CONVECTION IN A HORIZONTAL POROUS LAYER UNDERLYING A FLUID LAYER IN THE PRESENCE OF NON LINEAR MAGNETIC FIELD ON BOTH LAYERS

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Abstract: A linear stability analysis applied to a system consist of a horizontal fluid layer overlying a layer of a porous medium affected by a vertical magnetic field on both layers. Flow in porous medium is assumed to be governed by Darcy’s law. The Beavers-Joseph condition is applied at the interface between the two layers. Numerical solutions are obtained for stationary convection case using the method of expansion of Chebyshev polynomials. It is found that the spectral method has a strong ability to solve the multi-layered problem and that the magnetic field has a strong effect in his model.

1. INTRODUCTION

The onset of convection in a system consisting of a horizontal fluid layer overlying a layer of a porous layer when the system is heated from below has been considered first by Sun (1973) who showed that the critical Rayleigh number in the porous layer decreases continuously as the thickness of the fluid layer is increased. He used the shooting method to solve the linear stability equations. Nield (1977) formulated the problem with surface-tension effects at a deformable upper surface and obtained asymptotic solutions for small wave numbers for a constant heat-flux boundary condition. Sun and Nield used Darcy’s law in formulating the equations for porous layer and Somerton and Catton (1982) used the Brinkman term in the equation of motion to solve the problem using Galerkin method. Chen and Chen (1988) considered the problem with temperature and salinity gradients existing in both layers. Their investigation assumed stationary instability from the outset and they used a

Key Words:
shooting technique based on fourth order Runge-Kutta approximations for integration of all differential equations. Chen et. al (1991) studied the problem with anisotropic permeability and thermal diffusivity in the porous layer. Flow in porous layer was assumed to be governed by Darcy’s law. The linearized stability equations were solved using shooting method. In the present study, we shall emulate the work of Chen and Chen (1988) in the presence of a vertical magnetic field. i.e. we shall consider the onset of thermal convection in a horizontal porous layer, affected by a vertical magnetic field, superposed by a fluid layer. The flow in porous layer is assumed to be governed by Darcy’s law. The linear stability equations are solved using expansion of Chebyshev polynomials. This method has been used by Abdullah (1991) in the study of the Benard problem in the presence of a non-linear magnetic fluid and by Lindsay and Ogden (1992) in the implementation of spectral methods resistant to the generation of spurious eigenvalues. Lamb (1994) used this method to investigate an eigenvalue problem arising from a model discussing the instability in the earth’s core. The method possesses excellent convergence characteristics and effectively exhibits exponential convergence rather than finite power convergence.

2. MATHEMATICAL FORMULATION

Let $L_1$ and $L_2$ be two horizontal layers such that the bottom of the layer $L_1$ touches the top of the layer $L_2$. A right handed system of Cartesian coordinates $(x_i, \ i=1,2,3)$ is chosen so that the interface is the plane $x_3 = 0$, the top boundary of $L_1$ is $x_3 = d_f$ and the lower boundary of $L_2$ is $x_3 = -d_m$. Suppose that the upper layer $L_1$ is filled with an incompressible thermally and electrically conducting viscous fluid field whereas the lower layer $L_2$ is occupied by a porous medium permeated by the fluid and is subjected to a constant vertical magnetic field.
Gravity acts in the negative direction and the porous medium is heated at its lower boundary. Convection takes place in which temperature driven buoyancy effects are damped by viscous effects. A stationary fluid with a thermal gradient in the $x_3$ direction (the so called “conduction solution”) is one possible solution to this problem and so it is natural to investigate its stability.

The fluid flow in the porous layer $L_2$, with thickness $d_m$, is governed by Darcy's law, whereas the fluid flow in the upper layer $L_1$, with thickness $d_f$, is governed by Navier-Stokes equations. Convection is driven by the temperature dependence of the fluid density. Typically, the Oberbeck-Boussinesq approximation is made in which local thermal equilibrium, heating from viscous dissipation, radiation effects etc. are ignored as are variations in fluid density except where they occur in the momentum equation. Let $T$ denote the Kelvin temperature of the fluid and $T_0$ be a constant reference Kelvin temperature. For the purpose of this work, the fluid density $\rho_f$ is related to $T$ by

$$\rho_f = \rho_0 [1 - \alpha(T - T_0)]$$  \hspace{1cm} (2.1)

where $\rho_0$ is the density of the fluid at $T_0$ and $\alpha$ (suppose constant) is the coefficient of volume expansion of the fluid.

Let $V_m, H_m, B_m, J_m$ and $E_m$ be respectively the velocity of the fluid in the porous medium layer, magnetic field, magnetic induction, current density and electric field. the incompressibility of the fluid and the non-existence of magnetic monopoles require that $V_m$ and $B_m$ are both solenoidal vectors. Hence

$$\text{div } V_m = 0,$$  \hspace{1cm} (2.2)

$$\text{div } B_m = 0,$$  \hspace{1cm} (2.3)
Suppose that the magnetization in the fluid is directly proportional to the applied field and that the fluid behaves like an Ohmic conductor so that $H_m, B_m, J_m$ and $E_m$ are connected by the relations

$$H_m = \rho \phi B_m, \quad \phi = \frac{1}{B} \frac{\partial \xi}{\partial B}, \quad \xi = \xi(\rho, B)$$

$$J_m = \sigma (E_m + V_m \times B_m)$$  \hspace{1cm} (2.4)

and the Maxwell equations

$$\text{curl } E_m = -\frac{\partial B_m}{\partial t},$$

$$J_m = \frac{1}{4\pi} \text{curl } H_m,$$  \hspace{1cm} (2.5)

where $\mu_m$ (constant) is the magnetic permeability, $\sigma$ is the electrical conductivity and the displacement current has been neglected in the second of these Maxwell equations as is customary in situation when free charge is instantaneously dispersed. On taking the curl of equations (2.5)$_2$ and replacing the electric field by the Maxwell equation (2.5)$_1$, the magnetic field $H_m$ is now readily seen to satisfy the partial differential equation

$$\eta_m \text{curl curl } H_m = -\frac{\partial H_m}{\partial t} + \text{curl}(V_m \times H_m)$$

$$\hspace{1cm} (2.6)$$

where $\eta_m = (4\pi \mu_m \sigma)^{-1}$ is the electrical resistivity. Equation (2.6) is now reworked using standard vector identities to yield

$$\frac{\partial H_m}{\partial t} = (H_m \cdot \nabla) V_m - (V_m \cdot \nabla) H_m + \eta_m \text{curl } J_m.$$  \hspace{1cm} (2.7)

The relation (2.4) and (2.5) can be used to recast the Lorentz force $J \times B$ into

$$J_m \times B_m = \frac{1}{4\pi} (\text{curl } H_m) \times \left( \frac{H_m}{\rho \phi} \right) = \frac{1}{4\pi} \left[ H_m \cdot \left( \nabla \frac{H_m}{\rho \phi} \right) - \nabla \left( \frac{H_m^2}{2 \rho \phi} \right) \right]$$  \hspace{1cm} (2.8)
The field equations for this problem are written separately for the overlying fluid layer and porous medium layer. The governing equations for the fluid layer are

\[
\rho_0 \left( \frac{\partial V_f}{\partial t} + V_f \cdot \nabla V_f \right) = -\nabla P_f + \mu \nabla^2 V_f + \rho_f g + \frac{1}{4\pi} \mathbf{H}_f \cdot \nabla \left( \frac{\mathbf{H}_f}{\rho \phi} \right),
\]

\[
(\rho c_p)_f \left( \frac{\partial T_f}{\partial t} + V_f \cdot \nabla T_f \right) = k_f \nabla^2 T_f,
\]

\[
\frac{\partial H_f}{\partial t} = (\mathbf{H}_f \cdot \nabla) V_f - (V_f \cdot \nabla) H_f + \eta_f \text{curl} \mathbf{J}_f.
\]

where \( T_f \) is the Kelvin temperature of the fluid layer, \( P_f \) is the hydrostatic pressure, \( g \) is the acceleration due to gravity, \( \mu \) is the dynamic viscosity of the fluid, \( (\rho c_p)_f \) is the heat capacity per unit volume of the fluid at constant pressure and \( k_f \) is the thermal conductivity of the fluid. The governing equations for porous medium are given by

\[
\frac{\rho_0}{\phi} \frac{\partial V_m}{\partial t} = -\nabla P_m - \frac{\mu}{k} V_m + \frac{1}{4\pi} \mathbf{H}_m \cdot \nabla \left( \frac{\mathbf{H}_m}{\rho \phi} \right),
\]

\[
(\rho c)_m \frac{\partial T_m}{\partial t} + (\rho c_p)_m V_m \cdot \nabla T_m = k_m \nabla^2 T_m,
\]

\[
\frac{\partial H_m}{\partial t} = (\mathbf{H}_m \cdot \nabla) V_m - (V_m \cdot \nabla) H_m + \eta_m \text{curl} \mathbf{J}_m.
\]

where \( T_m \) is the Kelvin temperature of the porous medium layer, \( V_m \) is the solenoidal seepage velocity, \( P_m \) is the hydrostatic pressure, \( k \) is the permeability of the porous medium, \( \phi \) is its porosity, \( k_m \) is the overall thermal conductivity of the porous medium and \( (\rho c)_m \) is the overall heat capacity per unit volume of porous medium at constant pressure. In fact

\[
(\rho c)_m = \phi (\rho c_p)_f + (1 - \phi) (\rho c_p)_m
\]

where \( (\rho c_p)_m \) is the heat capacity per unit volume of porous substrate. The convection problem is completed by the specification of boundary conditions at the upper surface of the viscous fluid layer, at the interface between the fluid and porous medium layers
and at the lower boundary of the porous medium layer. Many combination of boundary conditions are possible but for comparison with Chen and Chen (1988) we shall assume that $x_3 = d_f$ is rigid and held at constant temperature $T_u$, whereas $x_3 = -d_m$ is assumed to be impenetrable and at constant temperature $T_m$. In terms of $w_f$ and $w_m$ the axial velocity components of the fluid in $L_1$ and $L_2$ respectively, these requirements leads to the three conditions

$$T_f(d_f) = T_u, \quad w_f(d_f) = 0, \quad \frac{\partial w_f(d_f)}{\partial x_3} = 0, \quad \frac{\partial H(d_f)}{\partial x_3} = 0. \quad (2.11)$$

on the top boundary of $L_1$ and the conditions

$$T_m(-d_m) = T_i, \quad w_m(-d_m) = 0, \quad \frac{\partial H(-d_m)}{\partial x_3} = 0. \quad (2.12)$$

on the lower boundary of $L_2$ where $H$ is the third component of the magnetic field.

The fluid/porous medium interface boundary conditions are based on the assumption that temperature, heat flux and normal fluid velocity are continuous across the interface. Thus

$$T_m(0) = T_f(0), \quad k_m \frac{\partial T_m(0)}{\partial x_3} = k_f \frac{\partial T_f(0)}{\partial x_3},$$

$$w_m(0) = w_f(0), \quad -P_f(0) + 2\mu \frac{\partial w_f(0)}{\partial x_3} = -P_m(0), \quad (2.13)$$

$$H_m(0) = H_f(0), \quad k_m \frac{\partial H_m(0)}{\partial x_3} = k_f \frac{\partial H_f(0)}{\partial x_3}.$$ 

this leaves two final conditions to be specified on the interface. One of these is related to the magnetic field which is

$$\frac{\partial H_f(0)}{\partial x_3} = 0, \quad \frac{\partial H_m(0)}{\partial x_3} = 0. \quad (2.14)$$

and the final one is due to Beavers and Joseph (1967) which has the form
\[
\frac{\partial u_f(0)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}} (u_f - u_m), \quad \frac{\partial v_f(0)}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}} (v_f - v_m),
\]

where \( u_f, v_f \) are the limiting tangential components of the fluid velocity as the interface is approached from the fluid layer \( L_1 \), whereas \( u_m, v_m \) are the same limiting components of tangential fluid velocity as the interface is approached from the porous layer \( L_2 \).

Suppose that the static solution is now perturbed so that the velocity, pressure, temperature and magnetic field in the fluid and porous layers are respectively

\[
v_f, \quad P_f + p_f, \quad T_0 - (T_0 - T_u) \frac{x_3}{d_f} + \theta_f, \quad He_{f} + h_m,
\]

and

\[
v_m, \quad P_m + p_m, \quad T_0 - \left( T_i - \frac{T_u}{T_0} \right) \frac{x_3}{d_m} + \theta_m.
\]

In the fluid layer we shall introduce the non-dimensional spatial coordinates \( \hat{x}_f \), time \( \hat{t}_f \), perturbed velocity \( \hat{v}_f \), pressure \( \hat{p}_f \), and temperature \( \hat{\theta}_f \) by the definitions

\[
x = d_f \hat{x}_f, \quad t_f = \frac{d_f^2}{\lambda_f} \hat{t}_f, \quad v_f = \frac{\lambda_f}{d_f} \hat{v}_f, \quad P_f = \frac{\mu \lambda_f}{d_f^2} \hat{p}_f, \quad \theta_f = |T_0 - T_u| \hat{\theta}_f. \tag{2.17}
\]

where \( \lambda_f \) is the thermal diffusivity of the fluid phase defined by \( \lambda_f = \frac{k_f}{(\rho c_p)_f} \).

A similar procedure is applied to the porous medium layer in which non-dimensional spatial coordinates \( \hat{x}_m \), time \( \hat{t}_m \), perturbed velocity \( \hat{v}_m \), pressure \( \hat{p}_m \), magnetic field \( \hat{h}_m \) and temperature \( \hat{\theta}_m \) are introduced by the definitions
\[ \begin{align*}
    x_m &= d_m \tilde{x}_m, & t_m &= \frac{d_m^2}{\lambda_m} \tilde{t}_m, & v_m &= \frac{\lambda_m}{d_m} \tilde{v}_m, \\
    p_m &= \frac{\mu \lambda_f}{K} \tilde{p}_m, & h_m &= \frac{H_m \lambda_m}{\eta_m} \tilde{h}_m, & \theta_m &= |T_0 - T_u| \tilde{\theta}_m.
\end{align*} \tag{2.18} \]

where \( \lambda_m \) is the thermal diffusivity of the porous medium layer defined by
\[ \lambda_m = \frac{k_m}{(\rho c_p)_f}. \]

Where \( \beta = \text{sign}(T_0 - T_u) = \text{sign}(T_1 - T_0) \) and hat superscript has been dropped although the variables are non-dimensional. By taking the \textit{curl curl} of the momentum equation in each layer then taking the third component of the equations in each layer.

We now look for a solution of the form
\[ \psi(t, x) = \psi(x_3) \exp[(rx_1 + qx_2) + \sigma \, t] \]

It follows from equations (2.33) and (2.34) that
\[ \begin{align*}
    \frac{\sigma_f}{Pr_f} (D_f^2 - a_f^2) w_f - \sigma_f Q P^{-1} m_f D_f h_f &= (D_f^2 - a_f^2)^2 w_f - Ra_f a_f^2 \theta_f - Q D_f^2 w_f, \\
    \sigma_f \theta_f &= w_f + (D_f^2 - a_f^2) \theta_f, \\
    \sigma_f P^{-1} m_f h_f &= (D_f^2 - a_f^2) h_f + D_f w_f, \\
    -\frac{Da}{\phi} \frac{\sigma_m}{Pr_m} (D_m^2 - a_m^2) w_m - \sigma_m Q P^{-1} m_m D_m h_m &= (D_m^2 - a_m^2)^2 w_m + Ra_m a_m^2 \theta_m - Q D_m^2 w_m, \\
    G_m \sigma_m \theta_m &= w_m + (D_m^2 - a_m^2) \theta_m, \\
    \sigma_m P^{-1} m_m h_m &= (D_m^2 - a_m^2) h_m + D_m w_m.
\end{align*} \tag{2.19} \]

where the parameters \( \varepsilon, \hat{a} \) and \( \hat{k} \) are defined by
\[ \begin{align*}
    \varepsilon &= \frac{\hat{d}}{\hat{k}}, & \hat{a} &= \frac{d_m}{d_f}, & \hat{k} &= \frac{k_m}{k_f}.
\end{align*} \]
and $G_m = \left( \frac{\rho c}{\rho c_p} \right)_f$, and

$$P_r_f = \frac{\nu}{\lambda_f}, \quad Ra_f = \frac{g \alpha a_f^3 |T_0 - T_u|}{\nu \lambda_f}, \quad P_m_f = \frac{\eta_f}{\lambda_f}.$$  

$$P_r_m = \frac{\nu}{\lambda_m}, \quad Da = \frac{K}{a_m^2}, \quad Ra_f = \frac{g \alpha k d_f^3 |T_f - T_0|}{\nu \lambda_m},$$  

$$P_m_m = \frac{\eta_m}{\lambda_m}, \quad Q = \frac{\mu H_m^2 d_m^2}{4 \rho_0 \pi \nu \eta_m}.$$  

where $a_m^2 = r_m^2 + q_m^2$, $a_f^2 = r_f^2 + q_f^2$ are non-dimensional wave numbers in the porous medium and fluid layers respectively and where $D(\cdot) = \frac{\partial}{\partial x_3}(\cdot)$, $a_f = \hat{d}a_m$,  

$\sigma_f = \frac{\hat{d}^2}{k} \sigma_m$. The final boundary conditions are:

Upper boundary $x_3 = 1$

$w_f = 0, \quad D_f w_f = 0, \quad \theta_f = 0, \quad D_f h_f = 0.$ \hspace{1cm} (2.20)

Middle boundary $x_3 = 0$

$$\theta_f = \varepsilon_\tau \theta_m, \quad D_f \theta_f = D_m \theta_m, \quad w_m = \varepsilon_\tau w_f,$$

$$h_f = \frac{k}{n} h_m, \quad D_f h_f = \frac{1}{\varepsilon_\tau n} D_m h_m,$$

$$\varepsilon_\tau \hat{d} \left( D_f w_f - \frac{\hat{d} \sqrt{Da}}{\alpha_{BJ}} D_f^2 w_f \right) = D_m w_m,$$ \hspace{1cm} (2.21)

$$\hat{d}^3 \varepsilon_\tau Da \left( D_f^3 w_f - 3 \varepsilon_\tau D_f w_f - \frac{\sigma_f}{P_f} D_f w_f \right) = -\left( \frac{Da \sigma_m}{\phi P_m} + 1 \right) D_m w_m.$$  

Lower boundary $x_3 = -1$
\[ w_m = 0, \quad \theta_m = 0, \quad D_m h_m = 0. \quad (2.22) \]

3. RESULTS AND DISCUSSION

The eigenvalue problem consist of a eight order ordinary differential equation in the fluid layer and a six order ordinary differential equation in the porous layer with 14 boundary conditions. This problem is solved using spectral method based on series expansion of Chebyshev polynomials. In producing the results, \( \sigma_f \) and \( \sigma_m \) are set to zero identically which corresponds to the stationary convection instability, where the relation between \( \sigma_f \) and \( \sigma_m \) are given by

\[ \sigma_f = \frac{\hat{d}^2}{\hat{k}} \sigma_m \]

Numerical results and stability curves are obtained for the problem, with thermal conductivity ratio \( \hat{k} = 1.43 \), Darcy number \( = 4 \times 10^{-6} \), Beavers-Joseph constant \( \alpha_{Bj} = 0.1 \) and for a variety reciprocal depth ratio ranging from 0.33 to 0.1. the results of this paper are illustrated in figures (1) – (4). They are qualitatively and quantitatively similar to those produced by Bukhari (1997) in the absence of magnetic field in both layers. Bukhari has showed that the numerical results produced by Chen and Chen (1988) have a large rounding error due to the method used which is a 4th order Runge-Kutta method and he showed that the spectral methods have a strong ability to solve the multi-layered problems and produces accurate results.

Figure (1) shows the relation between \( a_m \) and \( Ra_m \) for different values of the depth ratio of \( \hat{d} \) when the Chandrasekhar number \( Q = 100 \). It is clear from the figure that the Rayleigh number in the porous layer decreases continuously as the thickness of the layer increases. The results corresponding to \( Q = 500,1000,10000 \) are displayed in figures (2)- (4) respectively. It is clear from these figures that the
Rayleigh number increases as the Chandrasekhar number increases. i.e. the magnetic field has a stabilizing on the system.

REFERENCES


Euler-Maruyama Numerical solution of some stochastic functional differential equations

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Abstract

In this paper we study the numerical solutions of the stochastic functional differential equations of the following form

$$du(x,t) = f(x,t,u_t)dt + g(x,t,u_t)dB(t), \ t > 0$$

with initial data $u(x,0) = u_0(x) = \xi \in L^0_{F_0}([\tau,0]; \mathbb{R}^n)$.

Here $x \in \mathbb{R}^n$, ($\mathbb{R}^n$ is the $n$-dimensional Euclidean space),

$f : C([-\tau,0]; \mathbb{R}^n) \times \mathbb{R}^{n+1} \to \mathbb{R}^n$, $g : C([-\tau,0]; \mathbb{R}^n) \times \mathbb{R}^{n+1} \to \mathbb{R}^{n \times m}$,

$u(x,t) \in \mathbb{R}^n$ for each $t$, $u_t = u(x,t + \theta) : -\tau \leq \theta \leq 0 \in C([-\tau,0]; \mathbb{R}^n)$, and $B(t)$ is an $m$-dimensional Brownian motion.

Keywords: Euler-Maruyama, stochastic functional differential equations, local Lipschitz condition, linear growth condition, convergence theory.

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1-Introduction

The numerical solutions of the stochastic differential equations studied in many papers (see [1],[2],[3],[4], [5], [6],[7],[8],[9]). In this paper we study the Euler-Maruyama numerical solution of the SFDE

\[ du(x, t) = f(x, t, u_t)dt + g(x, t, u_t)dB(t), \quad t > 0 \]

with initial data \( u(x, 0) = u_0(x) = \xi \in L^p_{F_0}([\tau, 0]; R^n) \).

Here \( x \in R^n \) \((R^n\) is the \( \nu \) - dimensional Euclidean space),

\[ f : C([\tau, 0]; R^n) \times R^{\nu+1} \to R^n, \quad g : C([\tau, 0]; R^n) \times R^{\nu+1} \to R^{n \times m} \]

\( u(x, t) \in R^n \) for each \( t, u_t = u(x, t + \theta) : -\tau \leq \theta \leq 0 \in C([\tau, 0]; R^n) \), and \( B(t) \) is an \( m \)-dimensional Brownian motion (see [10],[11],[12],[13]). The initial data \( \xi \) is an \( F_0 \)-measurable \( C([\tau, 0]; R^n) \)-valued random variable such that \( E \| \xi \|^p < \infty \) for some \( p > 2 \). In the next section we introduce the Euler-Maruyama method for SFDEs, and we state our main result that the Euler-Maruyama numerical solutions convergence strongly to the exact solution if \( f \) and \( g \) satisfy local Lipschitz condition and the linear growth condition.

2- The Euler-Maruyama Method

Throughout this paper we use the following notations. Let

\[ \sup_x |u(x, t)| = \| u(\cdot, t) \|, \]

\( \| \cdot \| \) be the Euclidean norm in \( R^n \). If \( A \) is a vector or matrix, its transpose is denoted by \( A^T \). If \( A \) is a matrix, its trace norm is denoted by \( |A| = \sqrt{\text{trace}(A^TA)} \). Let \( R_+ = [0, \infty) \), and let \( \tau > 0 \). Denote by \( C([\tau, 0]; R^n) \) the family of continuous functions from \([\tau, 0]\) to \( R^n \) with norm

\[ \| \phi \| = \sup_{-\tau \leq \theta \leq 0} \| \phi(\cdot, \theta) \| \]
Let \((\Omega, F, \{F_t\}_{t \geq 0}, P)\) be a complete probability space with a filtration \(\{F_t\}_{t \geq 0}\) satisfying the usual condition (that is, it is increasing and right continuous, while \(F_0\) contains all \(P\)-null sets). Let \(B(t) = (B_1(t), ..., B_m(t))^T\) be an \(m\)-dimensional Brownian motion defined on the probability space. Let \(p > 0\), and denote by \(L^p_{F_0}([-\tau, 0]; R^n)\) the family of \(F_0\)-measurable \(C([-\tau, 0]; R^n)\)-valued random variables such that \(E \| \xi \|^p < \infty\). If \(u(x, t)\) is an \(R^n\)-valued stochastic process on \(t \in [-\tau, \infty)\), we let \(u_t = \{u(x, t + \theta) : -\tau \leq \theta \leq 0\}\) for \(t \geq 0\). Let

\[
f : C([-\tau, 0]; R^n) \times R^{\nu+1} \to R^n, \quad g : C([-\tau, 0]; R^n) \times R^{\nu+1} \to R^{n \times m}.
\]

In this paper we impose the following hypotheses.

**Assumption 2.1 (The local Lipschitz condition).** For each integer \(j \geq 1\), there is a right-continuous nondecreasing function \(\mu_j : [-\tau, 0] \to R_+\) such that

\[
\| f(., t, \phi) - f(., t, \psi) \|^2 \vee \| g(., t, \phi) - g(., t, \psi) \|^2 \leq \int_{-\tau}^0 \| \phi(., \theta) - \psi(., \theta) \|^2 d\mu_j(\theta)
\]

for those \(\phi, \psi \in C([-\tau, 0]; R^n)\) with \(\| \phi \| \vee \| \psi \| \leq j\), where the integral is of the Lebesgue Stieltjes type.

**Assumption 2.2 (The linear growth condition).** There is a constant \(K > 0\) such that

\[
\| f(., t, \phi) - g(., t, \psi) \|^2 \leq K(1 + \| \phi \|^2)
\]

for all \(\phi \in C([-\tau, 0]; R^n)\).

Consider the \(n\)-dimensional SFDE:

\[
du(x, t) = f(x, t, u_t)dt + g(x, t, u_t)dB(t), \quad t > 0 \quad (2.1)
\]

with initial data \(u(x, 0) = u_0(x) = \xi\). We impose the following condition on the initial data.

**Assumption 2.3.** \(\xi \in L^p_{F_0}([-\tau, 0]; R^n)\) for some \(p > 2\). We can therefore state the
following theorem.

Theorem 2.1. Under assumptions 2.1 - 2.3, for any $T > 0$ there is a constant $C > 0$ such that equation (2.1) has a unique continuous solution $u(x, t)$ on $t \geq -\tau$. Moreover, the solution has the property that

$$E(\sup_{-\tau \leq t \leq T} \| u(\cdot, t) \|^{p}) \leq 2^{(p+4)/2}(1 + E \| \xi \|^{p})e^{CT}. \quad (2.2)$$

In other words, pth moment of the solution is finite.

Let us now introduce a numerical scheme for the SFDE (2.1); we refer to it as the Euler Maruyama method. Let the step size $\Delta \in (0,1)$ be a fraction of $\tau$, namely $\Delta = \tau/N$ for some integer $N > \tau$. The discrete Euler-Maruyama approximate solution $\bar{u}(x, k\Delta), \ k \geq -N$ is defined as follows:

$$\left\{ \begin{array}{ll}
\bar{u}(x, k\Delta) = \xi(k\Delta), & -N \leq k \leq 0 \\
\bar{u}(x, (k+1)\Delta) = \bar{u}(x, k\Delta) + f(x, k\Delta, \bar{u}_{k\Delta})\Delta + g(x, k\Delta, \bar{u}_{k\Delta})\Delta B_{k}, & k \geq 0
\end{array} \right. \quad (2.3)$$

where $\Delta B_{k} = B((k+1)\Delta) - B(k\Delta)$ and $\bar{u}_{k\Delta} = \{\bar{u}_{k\Delta}(x, \theta) : -\tau \leq \theta \leq 0\}$ is a $C([-\tau, 0]; R^{n})$-valued random variable defined as follows:

$$\bar{u}_{k\Delta}(x, \theta) = \bar{u}_{k\Delta}(x, (k+1)\Delta) + \frac{\theta - i\Delta}{\Delta}[\bar{u}(x, (k+i+1)\Delta) - \bar{u}(x, (k+i)\Delta)]$$

for

$$i\Delta \leq \theta(i+1)\Delta, \ i = -N, -(N-1), ..., -1. \quad (2.4)$$

That is $\bar{u}(x, \cdot)$ is the linear interpolation of $\bar{u}(x, (k-N)\Delta), \bar{u}(x, (k-N+1)\Delta), ..., \bar{u}(x, k\Delta)$.

We can rewrite (2.4) as

$$\bar{u}_{k\Delta}(x, \theta) = \frac{\Delta - (\theta - i\Delta)}{\Delta} \bar{u}(x, (k+i)\Delta) + \frac{\theta - i\Delta}{\Delta} \bar{u}(x, (k+i+1)\Delta),$$

which yields
\[
\| \tilde{v}_{k\Delta}(., \theta) \| = \frac{\Delta - (\theta - i\Delta)}{\Delta} \| \tilde{v}(., (k + i)\Delta) \| + \frac{\theta - i\Delta}{\Delta} \| \tilde{v}(., (k + i + 1)\Delta) \| \\
\leq \| \tilde{v}(., (k + i)\Delta) \| \vee \| \tilde{v}(., (k + i + 1)\Delta) \| .
\]

We therefore have

\[
\| \tilde{v}_{k\Delta} \| = \max_{-N \leq i \leq 0} \| \tilde{v}(., (k + i)\Delta) \| \quad \text{for all } k \geq 0. \quad (2.5)
\]

In our analysis it will be more convenient to use continuous-time approximations. We hence introduce the \( C([-.\tau, 0]; R^n) \)-value step process

\[
\tilde{v}_t = \sum_{k=0}^{\infty} \tilde{v}_{k\Delta} 1_{[k\Delta, (k+1)\Delta)}(x, t), \quad t \geq 0, \quad (2.6)
\]

and we define the continuous Euler-Maruyama approximate solution as follows:

\[
v(x, t) = \begin{cases} 
\xi(x, t), & -\tau \leq t \leq 0 \\
\xi_0(x) + \int_0^t f(x, s, \tilde{v}_s)ds + \int_0^t g(x, s, \tilde{v}_s)dB(s), & t \geq 0.
\end{cases} \quad (2.7)
\]

It should be pointed out that the \( C([-.\tau, 0]; R^n) \)-value process \( \tilde{v}_t \) is simply defined by (2.6), but we do not define here an \( R^n \)-valued continuous process \( \bar{v}(x, t) \) from which \( \tilde{v}_t \) is then induced by

\[
\tilde{v}_t = \{ \bar{v}(x, t + \theta) : -\tau \leq \theta \leq 0 \}.
\]

It should also be pointed out that the reason why we do not use the linear interpolation of \( \bar{v}(x, k\Delta) \) as a continuous-time approximation for \( u(x, t) \), instead using \( v(x, t) \) from (2.7) is because the linear interpolation of \( \bar{v}(x, k\Delta) \) is not \( F_t \)-adapted. It follows from (2.7) that for any \( t \geq 0 \) that satisfy \( k\Delta \leq t \),

\[
v(x, t) = \xi_0(x) + \int_0^{k\Delta} f(x, s, \tilde{v}_s)ds + \int_0^{k\Delta} g(x, s, \tilde{v}_s)dB(s) + \int_{k\Delta}^t f(x, s, \tilde{v}_s)ds + \int_{k\Delta}^t g(x, s, \tilde{v}_s)dB(s) \\
= \bar{v}(x, k\Delta) + \int_{k\Delta}^t f(x, s, \tilde{v}_s)ds + \int_{k\Delta}^t g(x, s, \tilde{v}_s)dB(s). \quad (2.8)
\]
In particular, we observe that $v(x, k\Delta) = \overline{v}(x, k\Delta)$ for all $k \geq -N$. That is, the discrete and continuous Euler-Maruyama approximate solutions coincide at the gridpoints. It is then obvious that

$$\|\overline{v}_{k\Delta}\| \leq \|v_{k\Delta}\|, \text{ for all } k \geq 0. \quad (2.9)$$

Moreover, for any $t \geq 0$, let $[t/\Delta]$ be the integer part of $t/\Delta$. Then

$$\|\overline{v}_t\| = \|\overline{v}_{[t/\Delta]\Delta}\| \leq \|v_{[t/\Delta]\Delta}\| \leq \sup_{-\tau \leq s \leq t} \|v(., s)\|. \quad (2.10)$$

This property will be used frequently in what follows, without further explanation. To illustrate our numerical scheme, as well as to see why we call it the Euler-Maruyama method, let us consider a special SFDE

$$du(x, t) = F(x, t, D(u_t))dt + G(x, t, D(u_t))dB(t), \quad (2.11)$$

where

$$F : \mathbb{R}^{n+\nu+1} \to \mathbb{R}^n, \quad G : \mathbb{R}^{n+\nu+1} \to \mathbb{R}^{n \times m},$$

and is a linear operator from $C([-\tau, 0]; \mathbb{R}^n)$ to $\mathbb{R}^n$ given by

$$D(\phi) = \frac{1}{\tau} \int_{-\tau}^{0} \phi(x, \theta)d\theta,$$

$\phi \in C([-\tau, 0]; \mathbb{R}^n)$; that is, $D$ is an average operator. In this case, the discrete approximate solution (2.3) takes the following simple form

$$\left\{ \begin{array}{ll}
\overline{v}(x, k\Delta) = \xi(k\Delta), & \quad -N \leq k \leq 0 \\
\overline{v}(x, (k+1)\Delta) = \overline{v}(x, k\Delta) + F(x, k\Delta, D(\overline{v}_{k\Delta}))\Delta + G(x, k\Delta, D(\overline{v}_{k\Delta}))\Delta B_k, & \quad k \geq 0
\end{array} \right.$$

where
\[
D(\bar{v}_{k\Delta}) = \frac{1}{\tau^2} \int_{-\tau}^{0} \bar{v}_{k\Delta}(\theta)d\theta = \frac{1}{\tau} \sum_{i=-N}^{-1} \Delta \left[ \bar{v}(x, (k + i)\Delta) + \bar{v}(x, (k + i + 1)\Delta) \right] = \frac{1}{N} \left( \frac{1}{2} \bar{v}(x, (k - N)\Delta) + \bar{v}(x, (k - N + 1)\Delta) + ... + \bar{v}(x, (k - 1)\Delta) + \frac{1}{2} \bar{v}(x, k\Delta) \right).
\]

We see clearly from this simple form that the discrete approximate solution (2.3) is a natural generalization of the classical Euler-Maruyama numerical scheme for SDEs, and that is why we call (2.3) the Euler-Maruyama approximate solution. The primary aim of this paper is to establish the following main result.

Theorem 2.2. Under assumptions 2.1 - 2.3,

\[
\lim_{\Delta \to 0} E\left( \sup_{0 \leq \xi \leq T} \| u(., t) - v(., t) \|^2 \right) = 0 \text{ for all } T > 0. \quad (2.12)
\]

The proof of this theorem is very technical, so we present some lemmas.

Lemma 2.1. Let assumption 2.3 hold. Define \( \alpha : (0, T] \to R_+ \) by

\[
\alpha(z) = \sup_{t, \xi \in [-\tau, 0], \| \xi \| < z} E \| \xi(., t) - \xi(., s) \|^2.
\]

Then \( \alpha \) is nondecreasing and has the property that \( \alpha(z) \to 0 \) as \( z \to 0 \). Moreover,

\[
E \| \xi(., t) - \xi(., s) \|^2 \leq \alpha|t - s|. \quad -\tau \leq s \leq t \leq 0 \quad (2.13)
\]

Proof: From the definition of \( \alpha \) we see clearly that \( \alpha \) is nondecreasing and (2.13) holds. We therefore need only to show that \( \alpha(z) \to 0 \) as \( u \to 0 \). If this is not true, then

\[
\lim_{z \to 0} \alpha(z) = \varepsilon_0 > 0. \quad (2.14)
\]

From the definition of \( \alpha \) we observe that for each integer \( k \geq 1 \) we can find a pair of \( t_k \) and \( s_k \) in \([-\tau, 0]\) with \( |t_k - s_k| < \frac{1}{k} \) for which

\[
E \| \xi(., t_k) - \xi(., s_k) \|^2 \geq \frac{\varepsilon_0}{2}. \quad (2.15)
\]
Since \( \{t_k\} \) is a sequence in the bounded interval \([−τ, 0]\), it must have a convergent subsequence. Without any loss of generality, we may assume that \( \{t_k\} \) is already a convergent sequence, and that it converges to \( \bar{t} \in [−τ, 0] \). Clearly, \( \{s_k\} \) converges to \( \bar{t} \) too. Now, by the continuity of \( \xi(., .) \),

\[
\lim_{k \to \infty} \| \xi(., t_k) − \xi(., \bar{t}) \|^2 = 0
\]

almost surely.

Moreover

\[
\| \xi(., t_k) − \xi(., \bar{t}) \|^2 ≤ 2 \| \xi(., t_k) \|^2 + \| \xi(., \bar{t}) \|^2 ≤ 4 \| \xi \|^2,
\]

while (by assumption 2.3 and the Holder inequality)

\[
E \| \xi \|^2 ≤ (E \| \xi \|^p)^{2/p} < \infty.
\]

We can then apply the dominated convergence theorem to obtain

\[
\lim_{k \to \infty} E \| \xi(., t_k) − \xi(., \bar{t}) \|^2 = 0.
\]

Similarly, we can show that

\[
\lim_{k \to \infty} E \| \xi(., s_k) − \xi(., \bar{t}) \|^2 = 0.
\]

Consequently, we have

\[
\lim_{k \to \infty} E \| \xi(., t_k) − \xi(., s_k) \|^2 = 0,
\]

but this is in contradiction to (2.15). We therefore must have

\[
\lim_{z \to 0} \alpha(z) = 0
\]

the proof is therefore complete.

Lemma 2.2. Under assumption 2.2 and 2.3,

\[
E(\sup_{−τ ≤ t ≤ T} \| v(., t) \|^p) ≤ H, \text{ for all } T > 0,
\]

(2.16)
where \( H \) is a positive number dependent on \( \xi, K, p \) and \( T \), but independent of \( \Delta \).

**Proof.** By the Holder inequality, it is easy to see from (2.7) that

\[
\| v(.,t) \|^p < 3^{p-1} [\| \| x(\cdot) \|^p + t^{p-1} \int_0^t \| f(.,s,\overline{\mu}_s) \|^p ds] + \int_0^t g(.,s,\overline{\mu}_s) dB(s) \|^p.
\]

Hence, for any \( t_1 \in [0,T] \),

\[
E(\sup_{0 \leq t \leq t_1} \| v(.,t) \|^p) < 3^{p-1} [\| \| x(\cdot) \|^p + T^{p-1} \int_0^{t_1} \| f(.,s,\overline{\mu}_s) \|^p ds] + \int_0^{t_1} E(\sup_{0 \leq t \leq s} \| g(.,s,\overline{\mu}_s) dB(s) \|^p).
\] (2.17)

By assumption 2.2, we compute that

\[
E \int_0^{t_1} \| f(.,s,\overline{\mu}_s) \|^p ds \leq 2^{(p-2)/2} K^{p/2} E \int_0^{t_1} (1 + \| \overline{\mu}(.,s) \|^p) ds
\]

\[
\leq 2^{(p-2)/2} K^{p/2} [T + \int_0^{t_1} \| v(.,t) \|^p] ds.
\] (2.18)

We also compute, using the Burkholder-Davis-Gundy inequality,

\[
E(\sup_{0 \leq t \leq t_1} \| \int_0^t g(.,s,\overline{\mu}_s) dB(s) \|^p) \leq c_p E(\int_0^{t_1} g(.,s,\overline{\mu}_s) \|^2 ds)^{p/2} \leq c_p T^{(p-2)/2} E \int_0^{t_1} g(.,s,\overline{\mu}_s) \|^p ds,
\]

where \( c_p \) is a constant dependent only on \( p \). In the same way as (2.18) was obtained, we can then show that

\[
E(\sup_{0 \leq t \leq t_1} \| \int_0^t g(.,s,\overline{\mu}_s) dB(s) \|^p) \leq c_p (2T)^{(p-2)/2} K^{p/2} [T + \int_0^{t_1} E(\sup_{-\tau \leq t \leq s} \| v(.,t) \|^p) ds].
\] (2.19)

Substituting (2.18) and (2.19) into (2.17) yields

\[
E(\sup_{0 \leq t \leq t_1} \| v(.,t) \|^p) \leq 3^{p-1} E \| x(\cdot) \| + C_1 + C_2 \int_0^{t_1} E(\sup_{-\tau \leq t \leq s} \| v(.,t) \|^p) ds,
\] (2.20)

where \( C_1 \) and \( C_2 \) are two positive numbers dependent only on \( K,p \) and \( T \). We then derive the following inequalities:
\[
E(\sup_{-\tau \leq t \leq T} \| v(.,t) \|_P) \leq E(\| \xi \|_P) + E(\sup_{0 \leq t \leq T} \| v(.,t) \|_P)
\]
\[
(1 + 3^{p-1})E(\| \xi \|_P) + C_1 + C_2 \int_0^T E(\sup_{-\tau \leq t \leq s} \| v(.,t) \|_P)ds.
\]
(2.21)

By the Gronwall inequality we find that
\[
E(\sup_{-\tau \leq t \leq T} \| v(.,t) \|_P) \leq [(1 + 3^{p-1})E(\| \xi \|_P) + C_1 e^{C_3 T}],
\]
and hence the required assertion must hold.

Lemma 3.3. Let assumptions 2.1 - 2.3 hold, let \( T > 0 \). Then there is a nondecreasing function \( \beta : (0,T] \to \mathbb{R}_+ \) that has the property that \( \beta(z) = 0 \) as \( z \to 0 \), such that
\[
E(\| v(.,s + \theta) - \bar{v}_s(.,\theta) \|_P^2) \leq \beta(\Delta), \quad s \in [0,T], \quad \theta \in [-\tau,0].
\]
(2.22)

Proof. Fix \( s \in [0,T] \) and \( \theta \in [-\tau,0] \). Let \( k_s \) and \( k_\theta \) be the integers for which \( s \in [k_s \Delta, (k_s + 1)\Delta] \) and \( \theta \in [k_\theta \Delta, (k_\theta + 1)\Delta] \), respectively. (When \( \theta/\Delta \) is an integer, the choice for \( k_\theta \) may not be unique, but this will not affect the proof below.) Clearly, \( 0 \leq s - k_s \Delta < \Delta \) and \( 0 \leq \theta - k_\theta \Delta \leq \Delta \), so
\[
0 \leq s + \theta - (k_s + k_\theta)\Delta < 2\Delta.
\]
(2.23)
Moreover, it follows from (2.4) and (2.6) that
\[
\bar{v}_s(x,\theta) = \bar{v}_{k_s \Delta}(x,\theta) = \bar{v}(x,(k_s + k_\theta)\Delta) + \frac{\theta - k_\theta \Delta}{\Delta} [\bar{v}(x,(k_s + k_\theta + 1)\Delta) - \bar{v}(x,(k_s + k_\theta)\Delta)].
\]
Hence
\[
E(\| v(.,s + \theta) - \bar{v}_s(.,\theta) \|_P^2) \leq 2E(\| v(.,s + \theta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|_P^2)
\]
\[
+ 2E(\| \bar{v}(.,(k_s + k_\theta + 1)\Delta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|_P^2)
\]
(2.24)
If \( k_s + k_\theta \leq -1 \), then lemma 2.1,
\[
E(\| \bar{v}(.,(k_s + k_\theta + 1)\Delta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|_P^2) \leq \alpha(\Delta).
\]
If \( k_s + k_\theta \geq 0 \), and lemma 2.2 we compute from (2.3) that
\[ E \| \bar{v}(.,(k_s + k_\theta + 1)\Delta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|^2 = \Delta^2 E \| f(.,(k_s + k_\theta)\Delta, \bar{v}(.,(k_s + k_\theta)\Delta)) \|^2 \\
+ \Delta E \| g(.,(k_s + k_\theta)\Delta, \bar{v}(.,(k_s + k_\theta)\Delta)) \|^2 \leq 2\Delta K (1 + E \| \bar{v}(.,(k_s + k_\theta)\Delta) \|^2) \\
\leq 2\Delta K \{ (1 + E \| v(.,z) \|^2) \} \leq 2\Delta K (1 + H^{2/p}) \Delta, \]

where \( H \) is the constant specified in lemma 2.2. We hence always have

\[ E \| \bar{v}(.,(k_s + k_\theta + 1)\Delta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|^2 \leq 2K(1 + H^{2/p})\Delta + \alpha(\Delta). \]

Using this bounded in (2.24) gives

\[ E \| v(.,s+\theta) - \bar{v}(.,\theta) \|^2 \leq 2E \| v(.,s+\theta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|^2 + 4K(1 + H^{2/p})\Delta + 2\alpha(\Delta). \]  

(2.25)

To bound the first term on the right hand side, let us discuss the following possible cases.

Case 1: \( k_s + k_\theta \geq 0 \). It follows from (2.8) that

\[ v(x,s+\theta) - \bar{v}(x,(k_s + k_\theta)\Delta) = \int_{(k_s + k_\theta)\Delta}^{s+\theta} f(x,r,\bar{v}_r) \, dr + \int_{(k_s + k_\theta)\Delta}^{s+\theta} g(x,r,\bar{v}_r) \, dB(r). \]

By assumption 2.2 and lemma 2.2, we compute that

\[ E \| v(.,s+\theta) - \bar{v}(.,(k_s + k_\theta)\Delta) \|^2 \leq 2\Delta E \int_{(k_s + k_\theta)\Delta}^{s+\theta} \| f(.,r,\bar{v}_r) \|^2 \, dr + E \int_{(k_s + k_\theta)\Delta}^{s+\theta} \| g(.,r,\bar{v}_r) \|^2 \, dr \]

\[ \leq 6KE \int_{(k_s + k_\theta)\Delta}^{s+\theta} (1 + \| \bar{v} \|^2) \, dr \leq 6K \int_{(k_s + k_\theta)\Delta}^{s+\theta} (1 + E \| v(.,z) \|^2) \, dr \]

\[ \leq 6K \int_{(k_s + k_\theta)\Delta}^{s+\theta} \{ 1 + (E \sup_{-r \leq t \leq r} \| v(.,z) \|^p)^{2/p} \} \, dr \leq 12K(1 + H^{2/p})\Delta. \]  

(2.26)

Case 2: \( k_s + k_\theta = -1 \) and \( \Delta < s + \theta - (k_s + k_\theta)\Delta < 2\Delta \).

In this case,

\[ 0 \leq \Delta + (k_s + k_\theta)\Delta < s + \theta < 2\Delta + (k_s + k_\theta)\Delta = \Delta. \]
So

\[ E \| v(, s + \theta) - \overline{v}(, (k_s + k_\theta)\Delta) \|^2 = E \| v(, s + \theta) - \overline{v}(, -\Delta) \|^2 \]
\[ \leq E \| v(, s + \theta) - \xi_0(.) \|^2 + 2E \| \xi_0(.) - \xi(, -\Delta) \|^2. \]

It can be shown in the same way as in case 1 that

\[ E \| v(, s + \theta) - \xi_0(.) \|^2 \leq 4K(1 + H^{2/p})\Delta, \]

while by lemma 2.1,

\[ E \| \xi_0(.) - \xi(, -\Delta) \|^2 \leq \alpha(\Delta). \]

We therefore that

\[ E \| v(, s + \theta) - \overline{v}(, (k_s + k_\theta)\Delta) \|^2 \leq 8K(1 + H^{2/p})\Delta + 2\alpha(\Delta). \quad (2.27) \]

Case 3: \( k_s + k_\theta = -1 \) and \( 0 \leq s + \theta - (k_s + k_\theta)\Delta \leq \Delta \). In this case

\[-\Delta \leq (k_s + k_\theta)\Delta < s + \theta < \Delta + (k_s + k_\theta)\Delta = 0.\]

So

\[ E \| v(, s + \theta) - \overline{v}(, (k_s + k_\theta)\Delta) \|^2 = E \| \xi(, s + \theta) - \overline{v}(, -\Delta) \|^2 = E \| \xi(, s + \theta) - \xi(, -\Delta) \|^2. \]

By lemma 2.1, we then have

\[ E \| v(, s + \theta) - \overline{v}(, \theta) \|^2 \leq \alpha\Delta \quad (2.28) \]

Case 4: \( k_s + k_\theta \leq -2 \). In this case \( s + \theta \leq 0 \). So

\[ E \| v(, s + \theta) - \overline{v}(, (k_s + k_\theta)\Delta) \|^2 = E \| \xi(, s + \theta) - \xi(, (k_s + k_\theta)\Delta) \|^2. \]

By lemma 2.1 and (2.23), we then have

\[ E \| v(, s + \theta) - \overline{v}(, (k_s + k_\theta)\Delta) \|^2 \leq \alpha(2\Delta) \quad (2.29) \]
Combining the four cases above together, we can conclude that we always have

\[ E \| v(., s + \theta) - \overline{v}(., (k_s + k_\theta)\Delta) \|^2 \leq 12K(1 + H^{2/p})\Delta + 2\alpha(2\Delta) \]  

(2.30)

Now, we define \( \beta : (0, \tau] \to R_+ \) by

\[ \beta(z) = 28K(1 + H^{2/p})z + 6\alpha(2z). \]

Clearly, \( \beta \) is nondecreasing. Moreover, it follows from (2.25) and (2.30) that

\[ E \| v(., s + \theta) - \overline{v}_s(., \theta) \|^2 \leq \beta(\Delta). \]

Which is the required assertion. The proof is complete.

Proof of theorem 2.2

Let us now begin to prove theorem 2.2. We first note from theorem 2.1 and lemma 2.2 that there is a positive constant \( \overline{H} \) such that

\[ E(\sup_{-\tau \leq t \leq T} \| u(., t) \|^p) \vee E(\sup_{-\tau \leq t \leq T} \| v(., t) \|^p) \leq \overline{H}. \]  

(2.31)

Let \( j \) be sufficiently large integer. Define the stopping times

\[ p_j =: \inf\{t \geq 0 : \| u(., t) \| \geq j\}, \quad q_j =: \inf\{t \geq 0 : \| v(., t) \| \geq j\}, \quad \rho_j = p_j \land q_j, \]

where we set \( \inf \emptyset = \infty \). Let \( e(x, t) = u(x, t) - v(x, t) \) obviously,

\[ E[\sup_{0 \leq t \leq T} \| e(., t) \|^2] = E[\sup_{0 \leq t \leq T} \| e(., t) \|^2 1_{\{p_j \leq T \text{ or } q_j \leq T\}}]. \]

Recall the following elementary inequality:

\[ a^\gamma b^{1-\gamma} \leq \gamma a + (1 - \gamma)b, \forall a, b > 0, \gamma \in [0, 1]. \]

We thus have, for any \( \delta > 0 \),
\[ E[ \sup_{0 \leq t \leq T} || e(., t) ||^2 1_{\{\rho_j \geq T \text{ or } q_j \geq T\}} ] = E[ (\delta \sup_{0 \leq t \leq T} || e(., t) ||^{p})^{2/p} (\delta^{-2/(p-2)} 1_{\{\rho_j \geq T \text{ or } q_j \geq T\}})^{(p-2)/p} ] \leq \frac{2\delta}{p} E[ \sup_{0 \leq t \leq T} || e(., t) ||^p ] + \frac{p-2}{p^{\delta^2/(p-2)}} P(\rho_j \leq T \text{ or } q_j \leq T). \]

Hence
\[
E[ \sup_{0 \leq t \leq T} || e(., t) ||^2 ] \leq E[ \sup_{0 \leq t \leq T} || e(., t) ||^2 1_{\{\rho_j > 1\}} ] + \frac{2\delta}{p} E[ \sup_{0 \leq t \leq T} || e(., t) ||^p ] + \frac{p-2}{p^{\delta^2/(p-2)}} P(\rho_j \leq T \text{ or } q_j \leq T). \tag{2.32}
\]

Now
\[
P(q_j \leq T) = E[1_{\{\rho_j \leq T\}} \frac{|| u_{\rho_j}(., t) ||^p}{j^p}] \leq \frac{1}{j^p} E[ \sup_{-\tau \leq t \leq T} || u(., t) ||^p ] \leq \frac{\overline{H}}{j^p},
\]
using (2.31). Similarly, we have \( P(q_j \leq T) \leq \frac{\overline{H}}{j^p} \) Thus
\[
P(\rho_j \leq T \text{ or } q_j \leq T) \leq P(\rho_j \leq T) + P(q_j \leq T) \leq \frac{2\overline{H}}{j^p}.
\]

We also have
\[
E[ \sup_{0 \leq t \leq T} || e(., t) ||^2 1_{\{\rho_j > T\}} ] = E[ \sup_{0 \leq t \leq T} || e(., t \land \rho_j) ||^2 1_{\{\rho_j > T\}} ] \leq E[ \sup_{0 \leq t \leq T} || e(., t \land \rho_j) ||^2 ].
\]

Using these bounds in (2.32) yields
\[
E[ \sup_{0 \leq t \leq T} || e(., t) ||^2 ] \leq E[ \sup_{0 \leq t \leq T} || e(., t \land \rho_j) ||^2 ] + \frac{2\delta^2 \overline{H}}{p} + \frac{(p-2)2\overline{H}}{p^{\delta^2/(p-2)} j^p}. \tag{2.33}
\]

Now
\[
|| e(., t \land \rho_j) ||^2 = || u(., t \land \rho_j) - v(., t \land \rho_j) ||^2 = \int_0^{t \land \rho_j} [f(., s, u_s) - f(., s, \overline{u}_s) ds + \int_0^{t \land \rho_j} [g(., s, u_s) - g(., s, \overline{u}_s)] dB(s) ||^2 \leq 2[T \int_0^{t \land \rho_j} || f(., s, u_s) - f(., s, \overline{u}_s) ||^2 ds + \int_0^{t \land \rho_j} [g(., s, u_s) - g(., s, \overline{u}_s)] dB(s) ||^2].
\]

By the Doob martingale inequality we have, for any \( t_1 \leq T \),
\[ E[ \sup_{0 \leq t \leq t_1} \| e(., t \wedge \rho_j) \|^2 ] \leq 2[T \epsilon_t] \int_{0}^{t_1 \wedge \rho_j} \| f(., s, u_s) - f(., s, \bar{u}_s) \|^2 ds + 4 \int_{0}^{t_1 \wedge \rho_j} \| g(., s, u_s) - g(., s, \bar{u}_s) \|^2 ds \]
\[ = 4(T + 4)E \int_{0}^{t_1 \wedge \rho_j} \| f(., s, u_s) - f(., s, \bar{u}_s) \|^2 \wedge \| g(., s, u_s) - g(., s, \bar{u}_s) \|^2 ds \]

But, by assumption 2.1, we derive that, for \( s \in (0, t_1 \wedge \rho_j), \)

\[ \| f(., s, u_s) - f(., s, \bar{u}_s) \|^2 \leq 2 \| f(., s, u_s) - f(., s, v_s) \|^2 \leq \int_{\tau}^{0} \| u(., s + \theta) - u(., s + \theta) \|^2 d\mu_j(\theta) \]
\[ + 2 \int_{\tau}^{0} \| v(., s + \theta) - \bar{u}_s(., s) \|^2 d\mu_j(\theta) \leq 2 \int_{\tau}^{0} \| u(., s + \theta) - u(., s + \theta) \|^2 d\mu_j(\theta) \]
\[ + 2 \int_{\tau}^{0} \| v(., s + \theta) - \bar{u}_s(., s + \theta) \|^2 d\mu_j(\theta) \leq 2(\mu_j(0) - \mu_j(\tau)) \| u(., s) - u(., t) \|^2 \]
\[ + 2 \int_{\tau}^{0} \| v(., s + \theta) - \bar{u}_s(., s + \theta) \|^2 d\mu_j(\theta). \]

A similar result can be obtained for \( \| g(., s, u_s) - g(., s, \bar{u}_s) \|^2, \) so that

\[ E[ \sup_{0 \leq t \leq t_1} \| e(., t \wedge \rho_j) \|^2 ] \leq 8(T + 4)(\mu_j(0) - \mu_j(\tau))E \int_{0}^{t_1 \wedge \rho_j} \| e(., s) \|^2 ds + 8(T + 4)E \int_{0}^{t_1 \wedge \rho_j} \| v(., s + \theta) - \bar{u}_s(., s + \theta) \|^2 d\mu_j(\theta) \]
\[ \leq 8(T + 4)(\mu_j(0) - \mu_j(\tau)) \int_{0}^{t_1} E[ \sup_{0 \leq t \leq s} \| e(., s \wedge \rho_j) \|^2] ds + 8(T + 4) \int_{\tau}^{T} \| v(., s + \theta) - \bar{u}_s(., s + \theta) \|^2 d\mu_j(\theta) ds. \]

By lemma 2.3 we therefore find that

\[ E[ \sup_{0 \leq t \leq t_1} \| e(., t \wedge \rho_j) \|^2 ] \leq 8(T + 4)(\mu_j(0) - \mu_j(\tau)) \int_{0}^{t_1} E[ \sup_{0 \leq s \leq t} \| e(., s \wedge \rho_j) \|^2] ds + 8T(T + 4)(\mu_j(0) - \mu_j(\tau))\beta(\Delta). \]

The Gronwall inequality implies that

\[ E[ \sup_{0 \leq t \leq T} \| e(., t \wedge \rho_j) \|^2 ] \leq C_j\beta(\Delta), \]
\[ C_j = 8T(T + 4)(\mu_j(0) - \mu_j(-\tau)) \exp[8T(T + 4)(\mu_j(0) - \mu_j(-\tau))]. \]

Substituting this into (2.33) gives

\[ E[ \sup_{0 \leq t \leq T} \| e(\cdot, t) \|^2 ] \leq C_j \beta(\Delta) + \frac{2^{p+1}\overline{H}}{p} + \frac{(p-2)2\overline{H}}{p^{2/(p-2)}j^p}. \tag{2.36} \]

Given \( \varepsilon > 0 \) we can now choose \( \delta \) sufficiently small for \( (2^{p+1}\delta\overline{H})/p < \varepsilon/3 \), then choose \( j \) sufficiently large for \( \frac{(p-2)2\overline{H}}{p^{2/(p-2)}j^p} < \varepsilon/3 \) and finally choose \( \Delta \) so that \( C_j \beta(\Delta) < \varepsilon/3 \). Thus (2.36),

\[ E[ \sup_{0 \leq t \leq T} \| e(\cdot, t) \|^2 ] < \varepsilon, \]

as required.

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UNDERSTANDING OF NAVIER-STOKES EQUATIONS
VIA A MODEL FOR BLOOD FLOW

Joon Hyuck Choi, Nam Lyong Kang and Sang Don Choi

ABSTRACT: A pedagogic model for blood flow is introduced to help medicine majors understand a simplified version of Navier-Stokes equations which is known to be a good tool for interpreting the phenomena in blood flow. The pressure gradient consists of a time-independent part known as Hagen-Poiseuille's gradient and a time-dependent part known as Sexl's, and the model formula for the volume rate of blood flow is reduced to a very simple form. For demonstration, the blood rate in human aorta system is analyzed in connection with the time-dependence of pressure gradient. It is shown for Sexl's part that the flow rate lags the pressure gradient by $\pi/2$, which is thought to be due to the relaxation process involved.\(^1\)

1. INTRODUCTION

Navier Stokes equations[NSE] \([1]\) are a model example of Newton's law of motion and is a good tool for interpreting some interesting phenomena appearing in engineering flows. Nevertheless, the coverage is dealt with in rather limited scheme in physiology classes, since equations are difficult to solve analytically with few exceptions. Nowadays, however, due to the widespread use of computers, obtaining any numerical solutions is feasible. The most simple one will be finite element method[2] which includes several versions. Another reason for limiting the coverage lies in difficulty in finding easy and interesting examples beyond Hegen-Poiseuille's law [HP] for quasi-static pressure gradient[3]. In 1930 Sexl [4] introduced an example with sinusoidally varying pressure gradient and with no-slip condition for viscous mechanical fluids flowing in a circular duct. The solution is given in a Bessel function with complex arguments, and thus has drawn little attention among applied scientists who usually dislike mathematics. In 1956 Uchida [5] paved the way for easy

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**Key words**: Navier–Stokes equation, Blood Flow
access to this model by giving numerical calculations.

Blood is a good example of viscous fluid and blood tubes are
the counterpart of mechanical tubes in approximation. And thus
the blood flow in veins can be dealt with as viscous flow with
constant pressure gradient and that in arteries can be approxi-
mated to that with sinusoidal pressure gradient. The former one,
which shall be called the primitive HP stating that rate of flow in
a pipe with circular cross-section is proportional to the fourth
power of the radius of the pipe, is covered in regular physiology
classes. But the latter one is not covered, to the knowledge of
the present authors, since the Bessel functions of the first and
second kinds are not so popular even among physiology
professors.

This pedagogic article introduces a model for blood flow which
helps medicine majors understand the NSE. The model consists of
the traditional constant pressure gradient and sinusoidal pressure
gradient in blood tubes.

2. A MODEL FOR BLOOD FLOW

For incompressible fluids, the NSE in cylindrical coordinates
\((r, \theta, z)\) are given as [6]

**r-component:**

\[
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta v_r}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_r^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r \tag{1}
\]

**\(\theta\)-component:**

\[
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] + \rho g_\theta \tag{2}
\]

**z-component:**
\[
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\
\quad = -\frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z
\]  
(3)

where \( p \) and \( v \), respectively, are the pressure and velocity and \( \rho \) is the mass density, \( \mu \) the coefficient of viscosity, and \( \vec{g} \) the acceleration due to gravity.

For viscous flows in a circular pipe of radius \( R \), we can put
\[
v_r = 0, \quad v_\theta = 0, \quad v_z = v_z(r, t)
\]  
(4)
in Eqs. (1)-(3) along with the following boundary conditions
\[
\left( \frac{\partial v_z}{\partial r} \right)_{r=0} = 0
\]  
(5a)
\[
v_z(r = R) = 0
\]  
(5b)
And choosing the pressure gradient with a constant term and a sinusoidal term as \([3, 4]\)
\[
-\frac{1}{\rho} \frac{\partial p}{\partial z} = A_0 + A_1 \sin \omega t = A_0 + A_1 \text{Im}(e^{i\omega t})
\]  
(6)
the NSE for \( v_z \) becomes
\[
\frac{\partial v_z}{\partial t} = A_0 + A_1 \text{Im}(e^{i\omega t}) + \nu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right) + g_z
\]  
(7)
where \( A_0 \) and \( A_1 \) have dimension of acceleration, \( \text{Im} \) denotes "the imaginary part of" and \( \nu = \mu/\rho \) is the coefficient of kinetic viscosity. Eq. (7) is a simplified version of NSE for physiology teaching. Note that this model is reduced to Sexl’s if \( A_0 = 0 \) and to Poiseuille’s if \( A_1 = 0 \). The solution \( v_z(r, t) \) of the partial differential equation (7) can be obtained in the following form :
\[
v_z(r, t) = v_z^{(1)}(r) + \text{Im} \left[ e^{i\omega t} v_z^{(2)}(r) \right]
\]  
(8)
Then, substituting this trial form into Eq. (7) and applying the boundary conditions (5a–5b), we have \([7]\)
\[
v_z(r, t) = \rho \frac{A_0 + g_z}{4\mu} (R^2 - r^2) - \frac{A_1}{\omega} \text{Re} \left[ \left( 1 - \frac{J_0(\sqrt{-i\omega/\nu} \cdot r)}{J_0(\sqrt{-i\omega/\nu} \cdot R)} \right) e^{i\omega t} \right]
\]  
(9)
and the expression for volume rate of flow, defined by \([8]\)
\[ Q(t) = \int_0^R v_z(r, t) 2\pi r dr, \] is given as

\[ Q(t) = \text{Poiseuille's part} + \text{Sexl's part} \]

\[ = \frac{\pi R^4 \rho (A_0 + g_z)}{8 \mu} - \frac{\pi R^2 A_1}{\omega} \text{Re} \left[ e^{i\omega t} \left( 1 - \frac{2}{J_1(R^*)} \frac{J_0(R^*)}{J_1(R^*)} \right) \right]. \tag{10} \]

Here \( \text{Re} \) denotes "the real part of" and \( R^* = R \sqrt{-i\omega \rho/\mu} \), which shall be called the reduced radius hereafter, and \( J_0(R^*) \) and \( J_1(R^*) \) are the Bessel functions of the first kind of order zero and order one, respectively [9–10]. Note that the argument \( R^* \) is complex. Here the first part is Hagen–Poiseuille's result with inclusion of the gravitational effect and the second one is the counterpart obtained from Sexl's velocity distribution.

If we are interested in blood flow in the human system, we can take the asymptotic approximation \( J_1(R^*)/J_0(R^*) \approx \tan(R^*) \) since \( |R^*| \approx 10 \) which is large enough for this criterion to apply. We then have the Sexl term simplified for large \( R^* \) as

\[ \text{Sexl's part} = -\left( \frac{\pi R^2 A_1}{\omega} \right) \text{Re} \left[ \left( 1 - 2 \frac{\tan R^*}{R^*} \right) e^{i\omega t} \right] \]

\[ = -\frac{\pi R^2 A_1}{\omega} \cos \omega t = \frac{\pi R^2 A_1}{\omega} \sin \left( \omega t - \frac{\pi}{2} \right) \tag{11} \]

for the human blood flow system since \( \text{Re}(2\tan R^*/R^*) \ll 1 \) [See below]. We see that \( Q(t) \) for Sexl's part is out of phase with the pressure gradient by \( \pi/2 \). The fact that \( Q(t) \) lags the pressure gradient by \( 90^\circ \) comes simply from the relaxation process involved.

\section*{3. Discussions and Concluding Remarks}

It is to be noted that Eq. (10) along with Eq. (11) holds for laminar flow. For that purpose, the Reynolds number \( R_e = 2R\rho \bar{v}/\mu \), \( \bar{v} \) being the average velocity, should be smaller.
than 2300. Otherwise, the transition and/or turbulence will be set up.

The above can be summarized as follows: The combined Poiseuille-Sexl's formula for rate of flow in human blood systems is reduced to

$$Q(t) = \frac{\pi R^4}{8\mu} \rho (A_0 + g_z) - \frac{\pi R^2}{\omega} A_1 \cos \omega t \quad (12)$$

in the above approximation.

In order to get into details, considering only Poiseuille's part and neglecting the gravitational effects, we take the following experimental data for normal human aorta system. $\rho = 1.05 \times 10^3 \text{kg/m}^3$, $\mu = 4 \times 10^{-3} \text{Pa} \cdot \text{s}$, $R = 0.01 \text{m}$, and $\bar{v} = 0.4 \text{m/s}$ [11]. We then have $R_e = 2100$, implying that the blood flow in the aorta is laminar. Taking the angular frequency of the heart beat $\omega = 2\pi f = 7.5 \text{Hz}$, $f = 1.2 \text{Hz}$ being the frequency, we obtain $|R^*| \approx 10$ which is large enough for the criteria of our approximation to hold. Thus we have

$$Q(t) = 9 \times 10^{-4} (A_0 + g_z - 0.05 A_1 \cos \omega t) \left[\text{L}^3/\text{s}\right], \quad (13)$$

where $A_0$ and $A_1$ are given in the SI-unit [m/s²].

We now compare the Poiseuille term and Sexl term for the human aorta system. In order to adopt this model, the two factors $A_0$ and $A_1$ should compete with each order in the almost same order. Note that we cannot claim that our $A_0$ is identical with that in Hagen-Poiseuille's formula. $A_0$ and $A_1$ can be obtained by fitting the theory to the available experiment. It is regretful that the fitting cannot be accomplished due to lack of experimental data. We will give only qualitative analysis instead.

We consider the aorta of length $L$ and assume that the oscillation disappears at $z = L$. We further assume that $g_z = 0$, which means that the system is laid in the horizontal plane. Then the pressure can be expressed as

$$P_z(z) = P_0 - A_0 z + A_1 (L - z) \sin \omega t \quad (14)$$

which yields our pressure gradient $\partial P_z/\partial z = -(A_0 + A_1 \sin \omega t)$. 
Roughly we have $\rho(A_0 - 0.05 A_1 \sin \omega t) \approx 100 \text{ Pa/m}$ and $Q(t) = 10^{-4} \text{ m}^3/\text{s}$ [14]. Thus once either $A_0$ or $A_1$ is known, the whole behavior of the system can be exactly analyzed. A rough pictorial analysis is shown in the following figures.

![Diagram](image)

**Figure 1.** Pressure gradient and flow rate

\[
\frac{Q(t)}{9 \times 10^{-4}} = A_0
\]

0.05$A_1$

-0.05$A_1$

$\omega t$

$\pi$

$2\pi$

**Figure 2.** Flow rate versus $\omega t$

In conclusion, the flow rate of the human circulatory system is
also affected by Sexl's pressure gradient, which is physical as expected. So far we have introduced a simple model for flow rate on the basis of the simplified version of Navier-Stokes equation. We hope the medicine majors would be helped in understanding the Naviers-Stokes equation via this model. This model theory will be helpful in investigating other similar problems, too. One possible problem will be the flood flow in a pipe of elliptical cross-section [6]. Another interesting problem will be in the blood flow in curved vessels [13]. If Sexl's part is combined with these models, more meaningful results will be obtained. These works are in progress and will be reported later.

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MARK SEQUENCES IN 3-PARTITE 2-DIGRAPHS

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Abstract. A 3-partite 2-digraph is an orientation of a 3-partite multi-graph that is without loops and contains at most two edges between any pair of vertices from distinct parts. Let D(X, Y, Z) be a 3-partite 2-digraph with |X| = l, |Y| = m, |Z| = n. For any vertex v in D(X, Y, Z), let \(d^+_v\) and \(d^-_v\) denote the outdegree and indegree respectively of v. Define \(p_x = 2(m + n) + d^+_x - d^-_x\), \(q_y = 2(l + n) + d^+_y - d^-_y\) and \(r_z = 2(l + m) + d^+_z - d^-_z\) as the marks (or 2-scores) of x in X, y in Y and z in Z respectively. In this paper, we characterize the marks of 3-partite 2-digraphs and give a constructive and existence criterion for sequences of non-negative integers in non-decreasing order to be the mark sequences of some 3-partite 2-digraph.

1. INTRODUCTION

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Suppose \(V = \{v_1, v_2, \ldots, v_n\}\) be the vertex set of an oriented graph, and let \(d^+_v\) and \(d^-_v\) denote the outdegree and indegree respectively of a vertex v. Avery [1] defined \(a_v = n - 1 + d^+_v - d^-_v\), the score (or 1-score) of v, so \(0 \leq a_v \leq 2n - 2\). Then, the sequence \([a_1, a_2, \ldots, a_n]\) in non-decreasing order is called the score sequence of the oriented graph.

Key Words:
Avery obtained the following criterion for score sequences in oriented graphs.

**Theorem 1.1[1].** A non-decreasing sequence of non-negative integers \([a_1, a_2, \ldots, a_n]\) is the score sequence of an oriented graph if and only if
\[
\sum_{i=1}^{k} a_i \geq k(k-1), \quad \text{for} \quad 1 \leq k \leq n,
\]
with equality when \(k = n\).

An \(r\)-digraph is an orientation of a multi-graph that is without loops and contains at most \(r\) edges between any pair of distinct vertices. Clearly, 1-digraph is an oriented graph and complete 1-digraph is a tournament.

Suppose \(V = \{v_1, v_2, \ldots, v_n\}\) be the vertex set of a 2-digraph, and let \(d^+_v\) and \(d^-_v\) denote the outdegree and indegree respectively of a vertex \(v\). Define \(p_v = 2n - 2 + d^+_v - d^-_v\), the mark (or 2-score) of \(v\), so \(0 \leq p_v \leq 4n - 4\). Then, the sequence \([p_1, p_2, \ldots, p_n]\) in non-decreasing order is called the mark sequence of the 2-digraph.

The following result is given by Pirzada and Samee [4].

**Theorem 1.2[4].** A non-decreasing sequence of non-negative integers \([p_1, p_2, \ldots, p_n]\) is the mark sequence of a 2-digraph if and only if
\[
\sum_{i=1}^{k} p_i \geq 2k(k-1), \quad \text{for} \quad 1 \leq k \leq n,
\]
with equality when \(k = n\).

Some stronger inequalities for marks in 2-digraphs can be found in Pirzada and Naikoo [3].

The scores of oriented bipartite graphs have been characterized by Pirzada et al. [5] and those of marks in bipartite 2-digraphs by Samee et al. [6].
An oriented 3-partite graph is the result of assigning a direction to each edge of a simple 3-partite graph. Thus, it has no loops or parallel arcs. Suppose $U = \{u_1, u_2, \ldots, u_p\}$, $V = \{v_1, v_2, \ldots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_r\}$ be the parts of an oriented 3-partite graph, and let $d_{u}^{+}$ ($d_{v}^{+}$ and $d_{w}^{+}$) and $d_{u}^{-}$ ($d_{v}^{-}$ and $d_{w}^{-}$) be the outdegree and indegree respectively of vertex $u$ in $U$ ($v$ in $V$ and $w$ in $W$). Define $a_u = q + r + d_u^+ - d_u^-$, $b_v = p + r + d_v^+ - d_v^-$ and $c_w = p + q + d_w^+ - d_w^-$, the scores (or 1-scores) of $u$, $v$ and $w$ respectively. So, $0 \leq a_u \leq 2(q + r)$, $0 \leq b_v \leq 2(p + r)$ and $0 \leq c_w \leq 2(p + q)$. Then, the sequences $[a_1, a_2, \ldots, a_p]$, $[b_1, b_2, \ldots, b_q]$ and $[c_1, c_2, \ldots, c_r]$ in non-decreasing order are called the score sequences of the oriented bipartite graph.

The next result is due to Pirzada and Merajuddin [2].

**Theorem 1.3[2].** Let $A = [a_1, a_2, \ldots, a_p]$, $B = [b_1, b_2, \ldots, b_q]$ and $C = [c_1, c_2, \ldots, c_r]$ be the sequences of non-negative integers in non-decreasing order. Then, $A$, $B$ and $C$ are the score sequences of some oriented 3-partite graph if and only if

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j + \sum_{k=1}^{r} c_k \geq 2(lm + mn + nl),$$

for $1 \leq l \leq p$, $1 \leq m \leq q$ and $1 \leq n \leq r$,

with equality when $l = p$, $m = q$ and $n = r$.

A 3-partite 2-digraph is an orientation of a 3-partite multi-graph that is without loops and contains at most two edges between any pair of vertices from distinct parts. Suppose $X = \{x_1, x_2, \ldots, x_l\}$, $Y = \{y_1, y_2, \ldots, y_m\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$ be the parts of a 3-partite 2-digraph $D( X, Y, Z )$, and let $d_x^+$ ($d_y^+$ and $d_z^+$) and $d_x^-$ ($d_y^-$ and $d_z^-$) be the outdegree and indegree respectively of vertex $x$ in $X$ ($y$ in $Y$ and $z$ in $Z$). Define $p_x = 2(m + n) + d_x^+ - d_x^-$, $q_y = 2(l + n) + d_y^+ - d_y^-$ and $r_z = 2(l + m) + d_z^+ - d_z^-$, the marks (or 2-scores) of
x, y and z respectively. So, \( 0 \leq p_x \leq 4(m + n) \), \( 0 \leq q_y \leq 4(l + n) \) and \( 0 \leq r_z \leq 4(l + m) \). Then, the sequences \( P = [p_1, p_2, \ldots, p_l] \), \( Q = [q_1, q_2, \ldots, q_m] \) and \( R = [r_1, r_2, \ldots, r_n] \) in non-decreasing order are called the mark sequences of \( D(X, Y, Z) \). We can interpret a 3-partite 2-digraph as a result of competition between three teams in which each player of one team plays against everyone on the other two teams at most twice, with ties (draws) being allowed. A player receives two points for each win, and one point for each tie, and with this marking system, player \( x \) (y and z) receives a total of \( p_x \) (\( q_y \) and \( r_z \)) points. The sequences \( P, Q \) and \( R \) of non-negative integers in non-decreasing order are said to be realizable if there exists a 3-partite 2-digraph with mark sequences \( P, Q \) and \( R \).

2. CRITERIA FOR REALIZABILITY

If \( u \) and \( v \) are two vertices from distinct parts \( X, Y, Z \) of a 3-partite 2–digraph \( D(X, Y, Z) \), then we have one of the following six possibilities.

(i) Exactly two arcs directed from \( u \) to \( v \), and no arc directed from \( v \) to \( u \), and this is denoted by \( u(2-0)v \), see Figure 1(a).

(ii) Exactly two arcs directed from \( v \) to \( u \), and no arc directed from \( u \) to \( v \), and this is denoted by \( u(0-2)v \), see Figure 1(b).

(iii) Exactly one arc directed from \( u \) to \( v \), and exactly one arc directed from \( v \) to \( u \), and this is denoted by \( u(1-1)v \), and is called a pair of symmetric arcs between \( u \) and \( v \), see Figure 1(c).

(iv) Exactly one arc directed from \( u \) to \( v \), and no arc directed from \( v \) to \( u \), and this is denoted by \( u(1-0)v \), see Figure 1(d).
(v) Exactly one arc directed from \( v \) to \( u \), and no arc directed from \( u \) to \( v \), and this is denoted by \( u(0-1)v \), see Figure 1(e).

(vi) No arc directed from \( u \) to \( v \), and no arc directed from \( v \) to \( u \), and this is denoted by \( u(0-0)v \), see Figure 1(f).

![Figure 1](image1)

Figure 2 shows a 3-partite 2-digraph with mark sequences \([10,11], [8,9], [8,9,9]\).

![Figure 2](image2)
A triple in a 3-partite 2-digraph is an induced 2-subdigraph with one vertex from each part, and is of the form \( x(a_1-a_2)y(b_1-b_2)z(c_1-c_2)x \), where for \( 1 \leq i \leq 2 \), \( 0 \leq a_i, b_i, c_i \leq 2 \) and \( 0 \leq \sum_{i=1}^{2} a_i, \sum_{i=1}^{2} b_i, \sum_{i=1}^{2} c_i \leq 2 \).

In a 3-partite 2-digraph, an oriented triple is an induced 1-subdigraph with one vertex from each part. An oriented triple is said to be transitive if it is of the form \( x(1-0)y(1-0)z(0-1)x \), or \( x(1-0)y(0-1)z(0-0)x \), or \( x(1-0)y(0-0)z(0-1)x \), or \( x(1-0)y(0-0)z(0-0)x \), or \( x(0-0)y(0-0)z(0-0)x \), otherwise it is intransitive. A 3-partite 2-digraph is said to be transitive if every of its oriented triple is transitive. In particular, a triple \( C \) in a 3-partite 2-digraph is transitive if every oriented triple of \( C \) is transitive.

First, we have the following observation.

**Theorem 2.1.** Let \( D \) and \( D' \) be two 3-partite 2-digraphs with the same mark sequences. Then, \( D \) can be transformed to \( D' \) by successively transforming (i) appropriate oriented triples in one of the following ways,

either (a) by changing an intransitive oriented triple \( x(1-0)y(1-0)z(1-0)x \) to a transitive oriented triple \( x(0-0)y(0-0)z(0-0)x \), which has the same mark sequences, or vice versa,

or (b) by changing an intransitive oriented triple \( x(1-0)y(1-0)z(0-0)x \) to a transitive oriented triple \( x(0-0)y(0-0)z(0-1)x \), which has the same mark sequences, or vice versa,

or (ii) by changing a pair of symmetric arcs \( x(1-1)y \) to \( x(0-0)y \), which has the same mark sequences, or vice versa.

**Proof.** This result follows from Theorem 2.2[1].

The following result follows from Theorem 2.1.
Corollary 2.1. Among all the 3-partite 2-digraphs with given mark sequences, those with the fewest arcs are transitive.

A transmitter is a vertex with indegree zero. We assume without loss of generality that transitive 3-partite 2-digraphs have no pair of symmetric arcs. For, if there is a pair of symmetric arcs $x(1-1)y$ then it can be changed to $x(0-0)y$ with the same mark sequences. Thus, in a transitive 3-partite 2-digraph with mark sequences $P = [p_1, p_2, \ldots, p_l]$, $Q = [q_1, q_2, \ldots, q_m]$ and $R = [r_1, r_2, \ldots, r_n]$, any of the vertex with mark $p_i$, or $q_m$, or $r_n$ can act as a transmitter.

The following result provides a useful recursive test whether the sequences of non-negative integers form the mark sequences of some 3-partite 2-digraph.

Theorem 2.2. Let $P = [p_1, p_2, \ldots, p_l]$, $Q = [q_1, q_2, \ldots, q_m]$ and $R = [r_1, r_2, \ldots, r_n]$ be the sequences of non-negative integers in non-decreasing order with $p_l \geq 2(m + n)$, $q_m \leq 4(l + n) - 2$ and $r_n \leq 4(l + m) - 2$. Let $P'$ be obtained from $P$ by deleting one entry $p_l$, and let $Q'$ and $R'$ be obtained as follows.

(i) If $p_l \geq 3(m + n)$, then reducing $4(m + n) - p_l$ largest entries of $Q$ and $R$ by one each,

or (ii) If $p_l < 3(m + n)$, then reducing $3(m + n) - p_l$ largest entries of $Q$ and $R$ by two each and $p_l - 2(m + n)$ remaining entries by one each.

Then, $P$, $Q$ and $R$ are the mark sequences of some 3-partite 2-digraph if and only if $P'$, $Q'$ and $R'$ are.

Proof. Let $P'$, $Q'$ and $R'$ be the mark sequences of some 3-partite 2-digraph $D'$ with parts $X'$, $Y'$ and $Z'$. If $Q'$ and $R'$ be obtained from $Q$ and $R$ as in (i), then a 3-partite 2-digraph $D$ with mark sequences $P$, $Q$ and $R$ can be obtained by adding a vertex $x$ in $X'$ such that $x(1-0)y$ for those vertices $v$ of $Y'$ and $Z'$
whose marks are reduced by one in going from $P$, $Q$ and $R$ to $P', Q'$ and $R'$, and $x(2−0)v$ for those vertices $v$ of $Y'$ and $Z'$ whose marks are not reduced in going from $P$, $Q$ and $R$ to $P', Q'$ and $R'$.

If $Q'$ and $R'$ be obtained from $Q$ and $R$ as in (ii), then again a 3-partite 2-digraph $D$ with mark sequences $P$, $Q$ and $R$ are obtained by adding a vertex $x$ in $X'$ such that $x(1−0)v$ for those vertices $v$ of $Y'$ and $Z'$ whose marks are reduced by one in going from $P$, $Q$ and $R$ to $P', Q'$ and $R'$.

Conversely, suppose $P$, $Q$ and $R$ be the mark sequences of a 3-partite 2-digraph $D$ with parts $X$, $Y$ and $Z$. By Corollary 2.1, any of the vertex $x$, or $y$, or $z$ with mark $p_i$, or $q_i$, or $r_i$ respectively can be a transmitter. Let the vertex $x$ with mark $p_i$ be a transmitter. Clearly, $p_i ≥ 2(m + n)$, $q_m ≤ 4(l + n) − 2$ and $r_n ≤ 4(l + m) − 2$ because (a) if $p_i < 2(m + n)$, then by deleting $p_i$ we have to reduce more than $m + n$ entries from $Q$ and $R$, which is absurd, (b) if $q_m > 4(l + n) − 2$ and $r_n > 4(l + m) − 2$, then on reduction $q'_m = q_m − 1 > 4(l + n) − 3 = 4(l − 1 + n) + 1$, or $q'_m = q_m − 2 > 4(l + n) − 4 = 4(l − 1 + n) + 1$, or $r'_n = r_n − 1 > 4(l + m) − 3 = 4(l − 1 + m) + 1$, or $r'_n = r_n − 2 > 4(l + m) − 4 = 4(l − 1 + m)$, which in all cases is impossible.

(i) If $p_i ≥ 3(m + n)$, let $V$ be the set of $4(m + n) − p_i$ vertices of largest marks in $Y$ and $Z$, and let $W = (Y ∪ Z) − V$. Construct $D$ such that $x(1−0)v$ for all $v ∈ V$, and $x(2−0)w$ for all $w ∈ W$. Clearly, $D−x$ realizes $P'$, $Q'$ and $R'$ (arranged in non-decreasing order).

(ii) If $p_i < 3(m + n)$, let $V$ be the set of $3(m + n) − p_i$ vertices of largest marks in $Y$ and $Z$, and let $W = (Y ∪ Z) − V$. Construct $D$ such that $x(1−1)v$
( or \(x(0-0)v\) for all \(v \in V\), and \(x(1-0)w\) for all \(w \in W\). Then, again \(D-x\) realizes \(P', Q',\) and \(R'\) ( arranged in non-decreasing order ).

Theorem 2.2 provides an algorithm for determining whether or not the sequences \(P, Q\) and \(R\) of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding 3-partite 2-digraph. Let \(P = [p_1, p_2, \ldots, p_l]\), \(Q = [q_1, q_2, \ldots, q_m]\) and \(R = [r_1, r_2, \ldots, r_n]\), where \(p_l \geq 2(m + n)\), \(q_m \leq 4(l + n) - 2\) and \(r_n \leq 4(l + m) - 2\), be the mark sequences of a 3-partite 2-digraph with parts \(X = \{x_1, x_2, \ldots, x_l\}\), \(Y = \{y_1, y_2, \ldots, y_m\}\) and \(Z = \{z_1, z_2, \ldots, z_n\}\) respectively. Deleting \(p_l\) and performing (i) or (ii) of Theorem 2.2 according as \(p_l \geq 3(m + n)\) or \(p_l < 3(m + n)\), we get \(Q' = [q_1', q_2', \ldots, q_m']\) and \(R' = [r_1', r_2', \ldots, r_n']\). If the marks of the vertices \(y_j\) and \(z_k\) were decreased by one in this process, then the construction yielded \(x_i(1-0)y_j\) and \(x_i(1-0)z_k\), and if these were decreased by two, then the construction yielded \(x_i(1-1)y_j\) and \(x_i(1-1)z_k\) ( or \(x_i(0-0)y_j\) and \(x_i(0-0)z_k\) ). For vertices \(y_s\) and \(z_t\) whose marks remained unchanged, the construction yielded \(x_i(2-0)y_s\) and \(x_i(2-0)z_t\). Note that if at least one of the conditions \(p_l \geq 2(m + n)\), \(q_m \leq 4(l + n) - 2\), or \(r_n \leq 4(l + m) - 2\) does not hold, then we delete \(q_m\), or \(r_n\) for which the conditions get satisfied and the same argument is used for defining arcs. If this process is applied recursively, then it tests whether or not \(P, Q\) and \(R\) are the mark sequences, and if \(P, Q\) and \(R\) are the mark sequences, then a 3-partite 2-digraph \(\Delta( P, Q, R )\) with mark sequences \(P, Q\) and \(R\) is constructed.

We illustrate this reduction and the resulting construction with the following example, beginning with the sequences \(P_1, Q_1\) and \(R_1\).

\(P_1 = [7, 12, 17]\) \(Q_1 = [6, 12]\) \(R_1 = [8, 11, 11]\)
P_2 = [7, 12] \quad Q_2 = [6, 11] \quad R_2 = [8, 10, 10]
\quad x_3(1-0)y_2, x_3(1-0)z_3, x_3(1-0)z_2, x_3(2-0)y_1, x_3(2-0)z_1

P_3 = [7] \quad Q_3 = [5, 9] \quad R_3 = [7, 8, 8]
\quad x_2(0-0)y_2, x_2(0-0)z_3, x_2(0-0)z_2, x_2(1-0)y_1, x_2(1-0)z_1

P_4 = [6] \quad Q_4 = [5] \quad R_4 = [5, 6, 6]
\quad y_2(0-0)z_2, y_2(0-0)z_1, y_2(1-0)x_1

P_5 = [5] \quad Q_5 = [4] \quad R_5 = [5, 6]
\quad z_3(1-0)x_1, z_3(1-0)y_1

\quad z_2(1-0)x_1, z_2(1-0)y_1

P_7 = \phi \quad Q_7 = [1] \quad R_7 = [3]
\quad x_1(0-0)y_1, x_1(0-0)z_1

P_8 = \phi \quad Q_8 = [0] \quad R_8 = \phi
\quad z_1(1-0)y_1

Figure 3

The next result follows by using the argument as in Theorem 2.2.
Theorem 2.3. \( P = [p_1, p_2, \ldots, p_l], \ Q = [q_1, q_2, \ldots, q_m] \) and \( R = [r_1, r_2, \ldots, r_n] \) be the sequences of non-negative integers in non-decreasing order with \( p_l \geq 2(m + n), \ q_m \leq 4(l + n) - 2 \) and \( r_n \leq 4(l + m) - 2 \). Let \( P' \) be obtained from \( P \) by deleting one entry \( p_l \), and \( Q' \) and \( R' \) be obtained as follows.

(i) If \( p_l \) is even, then reducing \( \frac{4(m + n) - p_l}{2} \) largest entries of \( Q \) and \( R \) by two each,

or

(ii) If \( p_l \) is odd, then reducing \( \frac{4(m + n) - p_l - 1}{2} \) largest entries of \( Q \) and \( R \) by two each, and reducing the largest among the remaining entries of \( Q \) and \( R \) by one.

Then, \( P, Q \) and \( R \) are the mark sequences of some 3-partite 2-digraph if and only if \( P', Q' \) and \( R' \) are.

Theorem 2.3 also provides an algorithm of checking whether or not the sequences \( P, Q \) and \( R \) of non-negative integers in non-decreasing order are the mark sequences, and for constructing a corresponding 3-partite 2-digraph. Let \( P = [p_1, p_2, \ldots, p_l], \ Q = [q_1, q_2, \ldots, q_m] \) and \( R = [r_1, r_2, \ldots, r_n] \), where \( p_l \geq 2(m + n), \ q_m \leq 4(l + n) - 2 \) and \( r_n \leq 4(l + m) - 2 \), be the mark sequences of a 3-partite 2-digraph with parts \( X = \{x_1, x_2, \ldots, x_l\} \), \( Y = \{y_1, y_2, \ldots, y_m\} \) and \( Z = \{z_1, z_2, \ldots, z_n\} \) respectively. Deleting \( p_l \) and performing (i) or (ii) of Theorem 2.3 according as \( p_l \) is even or odd, we get \( Q' = [q'_1, q'_2, \ldots, q'_m] \) and \( R' = [r'_1, r'_2, \ldots, r'_n] \).

If the marks of the vertices \( y_j \) and \( z_k \) were decreased by one in this process, then the construction yielded \( x_l(1-0)y_j \) and \( x_l(1-0)z_k \), and if these were decreased by two, then the construction yielded \( x_l(1-1)y_j \) and \( x_l(1-1)z_k \) (or \( x_l(0-0)y_j \) and \( x_l(0-0)z_k \)). For vertices \( y_s \) and \( z_t \) whose marks remained
unchanged, the construction yielded $x_{i}(2-0)y_{s}$ and $x_{i}(2-0)z_{t}$. Note that if at least one of the conditions $p_{l} \geq 2(m + n)$, or $q_{m} \leq 4(l + n) - 2$, or $r_{n} \leq 4(l + m) - 2$ does not hold, then we delete $q_{m}$, or $r_{n}$ for which the conditions get satisfied and the same argument is used for defining arcs. If this process is applied recursively, then it tests whether or not $P$, $Q$ and $R$ are the mark sequences, and if $P$, $Q$ and $R$ are the mark sequences, then a 3-partite 2-digraph $\Delta( P, Q, R )$ with mark sequences $P$, $Q$ and $R$ is constructed.

We illustrate this reduction and the resulting construction with the following example, beginning with the sequences $P_{1}$, $Q_{1}$ and $R_{1}$.

$P_{1} = [7, 9, 12]$  $Q_{1} = [8, 12]$  $R_{1} = [7, 9]$

$P_{2} = [7, 9]$  $Q_{2} = [8, 10]$  $R_{2} = [7, 7]$

$x_{3}(0-0)y_{2}, x_{3}(0-0)z_{2}, x_{3}(2-0)y_{1}, x_{3}(2-0)z_{1}$

$P_{3} = [7]$  $Q_{3} = [6, 8]$  $R_{3} = [5, 6]$

$x_{2}(0-0)y_{2}, x_{2}(0-0)y_{1}, x_{2}(0-0)z_{2}, x_{2}(1-0)z_{1}$


$y_{2}(0-0)x_{1}, y_{2}(0-0)z_{1}, y_{2}(2-0)z_{2}$

$P_{5} = [3]$  $Q_{5} = \phi$  $R_{5} = [2, 3]$

$y_{2}(0-0)x_{1}, y_{2}(0-0)z_{2}, y_{2}(0-0)z_{1}$

$P_{6} = [2]$  $Q_{6} = \phi$  $R_{6} = [2]$

$z_{2}(1-0)x_{1}$

$P_{7} = \phi$  $Q_{7} = \phi$  $R_{7} = [0]$

$x_{1}(0-0)z_{1}$. 


The next result gives a simple criterion for determining whether three sequences of non-negative integers in non-decreasing order are realizable as marks.

**Theorem 2.4.** Let \( P = [p_1, p_2, \ldots, p_l] \), \( Q = [q_1, q_2, \ldots, q_m] \) and \( R = [r_1, r_2, \ldots, r_n] \) be the sequences of non-negative integers in non-decreasing order. Then, \( P, Q \) and \( R \) are the mark sequences of some 3-partite 2-digraph if and only if

\[
\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k \geq 4(fg + gh + hf), \tag{2.4.1}
\]

for \( 1 \leq f \leq l \), \( 1 \leq g \leq m \) and \( 1 \leq h \leq n \), with equality when \( f = l \), \( g = m \) and \( h = n \).

**Proof.** A sub-3-partite 2-digraph induced by \( f \) vertices from the first part, \( g \) vertices from the second part and \( h \) vertices from the third part has a sum of marks \( 4( fg + gh + hf ) \). This proves the necessity.

For sufficiency, assume that \( P = [p_1, p_2, \ldots, p_l] \), \( Q = [q_1, q_2, \ldots, q_m] \) and \( R = [r_1, r_2, \ldots, r_n] \) are the sequences of non-negative integers in non-decreasing order satisfying the conditions (2.4.1) but are not mark sequences of any 3-partite 2-digraph. Let these sequences be chosen in such a way that \( l, m \) and \( n \) are the smallest possible and \( p_1 \) is the least with that choice of \( l, m \) and \( n \). We have the following two cases.
Case (a). Suppose equality in (2.4.1) holds for some \( f < l, \ g \leq m \) and \( h \leq n, \) so that

\[
\sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k = 4(fg + gh + hf).
\]

By the minimality of \( l, \ m \) and \( n, \ P_1 = [p_1, p_2, \ldots, p_l], \ Q_1 = [q_1, q_2, \ldots, q_g] \) and \( R_1 = [r_1, r_2, \ldots, r_h] \) are the mark sequences of some 3-partite 2-digraph \( D_1(X_1, Y_1, Z_1). \) Let \( P_2 = [p_{f+1} - 4(g + h), p_{f+2} - 4(g + h), \ldots, p_l - 4(g + h)], \ Q_2 = [q_{g+1} - 4(f + h), q_{g+2} - 4(f + h), \ldots, q_m - 4(f + h)] \) and \( R_2 = [r_{h+1} - 4(f + g), r_{h+2} - 4(f + g), \ldots, r_n - 4(f + g)]. \) Now,

\[
\sum_{i=1}^{f} (p_{f+i} - 4(g + h)) + \sum_{j=1}^{g} (q_{g+j} - 4(f + h)) + \sum_{k=1}^{h} (r_{h+k} - 4(f + g))
\]

\[
= \sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k - \left( \sum_{i=1}^{f} p_i + \sum_{j=1}^{g} q_j + \sum_{k=1}^{h} r_k \right)
\]

\[
- 4F(g + h) - 4G(f + h) - 4H(f + g)
\]

\[
\geq 4(\left( f + F \right) (g + G) + (g + G) (h + H) + (h + H) (f + F))
\]

\[
- 4(fg + gh + hf) - 4F(g + h) - 4G(f + h) - 4H(f + g)
\]

\[
= 4(fg + fG + Fg + FG + gh + gH + Gh + GH + hf + hF + Hf + HF
\]

\[
- fg - gh - hf - Fg - Fh - Gf - Gh - Hf - Hg)
\]

\[
= 4(FG + GH + HF),
\]

for \( 1 \leq F \leq l - f, \ 1 \leq G \leq m - g \) and \( 1 \leq H \leq n - h, \) with equality when \( F = l - f, \ G = m - g \) and \( H = n - h. \) So, by the minimality for \( l, \ m \) and \( n, \) the sequences \( P_2, Q_2 \) and \( R_2 \) form the mark sequences of some 3-partite 2-digraph \( D_2(X_2, Y_2, Z_2). \) Now, construct a new 3-partite 2-digraph \( D(X, Y, Z) \) as follows.

Let \( X = X_1 \cup X_2, \ Y = Y_1 \cup Y_2, \ Z = Z_1 \cup Z_2 \) with \( X_1 \cap X_2 = \emptyset, \ Y_1 \cap Y_2 = \emptyset, \ Z_1 \cap Z_2 = \emptyset. \) Let \( x_2(2-0)y_1, \ x_2(2-0)z_1, \ y_2(2-0)x_1, \ y_2(2-0)z_1, \ z_2(2-0)x_1 \) and
$z_2(2-0)y_1$ for all $x_i \in X_i$, $y_i \in Y_i$, $z_i \in Z_i$ where $1 \leq i \leq 2$, so that we get the 3-partite 2-digraph $D( X , Y , Z )$ with mark sequences $P$, $Q$ and $R$, which is a contradiction.

**Case (b).** Suppose that the strict inequality holds in (2.4.1) for $f \neq l$, $g \neq m$ and $h \neq n$. Assume that $p_1 > 0$. Let $P_1 = [p_1 - 1, p_2, \ldots, p_{l-1}, p_l + 1]$, $Q_1 = [q_1, q_2, \ldots, q_m]$ and $R_1 = [r_1, r_2, \ldots, r_n]$, so that $P_1$, $Q_1$ and $R_1$ satisfy the conditions (2.4.1). Thus, by the minimality of $p_1$, the sequences $P_1$, $Q_1$ and $R_1$ are the mark sequences of some 3-partite 2-digraph $D_1( X_1 , Y_1 , Z_1 )$. Let $p_{x_i} = p_i - 1$ and $p_{x_i} = p_i + 1$. Since $p_{x_i} > p_{x_i} + 1$, therefore there exists a vertex $v$ either in $Y_1$ or in $Z_1$ such that $x_i(0-0)v(2-0)x_1$ (or $x_i(1-1)v(2-0)x_1$), or $x_i(1-0)v(2-0)x_1$, or $x_i(2-0)v(2-0)x_1$, or $x_i(1-0)v(1-0)x_1$, or $x_i(2-0)v(1-0)x_1$, or $x_i(2-0)v(0-0)x_1$ (or $x_i(2-0)v(1-1)x_1$) in $D_1( X_1 , Y_1 , Z_1 )$, and if these are changed to $x_i(0-1)v(1-0)x_1$, or $x_i(0-0)v(1-0)x_1$, or $x_i(1-0)v(1-0)x_1$, or $x_i(0-0)v(0-0)x_1$, or $x_i(1-0)v(0-0)x_1$, or $x_i(1-0)v(0-1)x_1$ respectively, the result is a 3-partite 2-digraph with mark sequences $P$, $Q$ and $R$, which is a contradiction. This proves the result.

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Effect of variable viscosity on combined forced and free convection boundary-layer flow over a horizontal plate with blowing or suction

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Abstract: The effects of variable viscosity, blowing or suction on mixed convection flow of a viscous incompressible fluid past a semi-infinite horizontal flat plate aligned parallel to a uniform free stream in the presence of the wall temperature distribution inversely proportional to the square root of the distance from the leading edge have been investigated. The equations governing the flow are transformed into a system of coupled non-linear ordinary differential equations by using similarity variables. The similarity equations have been solved numerically. The effect of the viscosity temperature parameter, the buoyancy parameter and the blowing or suction parameter on the velocity and temperature profiles as well as on the skin-friction coefficient and the Nusselt number are discussed.

Keywords: Mixed convection; variable viscosity; blowing or suction.

1. Introduction:

Combined forced and free convection or "mixed" convection arises in many transport processes in natural and engineering applications. Atmospheric-boundary layer flow, heat exchangers, solar collectors, nuclear reactors and electronic equipment are examples in which the effect of buoyancy force on forced flow is significant.

In contrast to the problem of mixed convective flow along a vertical flat plate, less attention has been given to studies of buoyancy force effects
on laminar forced convection over a horizontal flat plate. Mori [1] and Sparrow and Minkowycz [2] were the first investigators to treat this problem. Since then, extensive studies which has been conducted by Schneider [3], Dey[4], Ramarchandran et.al.[5], Raju et.al.[6], De Hong et.al.[7], Afzal and Hussain [8], Merkin and Ingham [9], Risbeck et.al.[10], Schneider et.al. [11], Risbeck and Chen [12], Steinrück[13], Rudischer and Steinruck [14], Rudischer and Steinrück [14] and Magyari et.al. [15].

In all the above mentioned studies, the fluid viscosity was assumed uniform in the flow region. But it is known that these physical properties change significantly with temperature. To illustrate the need to include this viscosity temperature variation, we quote two examples of common fluids: the viscosity of carbon tetrachloride varies from 1.329 centipoise at $0^\circ C$ to 0.384 cp at $100^\circ C$; Olive oil has viscosities of 138 and 12.4 cp for respective temperature of $10^\circ C$ and $70^\circ C$ [16].

A survey of literature reveals that the combined effects of variable viscosity and buoyancy forces on mixed convection heat transfer over a semi-infinite horizontal plate in the presence of blowing or suction have not been studied yet.

In the present study it is proposed to investigate the mixed convection flow of a viscous incompressible fluid having viscosity depending on temperature from horizontal flat plate with a non-uniform temperature in the presence of blowing or suction.
2- Mathematical Formulation

Let us consider the steady two-dimensional laminar free-forced convective flow of a viscous incompressible fluid over a semi-infinite horizontal flat plate aligned parallel to a uniform free stream with velocity \( u_\infty \), density \( \rho_\infty \) and temperature \( T_\infty \) in the presence of the wall temperature distribution \( T_w(x) \sim x^{-1/2} \). The \( x \)-axis is measured from the leading edge along the plate and the \( y \)-axis is normal to it. We assume that property variation with temperature are limited to viscosity, and density. The temperature dependent density is taken in the buoyancy force term in the momentum equation only. Neglecting the viscous dissipation, and under Boussinesq approximation, the two-dimensional boundary layer equations for the mixed convection flow of fluid past a semi-infinite horizontal plate may be written as [17].

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{1},
\]

\[
-\frac{\partial \bar{u}}{\partial x} u + v \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho_\infty} \frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho_\infty} \frac{\partial}{\partial y} (\mu \frac{\partial \bar{u}}{\partial y}) \tag{2-a},
\]

\[
-\frac{1}{\rho_\infty} \frac{\partial \bar{p}}{\partial y} = g \beta (T - T_\infty) \tag{2-b},
\]

\[
-\frac{\partial \bar{T}}{\partial x} + v \frac{\partial \bar{T}}{\partial y} = \frac{1}{\rho_\infty c_p} \frac{\partial}{\partial y} (k \frac{\partial \bar{T}}{\partial y}) \tag{3},
\]

where \( \bar{u} \) and \( \bar{v} \) the velocity components in the \( x \) and \( y \) directions, respectively, \( \bar{\mu} \) is the viscosity of the fluid, \( T \) is the temperature of the fluid in the boundary layer, \( g \) is the acceleration due to gravity, \( \beta \) is the coefficient of volumetric expansion, \( \bar{p} \) is the pressure, \( k \) is the thermal conductivity and \( c_p \) is the specific heat at constant pressure.
The boundary conditions to be satisfied are given by

\[ \tilde{v} = 0, \quad \tilde{u} = 0, \quad T = T_\infty(x), \]  
\[ \tilde{v} \to \infty : \quad \tilde{u} \to u_\infty, \quad \tilde{p} \to 0, \quad T \to T_\infty. \]

For a viscous fluid, Ling and Dybbs [18] and Lai and Kulacki [19] suggest a viscosity \( \bar{\mu} \) dependence on temperature \( T \) of the form

\[ \bar{\mu} = \frac{\mu_\infty}{1 + \gamma(T - T_\infty)} \quad \text{or} \quad \frac{1}{\mu} = \alpha(T - T_\infty), \]

with \( \alpha = \frac{\gamma}{\mu_\infty}, \quad T_\infty = T_\infty - \frac{1}{\gamma}, \)

where \( \alpha \) and \( T_\infty \) are constants and their values depend on the reference state of the fluid. In general, \( \alpha < 0 \) for gases and \( \alpha > 0 \) for liquids.

Introducing the following non-dimensional variables into equation (1) – (3):

\[ x = \frac{x}{\ell}, \quad y = \frac{\sqrt{Re}}{\ell}, \quad T_\infty(x) = T_\infty + T^* / \sqrt{x} \]

\[ \eta = \frac{y}{\sqrt{x}}, \quad \psi = v_* \sqrt{Re} x f(\eta), \quad v_* = \frac{\mu_\infty}{\rho_\infty} \]

\[ T - T_\infty = T^* \frac{\theta(\eta)}{\sqrt{x}}, \quad \bar{P} - P_\infty = \rho u_\infty^2 p(\eta), \]

where \( R_e \) is the Reynolds number, \( T^* \) represents a characteristic temperature difference between the plate and free stream and \( l \) is a reference length, we get

\[ 2\mu f''' + 2\mu f'' + f f'' + \lambda \eta \theta = 0, \]  
\[ 2\theta'' + p, \quad (f' \theta + f \theta') = 0, \]

where \( \mu = \frac{\mu}{\mu_\infty} \), \( \mu_\infty \) is the viscosity of the ambient fluid, and the prime denotes differentiation with respect to \( \eta \).

Introducing equation (5) into equations (8) and (9), we have
\[ f''' - \frac{\theta'}{\theta - \theta_r} f'' - \frac{2(\theta - \theta_r)}{\theta_r} (f'f'' + \lambda \eta \theta) = 0, \quad (10) \]

\[ 2\theta'' + p_r (f'\theta + f'\theta') = 0, \quad (11) \]

where \( \theta_r \) is a constant viscosity-temperature parameter given by

\[ \theta_r = \frac{T_r - T_\infty}{T_w - T_\infty} = \frac{1}{\gamma(T_w - T_\infty)}, \quad (12) \]

\( P_r = \nu_w / k_w \) is the Prandtl number, and \( \lambda = \frac{g\beta \gamma^*}{\sqrt{R_e u_w^2}} \) is the mixed convection parameter.

Equation (7) transform the boundary conditions (4) into

\[ \eta = 0 : \quad f = f_w, \quad f' = 0, \quad \theta = 1, \quad (13) \]

\[ \eta \to \infty : \quad f' \to 1, \quad \theta = 0, \]

where \( f_w = -2\sqrt{xw} \) is the blowing (<0) or the suction (>0) parameter. Integrating equation (11) and using the boundary conditions (13) we obtain

\[ \theta'' + p_r f\theta = 0. \quad (14) \]

It is interesting to note that if \( f_w = 0 \), then \( \theta'(0) = 0 \) for all values of \( \theta_r, p, \) and \( \lambda \). This means that no heat transferred, except for the singularity at the leading edge \( x = 0 \) of the plate [3].

The shearing stress at the plate is defined by

\[ \tau_w = \left( \frac{\mu}{\partial y} \right)_{y=0}. \quad (15) \]

The local skin friction coefficient is defined by

\[ C_p = \frac{2\tau_w}{\rho_w u_w^2} = 2\sqrt{R_e} \frac{\theta_r}{(\theta_r - 1)} f''(0, \theta_r). \quad (15) \]
where $R_\infty$ is the local Reynolds number.

The local Nusselt number is defined by

$$N_\infty = \sqrt{R_\infty} \theta' (0, \theta_c)$$  \hspace{1cm} (16)

3. Solution and discussion

The coupled non-linear ordinary differential equations (10) and (11) along with the boundary conditions (13) are solved numerically by the fourth-order Runge-Kutta scheme with the Newton-Raphson iteration method. This numerical solution technique is similar to that described in ref. [20]. The accuracy of the numerical results can be verified by comparing our results taking $f_w = 0$ with that obtained by Pop and Gorla [21] where $n = 1$. It was found that the results are in good agreement as shown in table 1.

Samples of the resulting velocity and temperature profiles for $p_r = 0.72$ and different values of the mixed-convection parameter $\lambda$, the temperature parameter $\theta$, and the blowing or suction parameter $f_w$ are presented in Figs.1-3. The effect of the mixed-convection parameter $\lambda$ and the temperature parameter $\theta$, on the dimensionless velocity $f'$ and the dimensionless temperature $\theta$ in the presence of suction, $f_w = 0.5$ is shown in Fig.1. This figure shows that the velocity gradient at the wall and the overshoot of the velocity decreases while it is accompanied by a further location of the peak from the wall as $\lambda$ or $\theta$, decreases. But at a certain distance from the plate, $\eta = 2.12$ for $\lambda = 2$ and $\eta = 3.17$ for $\lambda = 0.5$ it is noticed that the velocity increases as $\theta$, decreases. Also, it is observed from Fig.1 that the temperature drops more quickly and the thermal boundary layer thickness becomes thinner when $\lambda$ or $\theta$, is higher. This means that the mixed-convection parameter and the viscosity temperature parameter
effects have a tendency to induce more flow near the plate at the expense of small reduction in temperature. Figure 2 displays the effects of $\lambda$ and $\theta$, in the presence of blowing, $f_\omega = -0.5$. It is found from this figure that the dimensionless velocity increases near the plate as $\lambda$ increases. But an opposite effect is noticed at a certain distance from the plate $\eta = 2.36$ for $\lambda = 2$ and $\eta = 3.31$ for $\lambda = 0.5$.

The effect of the viscosity temperature parameter $\theta$, on $f'$ in the presence of injection is similar to the suction case. Also, we see that from Fig. 2 that there are sharp rises in the temperature profiles near the wall which yields an overshoot of fluid temperature beyond the wall temperature, especially for small values of $\lambda$ and $\theta$. These temperature distributions are quite different from those profiles shown in Fig. 1. The influence of the blowing / suction parameter $f_\omega$ on $f'$ and $\theta$ is shown in Fig. 3. It can be seen from this figure that the maximum velocities are decreased with increasing $f_\omega$.

Also, we see that as $f_\omega$ increases from negative to positive values, the temperature gradient at the wall decreases from positive to negative, as predicted by Eq. (14). From table 2, one sees that in the presence of blowing or suction the skin-friction coefficient increases as either $\lambda$ or $\theta$, increases. For fixed values of $\lambda$ and $\theta$, the skin-friction coefficient increased with the increasing of the blowing parameter and decreased as the suction parameter was increased. Also, it is found from table 2 that the Nusselt number increases as the suction parameter increases while it increases as the blowing parameter increases.

4. Conclusion

The problem of mixed convection in laminar boundary layer flow along horizontal flat plate with variable viscosity in the presence of
blowing or suction is analyzed. The governing equations are transformed and solved numerically by means of the shooting technique. It was found that the velocity increases near the plate as $\lambda$ or $\theta$, increases, while the thermal boundary layer thickness decreases as $\lambda$ or $\theta$, increases. Also, it was found that in the presence of suction the velocity decreased as the suction parameter was increased and increased as the blowing parameter was increased. Furthermore it was found that the temperature increased as the suction parameter was increased. The opposite is true in the presence of blowing.

Table 1. Comparison of $f''(0)$ for various values of $\lambda$

<table>
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<tr>
<th>$\lambda$</th>
<th>Pop and Gorla</th>
<th>Present</th>
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<tr>
<td>0</td>
<td>0.3320</td>
<td>0.3320</td>
</tr>
<tr>
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<tr>
<td>0.5</td>
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<td>0.7740</td>
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<tr>
<td>1</td>
<td>1.0574</td>
<td>1.0545</td>
</tr>
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</table>
Table 2. Values of $f''(0)$ and $-\theta'(0)$ for various values of $\theta$, $\lambda$ and $f_*$ with $p_* = 0.72$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$f_*$</th>
<th>$\theta$</th>
<th>$f''(0)$</th>
<th>$-\theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.40122</td>
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</tr>
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<td>0.07199</td>
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<tr>
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Fig. 1. Velocity and temperature distribution for various values of $\phi$ and $\theta$.

Fig. 2. Velocity and temperature distribution for various values of $\phi$ and $\theta$. 
Fig. 3. Velocity and temperature distribution for various values of $f_u$. 
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EFFECT OF PARTITION AND SPECIES DIFFUSIVITY ON DOUBLE DIFFUSIVE CONVECTION OF WATER NEAR DENSITY MAXIMUM

S SIVASANKARAN AND P KANDASWAMY

ABSTRACT. The double diffusive convection of cold water in the vicinity of its density maximum in a rectangular partitioned enclosure of aspect ratio 5 with isothermal side walls and insulated top and bottom is studied numerically. A thin partition is attached to the hot wall. The species diffusivity of the fluid is assumed to vary linearly with concentration. The governing equations are solved by finite difference scheme. The effects of position and height of the partition, variable species diffusivity and enclosure width are analyzed for various hot wall temperatures. It has been found that adding partition on the hot wall reduces the heat transfer. The density inversion of the water has a great influence on the natural convection. When increasing species diffusivity parameter heat and mass transfer rate is decreased.

2000 Mathematics Subject Classification. 76R10, 76M20, 80A20.

Keywords. Convection, Density Maximum, Partition enclosure, Species diffusivity

1. INTRODUCTION

Double diffusive convection in complex geometry has attracted the interest of several researchers. Double diffusive convection occurs in a very wide range of fields such as Oceanography, Astrophysics, Geology, Biology, Chemical process etc. Most works about cavities of complex geometry deal with partitions fitted to insulated walls. Cavities with baffles on their active walls have been less studied. Heat transfer in partially divided enclosures has received attention primarily due to its many engineering and physical applications such as in the design of energy efficient building, reduction of heat loss from flat plate solar collectors, natural gas storage tank, crystal manufacturing and metal solidification processes. This study describes the effect of a thin baffle at the hot wall of a water filled, differentially heated rectangular enclosure. Braga and Viskanta [1992] made an experimental and theoretical investigation of transient natural convection heat transfer of water near its maximum density in a rectangular cavity. They found two counter rotating eddies in all experiments caused by density inversion. Dag tekin and Oztop [2001] analyzed numerically the effect of positions and height of two partitions on natural convection within an enclosure. They found that the effect of the position of the partitions on the fluid flow more than that on heat transfer.

Frederick [1989] investigated numerically the natural convection of differentially heated cavity with a diathermal partition on its cold wall. He also described the results for inclined and rectangular cavity. He concluded that the partition causes convection suppression and heat transfer reductions up to 47% relative to the undivided cavity at the same Rayleigh number. Hyun and Lee [1990] investigated double
diffusive convection in a rectangular cavity with imposed temperature and concentration gradients in the horizontal direction. They found mean Sherwood number decreases monotonically as buoyancy ratio increases. Lin and Nansteel [1987] analyzed numerically the steady natural convection of cold water near the density extremum in a vertical annulus. They found that density inversion phenomena are altered substantially by curvature of the annulus. Mamou et al. [1996] studied both analytically and numerically double diffusive natural convection in a rectangular slot subject to uniform heat and mass fluxes along the vertical sides. They found a good agreement between the analytical predictions and the numerical simulation.

Robillard and Vasseur [1981] studied maximum density effect of water on laminar natural convection in a rectangular cavity with a convective boundary. Tasnim and Collins [2004] numerically investigated the effect of attaching a high conducting thin baffle on the hot wall of a square cavity. They concluded that adding baffle on the hot wall increase the rate of heat transfer about 31.46% compared with a wall without baffle. Vasseur and Robillard [1980] investigated the inversion of flow patterns caused by the maximum density of water at 4°C enclosed in rectangular cavities. Wang et al. [1999] studied numerically the natural convection in a partially divided rectangular enclosure and conclude that the location of the divider effect the heat transfer performance and height of the divider influences the heat transfer significantly. Zimmerman and Acharya [1987] numerically studied natural convection in an enclosure with finitely conducting baffles. They found out growing and spreading of the counter clockwise vortex in the entire rectangular cavity appears to be a slower process than in the case of square cavity.

There is a considerable number of numerical studies concerning on the double diffusive convection in two dimensional partitioned enclosures. However such studies are typically devoted to a linear density temperature dependent fluid. For a number of fluids, the density-temperature relation exhibits an extremum. Because the coefficient of thermal expansion changes sign at this extremum, simple linear relations for density as a function of temperature are inadequate near the extremum. The most common fluid, water has such behaviour. Sundaravadivelu and Kandaswamy [2000] analyzed the natural convection flow of pure water around its temperature of maximum density existing between temperatures 273 K and 285 K by using a non-linear temperature dependent equation for density in a square cavity. In the present study we continue to examine the effect of position and length of the baffle and species diffusivity in rectangular partitioned enclosure with different hot wall temperatures using a non linear density temperature profile.

2. MATHEMATICAL FORMULATION

The physical system under consideration is a two dimensional rectangular partitioned enclosure of width \( W \) and height \( H \) filled with water and some mass, Fig. 1. The vertical side walls of the enclosure are isothermal but maintained at different temperatures \( \theta_h \) (hot wall) and \( \theta_c \) (cold wall) with \( \theta_h > \theta_c \). The horizontal
walls are thermally insulated. A very thin adiabatic partition of length \( L_B \) attached on the hot wall at a height \( H_B \) from the bottom of the enclosure. The concentration level of mass is taken to be \( c_2 \) at \( y = 0 \) and \( c_1 \) at \( y = W \), where \( c_2 < c_1 \). The species diffusivity of the fluid \( D \) is assumed to vary linearly with concentration as \( D = D_0 \left[ 1 - a (c - c_0) \right] \), where \( a \) is the concentration coefficient of the diffusion and the subscript \( o \) refers to the reference state. A density-temperature 4th degree polynomial fit of the form \( \rho = \rho_o \left[ 1 - \sum_{i=1}^{4} (-1)^i \beta_i (\theta - \theta_o) - \beta_5 (c - c_2) \right] \) for pure water for the data available in the literature is found to be an ideal one with \( \rho_o = 9.9984 \times 10^2 \), \( \beta_1 = 6.8143 \times 10^{-5} \), \( \beta_2 = 9.9901 \times 10^{-6} \), \( \beta_3 = 2.7217 \times 10^{-7} \), \( \beta_4 = 6.7252 \times 10^{-9} \), and \( \beta_5 = 3 \times 10^{-3} \). A graphical representation of the relation provided in Fig. 2 gives a clear demonstration of the fit. Further there is no chemical reaction between the mass and the fluid. The cartesian coordinates \((x, y)\) with their corresponding velocity components \((u, v)\) are as indicated in Fig. 1.

The nondimensional vorticity-stream function formulation of the laminar two-dimensional incompressible flow of the fluid under consideration are

\begin{align}
\frac{\partial T}{\partial \tau} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial X} &= \frac{1}{Pr} \nabla^2 T \\
\frac{\partial C}{\partial \tau} + U \frac{\partial C}{\partial X} + V \frac{\partial C}{\partial X} &= \frac{1}{Sc} \left\{ (1 - \lambda C) \nabla^2 C - \lambda \left[ \left( \frac{\partial C}{\partial X} \right)^2 + \left( \frac{\partial C}{\partial Y} \right)^2 \right] \right\} \\
\frac{\partial \zeta}{\partial \tau} + U \frac{\partial \zeta}{\partial X} + V \frac{\partial \zeta}{\partial Y} &= \sum_{i=1}^{4} i(-1)^i \text{Gr}_i \frac{\partial T}{\partial Y} + \text{Gr}_5 \frac{\partial C}{\partial Y} + \nabla^2 \zeta \\
\nabla^2 \Psi &= -\zeta
\end{align}

where \( U = -\frac{\partial \Psi}{\partial Y}, \ V = \frac{\partial \Psi}{\partial X} \) and \( \zeta = \frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} \)

with the initial and boundary conditions

\begin{align}
\tau &= 0; \quad \zeta = \psi = 0; \quad T = C = 0; \quad 0 \leq X \leq Ar; \quad 0 \leq Y \leq 1 \\
\tau &> 0; \quad \Psi = \frac{\partial \Psi}{\partial X} = 0; \quad \frac{\partial T}{\partial X} = \frac{\partial C}{\partial X} = 0; \quad X = 0, \ Ar \ & \text{ on the baffle; } \quad 0 \leq Y \leq 1 \\
\Psi &= \frac{\partial \Psi}{\partial X} = 0; \quad T = 1, \ C = 0; \quad Y = 0; \quad 0 \leq X \leq Ar \\
\Psi &= \frac{\partial \Psi}{\partial X} = 0; \quad T = 0, \ C = 1; \quad Y = 1; \quad 0 \leq X \leq Ar
\end{align}

The following non-dimensional variables are used to transform the dimensional equations to non-dimensional form are

\begin{align}
X &= \frac{x}{W}, \quad Y = \frac{y}{W}, \quad U = \frac{u}{\nu/W}, \quad V = \frac{v}{\nu/W}, \quad \tau = \frac{t}{W^2/\nu}, \quad \Psi = \frac{\psi}{\nu}
\end{align}
\[ \zeta = \frac{\omega}{\nu/W^2}, \quad T = \frac{\theta - \theta_c}{\theta_h - \theta_c}, \quad C = \frac{c - c_0}{c_1 - c_2} \text{ where } \theta_h > \theta_c \text{ and } c_1 > c_2. \]

The non-dimensional parameters that appear in the equations are the thermal Grashof numbers \( Gr_i^T = (g\beta_i(\theta_h - \theta_c)W^3)/\nu^2, \ i = 1, 2, 3, 4, \) the species Grashof number \( Gr_C^T = (g\beta_s(c_1 - c_2)W^3)/\nu^2, \) the Prandtl number \( Pr = \nu/\alpha, \) the Schmidt number \( Sc = \nu/D, \) the aspect ratio \( Ar = H/W, \) and the species diffusivity parameter \( \lambda = a(c_1 - c_2), \) where \( g \) is gravity, \( \alpha \) is the thermal diffusivity, \( \beta \) is the coefficient of thermal expansion and \( \nu \) is the kinematic viscosity. The local Nusselt number and Sherwood number are defined by \( Nu = \frac{\partial T}{\partial Y} \big|_{Y=0} \) and \( Sh = \frac{\partial C}{\partial Y} \big|_{Y=1}, \) resulting in the average Nusselt number and Sherwood number as
\[ \overline{Nu} = \frac{1}{Ar} \int_0^{Ar} Nu \, dX \quad \text{and} \quad \overline{Sh} = \frac{1}{Ar} \int_0^{Ar} Sh \, dX. \]

3. THE METHOD OF SOLUTION

Numerical solution of the governing equations is obtained using finite difference method. An approximate solution of the equations is obtained at a finite number of grid points distributed over the rectangular enclosure, having the coordinates \( x = ih, y = ik, \) and at discrete times \( \tau = n\Delta \tau \) where \( i, j, n \) are integers and \( h, k \) are small increments. Here we find the solution for a \( 51 \times 251 \) square mesh with \( h = k. \) Knowing all quantities at a time \( \tau = n\Delta \tau \) (the initial condition corresponds to the special case \( n = 0 \)), an Alternating Direction Implicit (ADI) method is employed to find the temperature and velocity values at the interior grid points in the next time level \( (n+1)\Delta \tau. \) For this forward difference approximation is used for time derivatives and central difference approximation are used for all space derivatives. ADI method is a two step approach and requires minimal computer storage and is quite accurate. This approach involves the alternate use of two different finite difference approximations to the two dimensional problem in space. Solutions with more finer meshes \( (61 \times 301 - 101 \times 501) \) have produced in significant improvement results less than \( 1\% \) in velocity and temperature fields with a five fold increase in computer time. Hence the \( 51 \times 251 \) mesh was opted as the ideal one.

The method of Successive Over Relaxation (SOR) gives faster convergence than other relaxation methods. We fix the relaxation parameter to be \( 1.5. \) The velocity components \( U = -\frac{\partial \Phi}{\partial Y} \) and \( V = \frac{\partial \Phi}{\partial X} \) are then found using central difference approximations. After finding all the values at a particular level, the values at the higher levels are similarly computed. This computational cycle is repeated for each of the next levels and steady state solution is obtained when the convergent criteria \( |\Phi_{i,j,n} - \Phi_{i,j,n+1}| < \epsilon (=10^{-5}) \) for temperature, vorticity, species concentration and stream function have been met.
4. RESULTS AND DISCUSSIONS

The effect of concentration dependent species diffusivity on the double diffusive convective motion of cold water at temperatures around its density maximum is investigated numerically within a rectangular partitioned enclosure. The solute taken into consideration is assumed to possess the property of raising the temperature of maximum density. By invoking a fourth-order approximation for density-temperature-concentration relation, results were obtained for various values of parameters like, partition height and length, diffusivity parameter and width of an enclosure. Though the problem is unsteady the results are discussed after the final steady state is reached. The results of the investigations for cavity dimension namely width = 8\, mm are presented in the form of streamlines, isotherms, isoconcentration lines, velocity profiles and average Nusselt and Sherwood numbers for various hot wall temperatures between 279\, K and 285\, K. The cold wall is always maintained at temperature 273\, K.

Fig. 3(a) drawn to predict the fluid motion and the resulting temperature and concentration distribution for hot wall temperature (\(\theta_h\)) of 279\, K and \(\lambda = 0.001\). The stream lines shows a single counter clockwise rotating eddy and occupies the whole enclosure. This is found to be unusual when compared to common vertical flows where density is assumed as linearly varying with temperature. The resulting isotherms clearly show that the heat transfer mode is changed from conduction to convection. The upper region of the isotherms are pushed towards the cold wall and the lower region towards the hot wall. This results in a single buoyancy-induced cell.

When hot wall temperature is raised to 281\, K, Fig. 3(b), there exists a small clockwise rotating eddy near the top corner of hot wall side. This is due to density inversion. As the hot wall temperature increases the size of the hot wall side cell is increased because the maximum density plane is moved from hot wall side to cold wall. The heat and mass transfer process are found to be disturbed due to the resulting biconvex flow structure. For \(\theta_h = 283\, K\) the cell rotating in the clockwise direction in the above biconvex flows, is now found to grow in its size by suppressing its counterpart since the maximum density plane is very near to the cold wall. The same behaviour of flow was observed with increasing the species diffusivity parameter \(\lambda = 0.5\). Hence the graphs are plotted only for the case \(\lambda = 0.001\).

Figs. 4(a-c) show the streamlines, isotherms and isoconcentration lines for different hot wall temperatures with baffle at height \(H_B = 1/2\) and length \(L_B = 1/2\). It is clearly seen from these figures that the baffle affected the streamlines, isotherms and isoconcentration lines. The flow pattern is affected very much. The eddy in Fig. 4(a) circulates in counter clockwise direction and occupies the whole cavity. The movement of the eddy is downward along the hot wall and upward along the cold wall due to high density resulting from anomalous density behaviour. Difference in temperature, being the cause for such fluid motion found, is distributed across the isothermal walls of the cavity. The eddy for \(\theta_h = 279\, K\), occupies the whole cavity with two secondary eddies separated by baffle, that is, one above the baffle and another below the baffle. The concentration profiles are more deviated from the baffle when the density is decreasing. For \(\theta_h = 281\, K\), there exists a small vortex below the baffle as well as a
large clockwise rotating vortex appear above the baffle because the effect of density inversion. Still increasing the hot wall temperature $\theta_h = 283K$, two counter rotating cells exists the hot cell grows in size and suppressing its counterpart. The cold wall vortex also separate into two cells.

We observe that the hot wall side vortices grow in size with increasing temperature gradient by suppressing their respective cold wall side vortices. Therefore a transition in the heat flow behaviour from conduction to convection mode is switched on. Hence the average heat transfer rate is found to increase almost monotonically with increasing hot wall temperature above 283 K. Figs. 5(a & b) show that the streamlines pattern for different hot wall temperatures and $\lambda = 0.001$ with partition at $H_B = 1/4 & 3/4$ and $L_B = 1/2$. When baffle is at $H_B = 3/4$, we observed that the behaviour of flow, heat and concentration structures above the baffle is like as the structure below the baffle when baffle at $H = 1/4$. Figs. 6(a & b) show the streamlines for different hot wall temperature and $\lambda = 0.001$ with partition at $H_B = 1/2 & L_B = 1/4 & 3/4$. For the baffle length $L_B = 1/4$ there is no considerable effect on heat and mass transfer and there is a very small change in flow pattern is observed. Further increasing the baffle length $L_B = 3/4$, the flow pattern, isotherms and isoconcentration lines are more affected than the other two cases like $L_B = 1/4 & 1/2$. The flow structure is totally affected due to narrow gap and behaves like almost two separate regions.

The investigations are also carried out in a cavity with still higher dimension (20 mm) are reported. Figs. 7(a-c) show the streamlines, isotherms and isoconcentration lines for different hot wall temperature, $\lambda = 0.001$ and $L = 20$ mm with baffle at height $H_B = 1/2$ and length $L_B = 1/2$. From all these figures we observe that in general a high buoyancy force results inside the cavity, which in turn drives the fluid motion at a higher velocity. The resulting convection process is found to be more vigorous than in the case of 8 mm cavity. Their respective isotherms and isoconcentration lines are found to be distorted largely. The attraction of isotherms towards the cold wall and hence the formation of thermal boundary layer is clear in Fig. 7(a), showing that considerable amount of heat is propagated from hot wall to cold wall across an enclosure. As the cavity dimension is increased, circulation rate of the eddy becomes larger and the flow vigorous is increased significantly. We also found average Nusselt number is increased.

In order to evaluate how the presence of the partition affects the average Nusselt number along the hot wall, average Nusselt number is plotted as a function of partition length and height for different hot wall temperatures, see Figs. (8-10). Figs. 8(a-c) show the average Nusselt number for $\lambda = 0.001$ and different partition heights and lengths. We see that average Nusselt number decreases with increasing baffle length before density maximum and after the density maximum average Nusselt number is increased. Figs. 9(a-c) show the average Sherwood number for different baffle positions and $\lambda = 0.001$. When changing the position of the partition the effect on average Sherwood number is very small. As increasing the partition length we got some remarkable effects on average Sherwood number. Comparing these three figures, the range of variation of average Sherwood number is small for all values of $\lambda$. 
Average Nusselt and Sherwood numbers for different $\lambda$ values are depicted in the Figs. 10(a & b). As $\lambda$ is increased average Nusselt and Sherwood numbers decrease. There is no considerable variation in heat transfer rate after density inversion. Fig. 11 shows the time history of the average Nusselt number and Sherwood number for various hot wall temperatures. As time evolves the particles near the hot wall have higher temperature and so the heat transfer rate starts decreasing thus we get a sudden fall in the values of $\overline{Nu}$ as seen in the graph. Finally the steady state is reached and the $\overline{Nu}$ is seen to be constant. Average Nusselt number is decreased when increasing the hot wall temperature after one certain stage i.e., at the density maximum average Nusselt number is increased. The mid height velocity profiles are depicted in the Fig. 12 for different hot wall temperatures without partition, reveal the existence of multicellular counter acting flow behaviour from their respective bidirectional velocity distributions.

5. CONCLUSIONS

From the results it is concluded that the multiple fluid vortices exist inside the enclosure due to temperature of maximum density and the size of these vortices strongly depend on the hot wall temperature. The average heat transfer rate calculated is found to be an increasing function of hot wall temperatures. The partition reduces the heat transfer. The positions of the partition have more effects on fluid flow than heat transfer. It is clearly demonstrated that the density inversion of water has a great influence on the natural convection and heat and mass transfer rate is reduced around the density maximum region. The flow structure is totally different from the classical natural convection model which employ the linear density temperature relation. Heat and mass transfer rate is decreased when species diffusivity parameter is increased.
References


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Fig. 1 Physical Configuration

Fig. 2 Temperature dependence of density

(a) $\theta_h = 279$ K  (b) $\theta_h = 281$ K  (c) $\theta_h = 283$ K

Fig. 3 Streamlines, isotherms and isoconcentration lines for $\lambda=0.001$ and without baffle
Fig. 4 Streamlines, isotherms and isoconcentration lines for $\lambda=0.001$ and baffle at $H_B=L_B=1/2$.

Fig. 5 Streamlines for $\lambda=0.001$ and baffle at $L_B=1/2$.

Fig. 6 Streamlines for $\lambda=0.001$ and baffle at $H_B=1/2$.

Fig. 7 Streamlines, isotherms and isoconcentration lines for $\lambda=0.001$, $L=20\text{mm}$ & baffle at $H_B=L_B=1/2$. 
Fig. 8 Average Nusselt number for $\lambda = 0.001$
Fig. 9 Average Sherwood number for $\lambda = 0.001$
Effect of Partition and Species Diffusivity on Double Diffusive Convection of Water Near Density Maximum

Fig. 10 Average Nusselt and Sherwood number for different $\lambda$ & baffle at $H_B = L_B = \frac{1}{2}$.

Fig. 11 Time history of average Nusselt and Sherwood numbers for $\lambda=0.001$ and baffle at $H_B=L_B = 1/2$.

Fig. 12 Mid-height velocity profiles for $\lambda=0.001$
Existence Results for First Order Impulsive Functional Differential Equations in Banach Spaces

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ABSTRACT

In this paper we prove the existence of mild solutions of a first order impulsive initial value problems for functional differential equations in Banach spaces. The results are obtained by using the Leray-Schauder nonlinear alternative fixed point theorem.

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1.INTRODUCTION

This paper is concerned with the existence of mild solutions for the impulsive functional differential equations of the form

\begin{align}
  y' &= Ay + f(t, y_t), \quad t \in J = [0, b], \; t \neq t_k, \; k = 1, 2, \ldots, m \\
  y(t_k^+) &= I_k(y(t_k^-)), \quad k = 1, 2, \ldots, m \\
  y(t) &= \phi(t), \quad t \in [-r, 0]
\end{align}

where $A$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$, $f : J \times D \to E$ is a given function, where $D = \{ \psi : [-r, 0] \to E \}$ is continuous everywhere except for a finite number of points $s$ at which $\psi(s^-)$ and $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)$, $\phi \in C([-r, 0], E)$ and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $I_k \in C(E, E)(k = 1, 2, \ldots, m)$, are bounded functions, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

For any continuous function $y$ defined on the interval $[-r, b] - \{ t_1, t_2, \ldots, t_m \}$ and any $t \in J$, we denote by $y_t$ the element of $C([-r, 0], E)$ defined by

\[ y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0] \]
Here \( y_t(.) \) represents the history of the state from time \( t - r \), up to the present time \( t \).

Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in the biological or medical domain (see the monographs of Bainov and Simeonov\[2\], Lakshmikantham, Bainov and Simeonov\[10\], and Samoilenko and Perestyuk\[12\], and the papers of Agur, Cojocaru, Mazur, Anderson and Danon\[1\], Goldbeter, Li and Dupont\[6\]). Recently, an extension to functional differential equations with impulsive effects has been done by Yujun\[14\] by using the coincidence degree theory. For other results on functional differential equations, we refer to the monograph of Erbe, Kong and Zhang\[5\], Hale\[7\], Henderson\[8\].

The fundamental tools used in the existence proofs of all above-mentioned works are essentially fixed-point arguments, nonlinear alternative, topological transversality\[4\], topological degree theory\[11\], or the monotone method combined with upper and lower solutions\[9\].

In this paper is an extension to impulsive functional differential equations of the results\[3\]. My approach is here based on the Leray-Schauder nonlinear alternative fixed point theorem.

2. PRELIMINARIES

In this section, we give notations, definitions, and preliminary facts which are used throughout this paper.

\( C([-r, 0], E) \) is the Banach space of all continuous functions from \([-r, 0]\) into \( E \) with the norm

\[
\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}. 
\]

By \( C(J, E) \), we denote the Banach space of all continuous functions from \( J \) into \( E \) with the norm

\[
\|y\|_J = \sup\{|y(t)| : t \in J\}.
\]

Let \( B(E) \) be the Banach space of bounded linear operators from \( E \) into \( E \).

A measurable function \( y : J \to E \) is Bochner integrable if and only if \(|y|\) is Lebesgue integrable. (For properties of the Bochner integral, see Yosida \[13\].)

\( L'(J, E) \) denotes the Banach space of function \( y : J \to E \) which are Bochner integrable normed by

\[
\|y\|_{L'} = \int_0^b |y(t)| dt \quad \text{for all } y \in L'(J, E).
\]

**Definition 2.1.** A map \( f : J \times D \to E \) is said to be an \( L' - \text{Caratheodory} \) if

(i) \( t \to f(t, u) \) is measurable for each \( u \in D \);

(ii) \( u \to f(t, u) \) is continuous for almost all \( t \in J \);

(iii) for each \( k > 0 \), there exists \( g_k \in L'(J, R_+) \) such that

\[
|f(t, u)| \leq g_k(t), \text{ for all } \|u\| \leq k \text{ and for almost all } t \in J.
\]
In order to define the mild solution of (1.1)-(1.3) we shall consider the following space

\[ \Omega = \{ y : [-r,b] \rightarrow E : y_k \in C(J_k,E), k = 0,1,...,m \text{ and there exists } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k^+) = y(t_k^-), k = 1,2,...,m, y(t) = \phi(t), \text{ for all } t \in [-r,0] \} \]

Which is a Banach space with the norm

\[ \| y \|_\Omega = \max \{ \| y_k \|_\infty, k = 0,1,...,m \} , \]

where \( y_k \) is the restriction of \( y \) to \( J_k = [t_k, t_{k+1}], k = 0,1,...,m. \)

Next we define the mild solution of problem (1.1)-(1.3).

**Definition 2.2.** A function \( y \in \Omega \) is said to be a mild solution of (1.1)-(1.3) if \( y(t) = \phi(t) \), on \([-r,0]\), and the impulsive integral equation

\[
y(t) = \begin{cases} 
T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)ds, & t \in [0,t_1], \\
I_k(y(t_k^-)) + \int_{t_k}^t T(t-s)f(s,y_s)ds, & t \in J_k, \ k = 1,2,...,m.
\end{cases}
\]

is satisfied.

Our main result is based on the following

**Lemma 2.3.** (Nonlinear Alternative[4]). Let \( X \) be a Banach space with \( C \subset X \) closed and convex. Assume \( U \) is a relatively open subset of \( C \) with \( 0 \in U \) and \( G : U^* \rightarrow C \) is a compact map. Then either

(i) \( G \) has a fixed point in \( U^* \); or

(ii) there is a point \( u \in \partial U \) and \( \lambda \in (0,1) \) with \( u = \lambda G(u) \).

**Remark 2.4.** By \( U^* \) and \( \partial U \), we denote the closure of \( U \) and the boundary of \( U \), respectively.

We are now in a position to state and prove our existence result for IVP (1.1)-(1.3). For the study of this problem we first list the following hypotheses.

(H1) \( A \) is the infinitesimal generator of a linear bounded compact semigroup \( T(t) \), \( t \geq 0 \) and there exists \( M \geq 1 \) such that \( |T(t)|_{B(E)} \leq M \);

(H2) \( f : J \times D \rightarrow E \) is an \( L' \) - Caratheodory map;

(H3) there exists a continuous nondecreasing function \( \psi : [0,\infty) \rightarrow (0,\infty) \) and \( p \in L'(J,R_+) \) such that

\[ |f(t,u)| \leq p(t)\psi(\|u\|), \]

for a.e \( t \in J \) and each \( u \in D \) with

\[ \int_{t_{k-1}}^{t_k} p(s)ds < \int_{N_{k-1}}^{\infty} \frac{d\tau}{\psi(\tau)}, \ k = 1,2,...,m+1, \]

where \( N_0 = M\|\phi\| \), and for \( k=2,3,...,m+1 \), we have

\[ N_{k-1} = \sup_{y \in [-M_{k-2},M_{k-2}]} |I_{k-1}(y)|, \quad M_{k-2} = \Gamma_{k-1}^{-1} \left( M \int_{t_{k-2}}^{t_{k-1}} p(s)ds \right) \]
with
\[ \Gamma_l(z) = \int_{N_{l-1}}^{x} \frac{d\tau}{\psi(\tau)}, \quad z \geq N_{l-1}, \quad l \in \{1, 2, ..., m + 1\} \]

(H4) for each bounded \( B \subseteq C([-r, b], E) \) and the set
\[
\begin{cases}
T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)ds, & t \in [0, t_1], \\
I_k(y(t_k^-)) + \int_{t_k}^t T(t-s)f(s, y_s)ds, & t \in J_k, \quad k = 1, 2, ..., m_k; y \in B.
\end{cases}
\]
is relatively compact in \( E \).

3. MAIN RESULT

**Theorem 3.1.** Assume that hypotheses (H1)-(H4) hold. Then the IVP (1.1)-(1.3) has at least one mild solution \( y \in \Omega \).

**proof.** The proof is given in several steps.

**Step I.** Consider the problem (1.1)-(1.3) on \([-r, t_1]\)
\[
\begin{align*}
y'(t) &= Ay + f(t, y_t), \quad a.e \quad t \in J_0 \\
y_0 &= \phi
\end{align*}
\]
(3.1) (3.2)

We will show that the possible mild solutions of (3.1)-(3.2) and a priori bounded, that is, there exists a constant \( B_0 \) such that, if \( y \in \Omega \) is a mild solution on (3.1)-(3.2), then
\[ \sup\{|y(t)| : t \in [-r, 0] \cup [0, t_1]\} \leq B_0. \]

Let \( y \) be a mild solution to (3.1) – (3.2). Then for each \( t \in [0, t_1] \)
\[
y(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)ds.
\]

From (H3) we get
\[ |y(t)| \leq M\|\phi\| + M\int_0^t p(s)\psi(||y_s||)ds, \quad t \in [0, t_1] \]
(3.3)

we consider the function \( \mu_0 \) defined by
\[ \mu_0(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq t_1. \]

Let \( t^* \in [-r, t] \) be such that \( \mu_0(t) = |y(t^*)| \). If \( t^* \in [0, t_1] \), by the previous inequality (3.3), we have for \( t \in [0, t_1] \)
\[ \mu_0(t) \leq M\|\phi\| + M\int_0^t p(s)\psi(\mu_0(s))ds. \]
If $t^* \in [-r, 0]$ then $\mu_0(t) = \|\phi\|$ and the previous inequality holds if $M \geq 1$. Let us take the right-hand side of the above inequality as $v_0(t)$, then we have

$$v_0(0) = M\|\phi\| = N_0, \quad \mu_0(t) \leq v_0(t), \quad t \in [0, t_1]$$

and

$$v_0'(t) = Mp(t)\psi(\mu_0(t)), \quad t \in [0, t_1].$$

Using the nondecreasing character of $\psi$ we get

$$v_0'(t) \leq Mp(t)\psi(v_0(t)), \quad t \in [0, t_1].$$

This implies that for each $t \in [0, t_1]$ that

$$\int_{N_0}^{v_0(t)} \frac{dr}{\psi(r)} \leq M \int_0^{t_1} p(s)ds.$$  

In view of (H3), we get

$$|v_0(t^*)| \leq \Gamma_1^{-1} \left( M \int_0^{t_1} p(s)ds \right) = M_0$$

Since for every $t \in [0, t_1]$, $\|y_t\| \leq \mu_0(t)$, we have

$$\sup_{t \in [-r, t_1]} |y(t)| \leq \max \{\|\phi\|, M_0\} = B_0.$$  

We transform the problem into a fixed point problem. Consider the map $G : C([-r, t_1], E) \rightarrow C([-r, t_1], E)$ defined by

$$(Gy)(t) = \begin{cases} 
\phi(t) & t \in [-r, 0] \\
T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)ds, & t \in J_0.
\end{cases}$$

We shall show that $G$ satisfies the assumptions of Lemma 2.3. The proof will be given in several steps.

**Step 1.** $G$ maps bounded sets into bounded sets in $C(J_0, E)$.

Let $B_q = \{ y \in C(J_0, E) : \|y\|_{\infty} \leq q \}$ be a bounded set in $C(J_0, E)$ and $y \in B_q$, then for each $t \in J_0$, we have

$$(Gy)(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s, y_s)ds, \quad t \in J_0.$$
Thus, we have for each $t \in J_0$

$$|(Gy)(t)| \leq M\|\phi\| + M \int_0^t |f(s,y_s)|ds$$

$$\leq M\|\phi\| + M \int_0^t |g_q(s)|ds$$

$$\leq M\|\phi\| + M\|g_q\|_{L^1}.$$ 

**Step 2.** $G$ maps bounded sets in $C(J_0, E)$ into equicontinuous sets.

Let $r_1, r_2 \in J_0, r_1 < r_2$, and $B_q = \{y \in C(J_0, E) : \|y\|_\infty \leq q\}$ be a bounded set in $C(J_0, E)$. Let $y \in B_q$. Then

$$(Gy)(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)ds, \quad t \in J_0.$$

Hence,

$$|(Gy)(r_2) - (Gy)(r_1)| \leq |T(r_2) - T(r_1)||\phi(0)|$$

$$+ \left| \int_0^{r_2} [T(r_2-s) - T(r_1-s)]f(s,y_s)ds \right|$$

$$+ \left| \int_{r_1}^{r_2} T(r_1-s)f(s,y_s)ds \right|$$

$$\leq |T(r_2) - T(r_1)||\phi(0)|$$

$$+ \int_0^{r_2} |T(r_2-s) - T(r_1-s)|f(s,y_s)ds$$

$$+ M\int_{r_1}^{r_2} |g_q(s)|ds.$$

As $r_2 \to r_1$, the right hand side of the above inequality tends to zero. The equicontinuity for the cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ are obvious.

**Step 3.** $G : C(J_0, E) \to C(J_0, E)$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J_0, E)$. Then there is an integer $q$ such that $\|y_n\|_\infty \leq q$ for all $n \in \mathbb{N}$ and $\|y\|_\infty \leq q$, so $y_n \in B_q$ and $y \in B_q$.

From Dominated convergence theorem,

$$\|Gy_n - Gy\|_\infty \leq \sup_{t \in J_0} \left[ \int_0^t |T(t-s)||f(s,y_{ns}) - f(s,y_s)|ds \right] \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $G$ is continuous.

Set

$$U = \{y \in C([-r, t_1], E) : \|y\|_\infty < B_0 + 1\}$$
As a consequence of Step 2, Step 3, and in the view of (H4) together with the Arzelá-Ascoli theorem, we conclude that the map $G : U^* \to C([-\tau, t_1], E)$ is compact. From the choice of $U$ there is no $y \in \partial U$ such that $y = \lambda G(y)$ for any $\lambda \in (0, 1)$. As a consequence of Lemma 2.3, we deduce that $G$ has a fixed point $y_0 \in U^*$ which is a mild solution of (3.1)-(3.2).

**Step II.** Now consider the problem on $J_1 = [t_1, t_2]$. 

$$y' = Ay + f(t, y), \quad a.e \ t \in J_1,$$

$$y(t_1^+) = I_1(y(t_1^-))$$

Let $y$ be a mild solution to (3.4)-(3.5). Then for each $t \in [t_1, t_2]$

$$y(t) = I_1(y(t_1^-)) + \int_{t_1}^{t} T(t - s)f(s, y_s)ds.$$ 

Note that

$$|y(t_1^+)| \leq \sup_{t \in [-M_0, M_0]} |I_1(y_0(t^-))| = N_1.$$

From (H3), we get

$$|y(t)| \leq N_1 + M \int_{t_1}^{t} p(s)\psi(||y_s||)ds, \quad t \in [t_1, t_2].$$

we consider the function $\mu_1$ defined by

$$\mu_1(t) = \sup\{|y(s)| : t_1 \leq s \leq t\}, \quad t_1 \leq t \leq t_2.$$ 

Let $t^* \in [t_1, t]$ be such that $\mu_1(t) = |y(t^*)|$. Then we have for each $t \in [t_1, t_2]$

$$\mu_1(t) \leq N_1 + M \int_{t_1}^{t} p(s)\psi(\mu_1(s))ds.$$ 

Let us take the right hand side of the above inequality as $v_1(t)$, then we have

$$v_1(t_1) = N_1, \quad \mu_1(t) \leq v_1(t), \quad t \in [t_1, t_2]$$

and

$$v_1'(t) = Mp(t)\psi(\mu_1(t)), \quad t \in [t_1, t_2].$$

Using the nondecreasing character of $\psi$ we get

$$v_1'(t) \leq Mp(t)\psi(v_1(t)), \quad t \in [t_1, t_2].$$

This implies for each $t \in [t_1, t_2]$ that

$$\int_{N_1}^{v_1(t)} \frac{d\tau}{\psi(\tau)} \leq M \int_{t_1}^{t_2} p(s)ds.$$
In view of (H3), we obtain

$$|v_1(t^*)| \leq \Gamma_2^{-1} \left( M \int_{t_1}^{t_2} p(s)ds \right) = M_1$$

Since for every $t \in [t_1, t_2]$, $\|y_t\| \leq \mu_1(t)$, we have

$$\sup_{t \in [t_1, t_2]} |y(t)| \leq M_1.$$

A mild solution to (3.4)-(3.5) is a fixed point of the operator $G : C(J_1, E) \rightarrow C(J_1, E)$ defined by

$$G(y) = I_1(y(t_1^{-})) + \int_{t_1}^{t} T(t-s)f(s, y_s)ds$$

Set

$$U = \{y \in C(J_1, E) : \|y\|_\infty < M_1 + 1\}.$$

As in Step I, we can show that (with obvious modifications) $G : U^* \rightarrow C(J_1, E)$ is compact.

From the choice of $U$ there is no $y \in \partial U$ such that $y = \lambda G(y)$ for any $\lambda \in (0, 1)$.

As a consequence of Lemma 2.3, we deduce that $G$ has a fixed point $y_1 \in U^*$ which is a mild solution of (3.4)-(3.5).

**Step III.** We continue this process and taking account that $y_k \in C(J_k, E)$, $k=2, 3, ..., m$ to

$$y'(t) = Ay(t) + f(t, y_t), \quad a.e \ t \in J_k, \quad (3.7)$$

$$y(t_k^+) = I_k(y(t_k^-)). \quad (3.8)$$

Then

$$y(t) = \begin{cases} 
   y_0(t), & \text{if } t \in [-\tau, t_1] \\
   y_1(t), & \text{if } t \in (t_1, t_2) \\
   y_2(t), & \text{if } t \in (t_2, t_3) \\
   \vdots \\
   y_{m-1}(t), & \text{if } t \in (t_{m-1}, t_m) \\
   y_m(t), & \text{if } t \in (t_m, b) 
\end{cases}$$

is a mild solution to the problem (1.1)-(1.3).
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NUMERICAL SOLUTION FOR WOOD DRYING ON ONE-DIMENSIONAL GRID

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ABSTRACT. A mathematical modeling for the drying process of hygroscopic porous media, such as wood, has been developed in the past decades. The governing equations for wood drying consist of three conservation equations with respect to the three state variables, moisture content, temperature and air density. They are involving simultaneous, highly coupled heat and mass transfer phenomena. In recent, the equations were extended to account for material heterogeneity through the density of the wood and via the density variation of the material process, capillary pressure, absolute permeability, bound water diffusivity and effective thermal conductivity. In this paper, we investigate the drying behavior for the three primary variables of the drying process in terms of control volume finite element method to the heterogeneous transport model on one-dimensional grid.

1. INTRODUCTION

Drying is one of the most complex phenomena happened in engineering because of the simultaneous heat and mass transfer taking place in the course of the process. In the past decades, many researchers developed drying models of porous media, in particular, wood. Wood drying is more difficult than other porous media such as concrete and brick, because it has the anisotropic and heterogeneous characteristic. Although the investigation of the drying processes has been realized both experimentally and theoretically for centuries, the coupling of heat and mass transfer and other phenomena in drying is still a challenging problem.

In developing a drying model based on the continuous approach, Whitaker[12, 13] used the volume averaging technique to derive a system of macroscopic transport equations from a set of basic transport laws at microscopic level(pore scale) for the three phases(gas, liquid and solid). He assumed that a porous medium was to be equivalent

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to a continuum. A system of conservation equations for mass, energy and momentum was introduced using average state variables. His continuous model is considered as rigorous and the most advanced continuous model today.

Based on the Whitaker's theory, a mathematical modelling of the wood drying is developed by many authors, for example, Perré and Turner [4, 5, 7]. Also the numerical simulation for the drying process was developed using derived average conservation equations [1, 2, 3, 8, 11]. Among others, Perré and Turner employed the control volume finite element method (CVFEM) to solve the system of transport equations. The main advantage of CVFEM is that it ensures the conservation of mass and enthalpy through the boundary of each control volume as well as whole domain. However it has some difficulties due to mathematical complexity of the governing equations.

In this paper, we implement the heterogeneous and highly coupled heat and mass transfer modelled by Perré and Turner in the literature of CVFEM. In this model, the capillary pressure, the liquid and gas permeability, the effective diffusivity and effective thermal conductivity are very important coefficients. These parameters are normally functions of one or all state variables and depend on material characteristics. We compute this parameters by using the published values of corresponding primary state variables.

2. MATHEMATICAL MODELLING

In this section, we introduce the mathematical formulation which represents the process of the wood drying, based on the model developed by Whitaker [12] and later by Perré and Turner [5]. The volume averaging method is applied to derive a continuous drying model for wood at macroscopic level. The wood consists of three phases, free water, gas (air and water vapor) and solid. Solid includes the bound water.

The conservation equation is typically represented by the following:

$$\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \vec{v} - \vec{D} \nabla \varphi) = 0,$$

where $\varphi$ is the state variables, and $\vec{v}$ is velocity vector and $\vec{D}$ effective diffusivity tensor.

The process of the wood drying is governed by three macroscopic conservation equations [6, 10]. The equations were extended to account for material heterogeneity through the density of the wood and via density variation of the material properties, capillary pressure, absolute permeability, bound liquid diffusivity and effective thermal conductivity. The system of the three governing equations is summarized as follows:

**Liquid Mass Conservation:**

$$\frac{\partial}{\partial t}(\rho_0 X + \varepsilon g \rho_v) + \nabla \cdot (\rho_w \vec{v}_w + \rho_v \vec{v}_g - \rho_0 \vec{D}_b \nabla X_b) = \nabla \cdot (\rho_g \vec{D}_v \nabla w_v),$$
Energy Conservation:

\[
\frac{\partial}{\partial t} \left( \rho_0 (X_h w + h_a) + \varepsilon_g (\rho_v h_v + \rho_a h_a) - \int_0^{\rho_0 X_b} \Delta h_w d\rho - \varepsilon_g P_g \right) \\
+ \nabla \cdot (\rho_w h_w \bar{v}_w + (\rho_v h_v + \rho_a h_a) \bar{v}_g - h_b \rho_0 \bar{D}_b \nabla X_b) \\
= \nabla \cdot (\rho_g \bar{D}_v (h_v \nabla w_v + h_a \nabla w_a) + \bar{K}_\text{eff} \nabla T),
\]

Air Conservation:

\[
\frac{\partial}{\partial t} (\varepsilon_g \rho_a) + \nabla \cdot (\rho_a \bar{v}_g) = \nabla \cdot (\rho_g \bar{D}_v \nabla w_a).
\]

The primary variables in the system are the moisture content \( X \), the temperature \( T \) and the intrinsic phase air density \( \bar{\rho}_a \), where:

\[
\bar{\rho}_a = \varepsilon_g \rho_a.
\]

Here the moisture content \( X \) contains the bound water \( X_b \), i.e.,

\[
X = X_w + X_b, \quad X_w = \frac{\varepsilon_w \rho_w}{\rho_0}, \quad X_b = \min(X, X_{\text{fsp}}),
\]

where \( X_w \) is free water.

The remaining symbols are secondary variables and parameters, where \( \rho \) represents density, \( \varepsilon \) the volume fraction, \( \bar{v} \) the phase velocity, \( \omega \) the mass fraction, \( h \) the enthalpy and \( P \) the pressure. The subscript \( a, b, g, s, v, w \) represent the air, bound water, gas, solid(cell wall), vapor and water(or liquid) phases, respectively. The density of the wood is represented by \( \rho_0 \) and the effective vapor diffusivity, bound water diffusivity and effective thermal conductivity are represented by \( \bar{D}_v \), \( \bar{D}_b \) and \( \bar{K}_\text{eff} \), respectively and \( \bar{v}_w \) and \( \bar{v}_g \) are the liquid and gaseous phase velocities, respectively, given by the generalized Darcy's law:

\[
\bar{v}_\ell = -\bar{K}_\ell \frac{\bar{\kappa}_\ell}{\mu_\ell} \nabla \varphi_\ell, \quad \nabla \varphi_\ell = \nabla P_\ell - \rho_\ell \epsilon \chi,
\]

where \( \ell = w, g \) and \( \chi \) is the depth scalar.

For the sake of computation, we consider the one-dimensional virtual board sample which is flat-sawn board cut from a soft wood species, as depicted by Figure 1. Also the computational meshes are generated by dividing evenly spaced within the board. The board size is 0.04m and as a result of the symmetry, the computational domain is 0.02m. The density for each node ranges from 252Kg/m\(^3\) in the earlywood to 1016Kg/m\(^3\) in the latewood, where the average density is 529.64Kg/m\(^3\). The distribution of the density is shown in Figure 1.

Two end boundaries of the virtual board sample has the different type. The right end is external boundary and the left end is symmetric boundary. The boundary conditions
Figure 1. The virtual board description and the distribution of the density

proposed for the external drying surfaces are given as

\[ \mathbf{J}_w \cdot \hat{n} = k_m c M_v \ln \left( \frac{1 - x_{v\infty}}{1 - x_v} \right), \]

\[ \mathbf{J}_e \cdot \hat{n} = q(T - T_\infty) + h_v k_m c M_v \ln \left( \frac{1 - x_{v\infty}}{1 - x_v} \right), \]

\[ P_g = P_\infty, \]

where \( \mathbf{J}_w \) and \( \mathbf{J}_e \) represent the fluxes of liquid and energy at the boundary, respectively, \( \hat{n} \) is the outward unit normal vector, \( h_v \) and \( k_m \) the heat and mass transfer coefficients, respectively, \( x_v \) and \( x_{v\infty} \) the molar fractions of vapor at the exchange surface and in the air, respectively, and \( c \) the molar concentration. The pressure at the external drying surface is given at the atmospheric pressure. At symmetric boundary, all fluxes of liquid, heat, vapor and air are set to zero.

Also, we need some of initial conditions. The initial average moisture content \( \bar{X} \) and temperature \( T \) are given by 120% and 25°C, respectively. Then initial moisture content distribution has to be determined prior to the commencement of the drying process. The liquid saturation \( S_{wi}(i = 1, \ldots, N) \) at each node and the equilibrium capillary pressure \( P_{ceq} \) are calculated from the nonlinear system of \((N + 1)\) equations as follows:

\[ P_c(S_{wi}, \rho_{oi}, T) = P_{ceq}, \quad i = 1, \ldots, N \]

\[ \frac{\rho_w}{\sum_i \phi_i S_{wi} A_i} + X_{fsp} = \bar{X}, \]

where \( A_i \) is the area corresponding to the node \( i \). Once the values of \( S_{wi} \) have been determined using Newton iteration, the initial moisture content can be calculated using:

\[ X_i = \frac{\rho_w \phi_i S_{wi}}{\rho_{oi}} + X_{fsp}, \quad i = 1, \ldots, N \]
The computed moisture content ranged from 66% to 155%. In earlywood regions with lower density higher moisture contents occurs and in latewood regions with higher density moisture content is lower as depicted in Figure 2.

For the computation of drying process, we also need a suitable drying schedule. The dry and wet bulb temperatures were ramped up to their kiln operating values of 60°C and 40°C, respectively, over a period of 10 minutes.

3. CONTROL VOLUME FINITE ELEMENT METHODS

In this section we introduce CVFEM which is used to discretize the transport model. First, we recast three conservation equations to typical formulation as following:

\[ \frac{\partial}{\partial t} \Psi + \nabla \cdot \mathbf{J} = 0, \]

where \( \Psi \) represents conserved quantities \( \rho_0 X + \varepsilon_g \rho_v, \rho_0 (X h_w + h_s) + \varepsilon_g (\rho_v h_v + \rho_a h_a) - \int_{0}^{X_b} \Delta h_w d\rho - \varepsilon_g P_g \) or \( \varepsilon_g \rho_a \) and \( \mathbf{J} \) represents fluxes

\[ J_w = \rho_w \bar{v}_w + \rho_v \bar{v}_g - \rho_0 \bar{D}_b \nabla X_b - \rho_g \bar{D}_v \nabla w_v, \]

\[ J_e = \rho_w h_w \bar{v}_w + (\rho_v h_v + \rho_a h_a) \bar{v}_g - h_b \rho_0 \bar{D}_b \nabla X_b \]

\[ - \rho_g \bar{D}_v (h_v \nabla w_v + h_a \nabla w_a) - \bar{K}_{eff} \nabla T, \]

\[ J_a = \rho_a \bar{v}_g - \rho_g \bar{D}_v \nabla w_a. \]

Applying time discretization technique such as the backward Euler or the Crank-Nicolson schemes to (3.1), we have the following stationary equation at each time step

\[ (\Psi - \Psi^{(prev)})/\delta t + \nabla \cdot \mathbf{J} = 0, \]

where \( \Psi^{(prev)} \) means the value of the conserved quantity at the previous time step. \( \delta t \) means the time step size.
As shown in Figure 3, the computational domain is meshed with subinterval elements, and at each node the control volumes(CVs) are constructed. To obtain the discretized formulation of the stationary equation (3.2), we have integrating over the each CV
\[
\frac{\text{Area(CV)}}{\delta t} (\Psi_{pt} - \Psi_{pt}^{(prev)}) + \int_{CV} \nabla \cdot J \ dS = 0,
\]
where $\Psi_{pt}$, the value of $\Psi$ at the node point $pt$, is representative value of $\Psi$ in the CV, i.e.,
\[
\Psi_{pt} = \frac{1}{\text{Area(CV)}} \int_{CV} \Psi \ dS,
\]
and applying the Gauss divergence theorem:
\[
\alpha \Psi_{pt} + \sum_{f \in \mathcal{F}_{CV}} (J \cdot n)_f = \alpha \Psi_{pt}^{(prev)},
\]
where $\alpha = \frac{\text{Area(CV)}}{\delta t}$, $\mathcal{F}_{CV}$ is the set of two end-points of the CV and $n_f$ is outward unit normal vector. Also the term $(J \cdot n)_f$ is evaluated accurately at the end-point of the CV. In order to evaluate the approximated value of the flux, we use different method for the advection and the diffusion terms. The finite element shape functions are used for the evaluation of the diffusion terms. For examples, the bound liquid diffusivity and the gradient of the bound water gives:
\[
\bar{D}_b = \sum_{i=1}^{2} N_i \bar{D}_{bl}, \quad \nabla X_b = \sum_{l=1}^{2} \nabla N_l X_{bl},
\]
where the $N_l$ are the shape function for a subinterval element containing the end-point of the CV and $l$ denotes the nodes of this element. Also, for the advection terms, only
\( \vec{v}_w \) and \( \vec{v}_g \), the flux limiting method is used generally [9]. Then we have the \( 3N \) discrete analogue of the equations (3.2) as following:

\[
F_{pt}(x) := \alpha (\Psi_{pt}^{(n+1)} - \Psi_{pt}^{(n)}) + \alpha_{pt} J_{pt}^{(n+1)} + \alpha_W J_{W}^{(n+1)} + \alpha_E J_{E}^{(n+1)} = 0,
\]

where the superscript \( (n+1) \) means the current time level \( t^{(n+1)} \), \( (n) \) the previous time level \( t^{(n)} \), and time step size is \( \delta t = t^{(n+1)} - t^{(n)} \).

4. NUMERICAL RESULTS

As explained above, we find the numerical solution by solving the transport equations in terms of control volume finite element methods. The transport coefficients that are necessary for numerical computation were referred to Truscott and Turner [11].

The discretized formulation (3.3) with boundary conditions is highly coupled system of nonlinear equations. Then we use the inexact Newton iterative method to solve this system.

\[
x^{(0)}: \text{ Initial iterate} \\
\text{For } n = 0, 1, \ldots, \text{ until } \| F(x^{(n)}) \| < \text{ tolerance} \\
\text{Solve } \nabla_h F(x^{(n)}) \delta x^{(n)} = F(x^{(n)}) \\
x^{(n+1)} = x^{(n)} - \delta x^{(n)},
\]

where the matrix \( \nabla_h F(x) \) is the difference approximation to the Jacobian matrix \( F'(x) = \left( \frac{\partial F_j}{\partial x_i} \right) \), and we can use the Bi-CGSTAB method to solve linear system. We can also use the Block TDMA solver because the Jacobian matrix is block tridiagonal matrix.

Figure 4 shows the drying kinetics during 20 hours. The left figures plot the behaviors of the average values and the right figures plot the behaviors at the four positions. The position A is the symmetric boundary, and the position D is the external drying boundary and the position B(0.01) is located in the latewood which has higher density, and the position C(0.0175) is located in the earlywood which has lower density.

The free water begins to move the evaporative free surface at the beginning of drying and remains the constant rate of drying until the free water movement stops, which is called to the constant drying rate period or funicular state. After the funicular state, a drying front recedes toward the interior of wood and the drying rate begin to decrease, which is called to the falling rate drying period or pendular state.

Figure 5 shows the spatial evolution of moisture content, gas pressure, and temperature distribution during drying. The effect of growth rings on moisture distribution is evident during drying as expected. The moisture content of earlywood locations decreases steadily from the beginning of drying but one of latewood locations remains constant up to 48 hour. The surface moisture content comes to the equilibrium state after drying time 4 hour but the surface temperature does not come to the equilibrium
FIGURE 4. Average values (Left) and location values (Right) of Moisture contents, Temperature and Gas Pressure during 20 hours.
FIGURE 5. Spatial evolution of moisture content, temperature and gas pressure distribution during drying
state and approaches steadily to the dry bulb temperature due to the effect of the latent heat of evaporation. It exhibits an underpressure during the constant drying period, followed by a steady increase and overpressure during the falling rate drying period. The underpressure in wood occurs when the flow of free water moving towards the surface is greater than the air flow infiltrating from the outside due to lower permeability, in which it may cause the collapse of wood. The temperature begins to increase up to wet bulb temperature 40°C during the constant rate drying period, followed by approaching to the dry bulb temperature 60°C. It is noticed that the effect of growth rings on the temperature gas pressure distributions are not obvious.

5. Concluding Remarks

A numerical simulation for the process of the wood drying has some of difficulties, for instance, tightly coupled equations, highly non-linear equations, non-linear boundary conditions, steep moisture and pressure gradients, highly convective internal gaseous flows and outer and inner iteration stages make another stumbling block.

In this study we find a appropriate method, so called control volume finite element method, which is convenient to non-dimensionalise the system of equations, and makes an unstructured meshing for processing possible. Also it has a number of different alternatives for the exact shape of CV and is flexible for evaluating fluxes through faces.

A complete CVFE method is suitable for resolving nonlinear transport equations on one-dimensional meshes.

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