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STEADY-STATE TEMPERATURE ANALYSIS TO 2D ELASTICITY AND THERMO-ELASTICITY PROBLEMS FOR INHOMOGENEOUS SOLIDS IN HALF-PLANE

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ABSTRACT. The concept of temperature distribution in inhomogeneous semi-infinite solids is examined by making use of direct integration method. The analysis is done on the solution of the in-plane steady state heat conduction problem under certain boundary conditions. The method of direct integration has been employed, which is then reduced to Volterra integral equation of second kind, produces the explicit form analytical solution. Using resolvent- kernel algorithm, the governing equation is solved to get present solution. The temperature distribution obtained and calculated numerically and the relation with distribution of heat flux generated by internal heat source is shown graphically.

1. INTRODUCTION

The distribution of temperature is subjected to known temperature and/or heat flux conditions on the surface of solids. Steady-state heat conduction is happens when the heat conduction is constant, so that the special distribution of temperatures in the inhomogeneous solid does not change any further. The interest of researchers to study the analytical solution for elasticity and thermo-elasticity problems has grown very fast due to wide applications to real world. In particular, the models and methodologies which admit the dependence of material properties on inhomogeneous materials developed recently. Among various inhomogeneous materials, functionally graded materials (FGM) have attracted researchers in the past years, whose properties vary continuously from one surface to another. Except for few particular cases, it is impossible to get the analytical solution. To overcome this difficulty, some simplifications were employed.

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Kalynyak[1] used a method of direct integration of equation of equilibrium and continuity in terms of stresses for inhomogeneous cylindrical bodies. Makhovikov[2] discussed the solution of the heat conduction equation, with the use of procedure of solving plane boundary problems. Nohel[3] illustrated the solution of systems of first order differential equations. Porter[4] discussed the procedure of the solution to integral equations with difference kernels applied on finite intervals. Rychachivskyy[5], Tokovyy et al.[6] emphasized on solution of the two dimensional elasticity and thermo-elasticity problems for inhomogeneous (cylinders, strips) planes and semi-planes. Tokovyy et al.[7, 8] and Vigak et al.[9] extended the mentioned method for three dimensional temperature and thermal stress analysis of an inhomogeneous layer, by making use of direct integration method.
Tokovyy et al.[10, 11] used the same technique to study the construction of solution of the plane-quasi-static, non-axially- symmetric elasticity and thermo-elasticity problems for cylindrically anisotropic and radially inhomogeneous hollow cylinders and disks. Tokovyy et al.[12] suggested the procedure of the reduction of plane thermo-elasticity problems in inhomogeneous strip to integral Volterra type equation. Vasil'eva et al.[13] suggested approximate methods for the solution of the temperature field in inhomogeneous medium in one-dimensional case, using the replacement of inhomogeneous medium, by a quasi homogenous medium with effective heat transfer coefficients. Vigak[14, 15, 16] and his followers in [17] has been developed a method for solution of the elasticity and thermo-elasticity problems and used the direct integration method of the given equilibrium and compatibility equations.
Many engineering problems are concerned with the evaluation of the amount of heat transferred through surfaces of a solid. The present paper deals with the determination on temperature field due to generation of internal heat in inhomogeneous solids under steady-state temperature for a half-plane. Herein, we consider an application of the direct integration method. Solution of mentioned problem is reduced to the governing Volterra integral equation of second kind. Moreover, the resolvent-kernel algorithm is applied; the solutions of corresponding heat conduction problem for inhomogeneous solids appear in an explicit form.
In the present research article, an isotropic inhomogeneous solid in half-plane is considered and solution of the in-plane steady-state heat conduction problem is determined.
The key points of this article are as below:

- The governing heat conduction problem for semi-plane is formulated as a boundary value problem [18].
- The method of direct integration is applied to solve the stated heat conduction problem.
- The heat conduction problem is reduced to Volterra- type integral equation.
- Based on resolvent-kernel technique, temperature distribution of an isotropic inhomogeneous solid is derived.
- Being an iterative method, the convergence is established to perform the numerical calculation.

2. PROBLEM FORMULATION

Consider the two dimensional heat conduction problems for half-plane
$R = \{(x, y) \in (-\infty, \infty)\mathrm{X}[0, \infty)\}$ in the dimensionless Cartesian co-ordinate system (x, y).

Temperature and heat flow are important quantities in heat conduction problems. Heat flow in the solid depends on the distribution of temperature and that heat flow is always in the direction of decreasing temperature. Since, heat flux points in the direction of deceasing temperature, minus sign is included in order to make the heat flow a positive quantity. Assuming the properties of isotropic material, the problem is governed by the heat conduction equation

$$\frac{\partial}{\partial x}(K(x)\frac{\partial T(x,y)}{\partial x}) + \frac{\partial}{\partial y}(K(x)\frac{\partial T(x,y)}{\partial y}) = -q(x,y), \tag{2.1}$$

where, $K(x)$ is the thermal conductivity, $T(x,y)$ is the steady-state temperature distribution and $q(x,y)$ denotes the quantity of heat generated by internal heat generated in R. Eq. (2.1) presents the classical equation of quasi-static heat conduction equation when the thermal conductivity is constant.

$$\Delta T(x,y) = -W(x,y), \tag{2.2}$$

where,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

and

$$W(x,y) = \frac{q(x,y)}{k}$$

The temperature distribution can be in steady-state condition can be determined from Eq. (2.1) for $k(x)$ or from Eq. (2.2) for k = constant under boundary condition employed at boundary $x = 0$.

The imposed boundary conditions of the problems are

$$\begin{gathered} T(x,y) = T_0(y), \quad at \;\; x = 0; \\ \frac{\partial T(x,y)}{\partial x} + \alpha_0 T(x,y) = \Phi_0(y), \quad at \;\; x = 0; \\ \frac{\partial T(x,y)}{\partial x} = \beta_0, \quad at \;\; x = 0. \end{gathered} \tag{2.3}$$

Here,α_0, β_0 are constants, $\Phi_0(y)$ and $T_0(y)$ are given functions.

We assume that the temperature field, the density and heat fluxes of heat sources vanishes with $|y|$ when, x tends to infinity. When K is constant, the solution to Eq. (2.2) using condition Eq. (2.3), we have,

$$\iint_R W(x,y)dxdy = \int_{-\infty}^{\infty} [\Phi_0(y) - \alpha_0 T(x,y)]dy. \tag{2.4}$$

Let us consider the resultant temperature is zero.

$$\iint_R T(x,y)dxdy = 0.$$

The condition Eq. (2.4) holds for the case of inhomogeneous material, when $k = k(y)$,in Eq. (2.1).

Applying the Fourier integral transformation defined in [19]

$\bar{f}(x;\omega) = \int_{-\infty}^{\infty} f(x,y)\exp(-i\omega y)dy$;

to Eq. (2.1) and boundary conditions, we get, the second order ordinary differential equation

$$\frac{d^2\bar{T}(x;\omega)}{dx^2} - \omega^2\bar{T}(x;\omega) = \frac{-1}{K(x)}[\bar{q}(x;\omega) + \frac{dk(x)}{dx}\frac{d\bar{T}(x;\omega)}{dx}] \tag{2.5}$$

with boundary conditions,

$$\bar{T}(x;\omega) = \bar{T}_0(\omega), \qquad at\ x = 0; \tag{2.6}$$

$$\frac{\partial\bar{T}(x;\omega)}{\partial x} + \alpha_0\bar{T}(x;\omega) = \bar{\Phi}_0(\omega), \qquad at\ x = 0; \tag{2.7}$$

$$\frac{\partial\bar{T}(x;\omega)}{\partial x} = \beta_0, \qquad at\ x = 0; \tag{2.8}$$

where, ω is the Fourier integral transformation parameter, $i^2 = -1$ for sake of simplicity we are omitting parameter ω from the argument of functions.

A solution to expression (2.5) in R is given as,

$$\begin{aligned}\bar{T}(x) = P\exp(-|\omega|x) + \frac{1}{2|\omega|}\int_0^{\infty}\frac{\bar{q}(\xi)}{k(\xi)}\exp(-|\omega||x-\xi|)d\xi \\ +\frac{1}{2|\omega|}\int_0^{\infty}\frac{1}{k(\xi)}\frac{dk(\xi)}{d(\xi)}\frac{d\bar{T}(\xi)}{d(\xi)}\exp(-|\omega||x-\xi|)d\xi,\end{aligned} \tag{2.9}$$

here $|.|$ denotes the absolute value function and P is a constant.

$$\bar{T}(x) = [P - \frac{\bar{T}(0)}{2|\omega|k(0)}]\exp(-|\omega|x) + \frac{1}{2|\omega|}\int_0^\infty \frac{\bar{q}(\xi)}{k(\xi)}\exp(-|\omega||x-\xi|)d\xi + \int_0^\infty \bar{T}(\xi)K(x,\xi)d\xi,$$

where,

$$K(x,\xi) = \frac{-1}{2|\omega|}\frac{d}{d\xi}[\frac{1}{k(\xi)}\frac{dk(\xi)}{d\xi}\exp(-|\omega||x-\xi|)].$$

Here we apply the resolvent-kernel method to solve expression (2.9), this gives the analytical solution to the temperature distribution in R. The resolvent-kernel is determined as

$$\Re(x,\xi) = \sum_{i=0}^{\infty} K_{i+1}(x,\xi),$$

where, $K_1(x,\xi) = K(x,\xi)$,

$K_{i+1} = \int_0^\infty K(x,\eta)K_i(\eta,\xi)d\eta,\ n = 1,2,...$

Note that, the recurring kernels $K_{i+1} \longrightarrow 0$ as $i \longrightarrow \infty$. Consequently for natural number N,

$\Re(x,\xi) \approx \Re_N(x,\xi) = \sum\limits_{i=0}^{\infty} K_{i+1}(x,\xi).$

The temperature distribution appears as,

$$\bar{T}(x) = (P - \frac{\bar{T}(0)}{2|\omega|k(0)})\tau(x) + \theta(x), \tag{2.10}$$

here,

$\tau(x) = P\exp(-|\omega|x) + \int_0^\infty \exp(-|\omega|\xi)\Re(x,\xi)d\xi$;

$\theta(x) = \frac{1}{2|\omega|}\int_0^\infty \frac{\bar{q}(\xi)}{k(\xi)}[\exp(-|\omega||x-\xi|) + \Re(x,\xi)]d\xi$.

Using above mentioned techniques, Eq. (2.10) is very useful tool for the solution of Volterra integral equations.

2.1. **Determination of Unknown Constant P.** To determine the unknown constant P in Eq. (2.10) , we use either one of the conditions (2.6, 2.7) or (2.8).
Using condition (2.6) into Eq. (2.10)

$P = \frac{\bar{T}(\omega)}{\tau(0)}[1 + \frac{\tau(0)}{2|\omega|k(0)}\frac{dk(0)}{dx}] - \frac{\theta(0)}{\tau(0)}.$

Consequently using value of P in Eq. (2.10), yields,

$$\bar{T}(x) = \frac{\bar{T}_0(\omega) - \theta(0)}{\tau(0)}\tau(x) + \theta(x). \tag{2.11}$$

In case of condition (2.7) the constant P appears as,

$P = [\phi_0(y) - \frac{d\theta(0)}{dx} - \alpha_0\theta(0)][\frac{d\tau(0)}{dx} + \alpha_0\tau(0)]^{-1} + \frac{\bar{T}_0(\omega)}{2|\omega|k(o)}\frac{dk(0)}{dx}.$

Then the temperature is given as,

$$\bar{T}(x) = [\phi_0(y) - \frac{d\theta(0)}{dx} - \alpha_0\theta(0)][\frac{d\tau(0)}{dx} + \alpha_0\tau(0)]^{-1}\tau(x) + \theta(x).$$

Using boundary condition (2.8) into expression (2.10) , the constant P takes a form

$P = \beta_0(\frac{d\tau(0)}{dx})^{-1} + \frac{\bar{T}_0(w)}{2|w|k(o)}\frac{dk(0)}{dx} - \frac{d\theta(0)}{dx}(\frac{d\tau(0)}{dx})^{-1},$

then the temperature distribution is given as

$$\bar{T}(x) = (\beta_0 - \frac{d\theta(0)}{dx})(\frac{d\tau(0)}{dx})^{-1}\tau(x) + \theta(x).$$

3. Numerical Results and Discussion

Applying the formula of inverse Fourier transform

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(x;\omega)\exp(i\omega y)dw. \tag{3.1}$$

We can find the temperature field in half-plane R
To verify the obtained result to problem, consider the half-plane is heated by internal heat source

$$q(x,y) = q_0\delta(y)\delta(x - x_0), \tag{3.2}$$

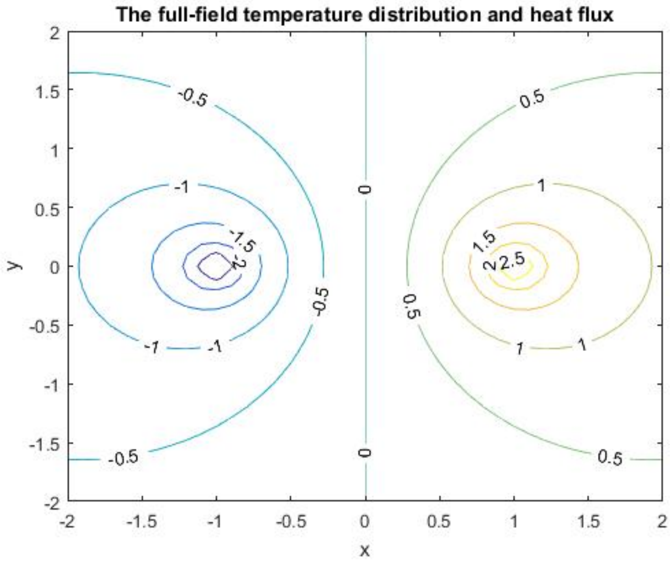

FIGURE 1. Full-field distribution of $\frac{k_0 T(x,y)}{q_0}$

where q_0 is constant, $\delta(.)$ is the Dirac-delta function. We consider the case, at constant temperature ($T_0 = 0$) the boundary $x = 0$ remains of constant temperature. Here the solution should be calculated on the basis of Eq. (2.10).
Consider, thermal conductivity is

$$k(x) = k_0 \exp(\gamma x),$$

where, k_0 and γ are constants. For $\gamma = o$, the coefficient of thermal conductivity is constant, that corresponds to homogeneous material. The resolvent- kernel $\Re(x, \xi) = 0$, thus Eq. (2.11) gives an exact analytical solution for temperature distribution in half-plane R,which gives

$\bar{T}(x) = \frac{1}{2|\omega|} \int_0^\infty \frac{\bar{q}(\xi)}{k(\xi)} [\exp(-|\omega||x - \xi|) - \exp(-|\omega|(x + \xi)] d\xi,$

further,

$$\frac{k_0}{q_o} \bar{T}(x) = \frac{1}{2|\omega|} [\exp(-|\omega||x - \xi|) - \exp(-|\omega|(x + \xi)].$$

Making the use of equation Eq. (3.1) to Eq. (3.2) yields the expression for temperature in homogeneous half-plane,

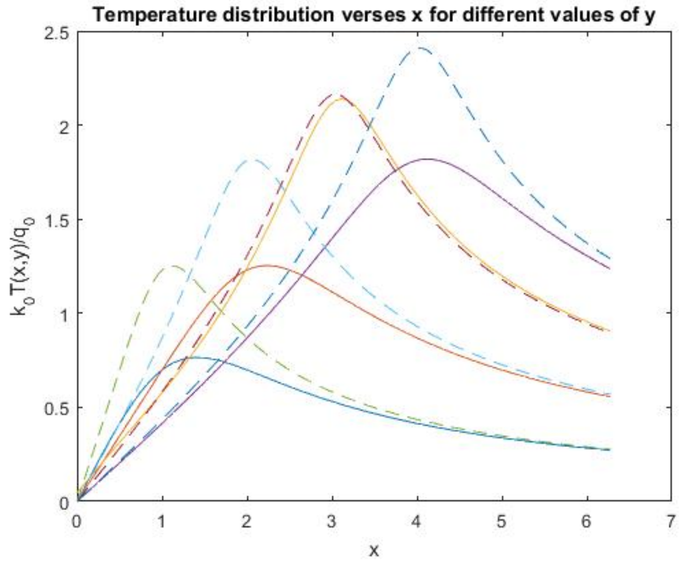

FIGURE 2. Temperature distribution verses coordinate x for $y = 0.5; 1.0$

$$\frac{k_0}{q_o}T(x,y) = \frac{1}{4\pi}\log\frac{(x+x_0)^2+y^2}{(x-x_0)^2+y^2}. \tag{3.3}$$

The full-field temperature distribution (3.3) and the corresponding heat flux for $x_0 = 0$ is depicted in Fig. 1.

Figure 2 shows the temperature distribution verses variable x, for $y = 0.5; 1.0$ and different values $x_0 = 1.0; 2.0; 3.0; 4.0$. The dotted graphs shows distribution for $y = 0.5$ and the dark graphs gives the same for $y = 1.0$. We can observed that, the temperature vanish when moving away from the heat source. When approching towards boundary $x = 0$, the temperature vanish faster than the opposite direction. For the determined temperature (3.3), the heat flux through the boundary $x = 0$ can be calculated as

$$\frac{k_0}{q_0}[\phi_0 - \alpha_0 T_0] = \frac{1}{\pi}\frac{x_0}{x_0^2+y^2}. \tag{3.4}$$

For a special case $T_0 = 0$, the distribution of heat flux (3.4) for different values of x_0 is shown in Fig. 3. It is observed that, the heat flux over the boundary $x = 0$ is distributed with maximum value at $y = 0$ which decreases as the heat source from the boundary.

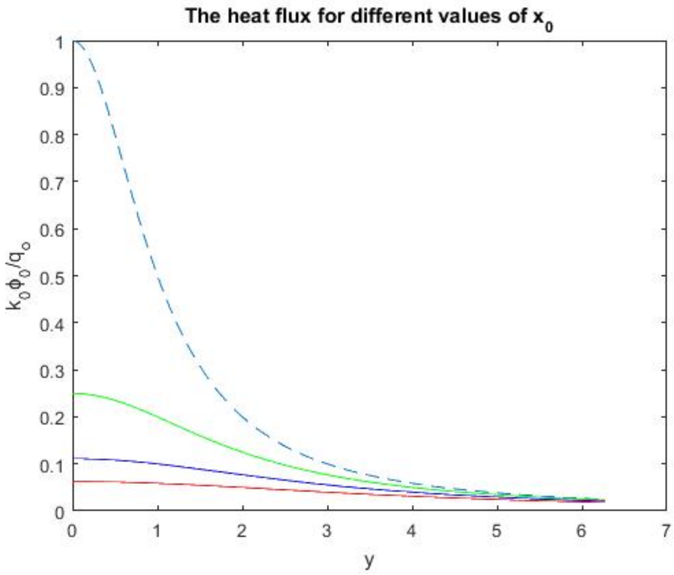

FIGURE 3. The heat flux (3.4) for different values of x_0

4. CONCLUSION

In the present research article, two dimensional heat conduction problems in an inhomogeneous half-plane is considered and determined the explicit form analytical solutions to steady-state temperature field. Using direct integration method solution has been found. The problem is reduced to Volterra integral equation of second kind and further applying the iterated kernel algorithm gives us with the analytical solutions to the temperature distribution, which is examined with the help of heat flux applied on the boundary. The fast convergence of the iterative procedure is happened to perform the numerical calculation. The proposed approach illustrates its efficiency in analyzing the problems for inhomogeneous solids. This technique is useful for finding stresses or displacements for an isotropic inhomogeneous half-plane, in case of the plane thermo-elasticity problems.

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