mathcal{X}) \), we have

\[
\| (\mathcal{F}x)(t) - (\mathcal{F}y)(t) \| \leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{q-1} \| \xi(s, x(s)) - \xi(s, y(s)) \| ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \| \xi(s, x(s)) - \xi(s, y(s)) \| ds + \sum_{0 < t_k < t} \| I_k(x(t_k)) - I_k(y(t_k)) \|
\]

\[
\leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k-1}^{t_k} (t_k - s)^{q-1} \delta \| x - y \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \delta \| x - y \| ds
\]

\[
+ \sum_{0 < t_k < t} \gamma I_k \| x - y \|, \quad k = 1, 2, ..., m
\]

\[
\| (\mathcal{F}x)(t) - (\mathcal{F}y)(t) \| \leq \left( \frac{(m+1)\delta T^q}{\Gamma(q+1)} + m\gamma I \right) \| x - y \|
\]

Thus, \( \mathcal{F} \) is a contraction mapping on \( \mathcal{PC}(\mathcal{J}, \mathcal{X}) \) with a contraction constant \( \left( \frac{(m+1)\delta T^q}{\Gamma(q+1)} + m\gamma I \right) \). By applying the Banach’s contraction mapping principle we deduce that the operator \( \mathcal{F} \) has a unique fixed point on \( \mathcal{PC}(\mathcal{J}, \mathcal{X}) \). Therefore, the ICP(1.1) has a unique solution which completes the proof. \( \square \)
Example 3.1. Consider the following fractional ICP

\[
\begin{cases}
\frac{d^{\frac{3}{2}}}{dt^{\frac{3}{2}}} x(t) = \frac{|x(t)|}{(1+e^t)(1+|x(t)|)}, & t \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}, \\
x(0) = 0, \\
\Delta x\left(\frac{1}{2}\right) = \frac{1}{3}|x(\frac{1}{2})|.
\end{cases}
\] (3.2)

Set \( q = \frac{2}{3} \), for \((t, x) \in [0, 1] \times [0, +\infty)\), we can define \( \xi(t, x) = \frac{x}{(1+e^t)(1+x)} \). Also, for \( t \in [0, 1] \) we have \( x(t) = \frac{1}{1+e^t} \), and \( I_k(x(t_k)) = \frac{1}{2} x(\frac{1}{2}) \), \( K = 1 \). By Theorem (3.1), we have

\[
|\xi(t, x) - \xi(t, y)| = \frac{1}{(1+e^t)} \left| \frac{x}{1+x} - \frac{y}{1+y} \right|, \quad t \in [0, 1]
\]

\[
\leq \frac{1}{2} \left| \frac{x}{1+x} - \frac{y}{1+y} \right|
\]

\[
\leq \frac{1}{2} |x - y| \Rightarrow \delta_1 = \frac{1}{2}
\]

And,

\[
|\xi(t, x)| = \left| \frac{x}{(1+e^t)(1+x)} \right|, \quad t \in [0, 1]
\]

\[
\leq \frac{1}{2} \left| \frac{x}{1+x} \right| \leq \frac{1}{2} |x| \Rightarrow \delta_1 = \frac{1}{2}, q_1 = 1, q_2 = 0
\]

Next,

\[
|I(x) - I(y)| = \frac{1}{9} |x(\frac{1}{2}) - y(\frac{1}{2})| \Rightarrow \gamma_1 = \frac{1}{9},
\]

\[
|I(x)| = \frac{1}{9} |x(\frac{1}{2})| \Rightarrow \gamma_1 = \frac{1}{9}, q_2 = 1, q_2 = 0
\]

Obviously, it is not difficult to see that all assumptions in Theorem (3.1) are satisfied. Therefore, our results can be used to solve the problem (3.2).

CONCLUSION

We established sufficient conditions for existence of a solution for the ICP (1.1) by using Schaefer’s fixed point theorem, Banach contraction mapping principle besides to topological technique of approximate solutions. Moreover, we studied some of topological properties for the set of solutions. Finally, an example was presented to clarify our results.

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REFERENCES

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