Comment on: "Hall and Ion-Slip effects on magneto-micropolar fluid with combined forced and free convection in boundary layer flow over a horizontal plate"

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In recent paper by Seddeek and Abdelmeguid [1] the effects of Hall and ion-slip currents on the steady magneto-micropolar of a viscous incompressible and electrically conducting fluid over a horizontal plate is studied numerically using the shooting method. It is shown here that the results obtained for the rate of heat transfer at the wall are incorrect. The dimensionless energy equation (15) and the transformed boundary conditions (16) in ref.[1] are given by

\[ \frac{2}{Pr} \phi'' + f \phi' + f' \phi = 0, \]

\[ f(0) = f'(0) = 0, \ f'(\infty) = 1, \ \phi(0) = 1, \ \phi(\infty) = 0. \]

From Eqns. (15) and (16), one obtains \( \phi'(0) = 0 \), for both Newtonian fluid [2] and micropolar fluid [3].

This result indicates that there is no local heat transfer at the plate surface for all \( x > 0 \). Nevertheless, although dissipation has been neglected the temperature of the fluid is changed during the flow process. The paradox is resolved by recalling that the similarity solution requires a singular behavior of the wall temperature at \( x = 0 \), cf. equation (11) in ref.[1]. Thus all the heat necessary to change the fluid temperature must be transferred in the singular point \( x = 0 \), which is the leading edge of the plate [2]. However in the paper by Seddeek and Abdelmeguid [1], the
values of $\phi'(0) \neq 0$ as shown in tables 1-5 in ref.[1], which contradicts the analytical solution, $\phi'(0) = 0$.

Since Eqns. (12)-(15) in ref.[1] are coupled, then all the results obtained for $f', \phi, h, g, h'(0), g'(0)$ and $f''(0)$ are also incorrect.

An example to show that the results obtained in ref. [1] are incorrect can be explained as follows: The coupled non-linear ordinary differential equations (12)-(15) with the boundary conditions (16) in ref.[1] are solved numerically using the shooting method. In order to check the accuracy of our results, we have compared it when neglecting the magnetic field with those obtained by Hassanien [3] for different values of $\Delta = 0.5, 1.5$ and 4.5 with $\lambda = \Omega = 0.5, B = 0.01$ and $\rho_s = 0.7$. Our results are in agreement with those obtained by Hassanien [3]. Then we calculated the values of $f''(0), h'(0), -g'(0)$ and $-\phi'(0)$ with $\beta_r = 0.4, M = 0.3, \Omega = 0.5$ as shown in the following table:

<table>
<thead>
<tr>
<th>$\rho_s$</th>
<th>$f''(0)$</th>
<th>$h'(0)$</th>
<th>$-g'(0)$</th>
<th>$-\phi'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.229358</td>
<td>0.007381</td>
<td>0.055351</td>
<td>$O(10^{-5}) \sim 0$</td>
</tr>
<tr>
<td>0.8</td>
<td>0.251879</td>
<td>0.010420</td>
<td>0.055763</td>
<td>$O(10^{-5}) \sim 0$</td>
</tr>
<tr>
<td>14</td>
<td>0.260535</td>
<td>0.011747</td>
<td>0.05691</td>
<td>$O(10^{-5}) \sim 0$</td>
</tr>
<tr>
<td>2</td>
<td>0.293218</td>
<td>0.010028</td>
<td>0.060819</td>
<td>$O(10^{-5}) \sim 0$</td>
</tr>
<tr>
<td>4</td>
<td>0.315599</td>
<td>0.006824</td>
<td>0.063193</td>
<td>$O(10^{-5}) \sim 0$</td>
</tr>
</tbody>
</table>

From this table we see that the values of $\phi'(0)$ are of order $O(10^{-5}) \sim 0$ which is clearly in agreement with the analytical solution $\phi'(0) = 0$.

References

CODING THEOREMS ON A GENERALIZED INFORMATION MEASURES.

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Abstract: In this paper a generalized parametric mean length $L(P^v, R)$ has been defined and bounds for $L(P^v, R)$ are obtained in terms of generalized R-norm information measure.

Key words and phrases: Parametric mean length, Entropy, Holder’s inequality.

1. Introduction

Consider the model $A$ given below for a finite scheme random experiment having $(A_1, A_2, ..., A_n)$ as the complete system of events with respective probabilities $P = (p_1, p_2, ..., p_n)$, $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Denote

(1.1) $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ p_1 & p_2 & \cdots & p_n \end{bmatrix}$

We call the scheme (1.1) as a finite information scheme. Every finite scheme describes a state of uncertainty. Shannon [5] introduced a quantity which in a reasonable way, measures the amount of uncertainty (entropy) associated with a given finite scheme. This measure is given by

(1.2) $H(P) = -\sum_{i=1}^{n} p_i \log p_i$

can serve as a very suitable measure of entropy of the finite scheme (1.1). Throughout the paper, logarithms are taken to base $D (D \geq 2)$.

Let $X = (x_1, x_2, \ldots, x_n)$ be the finite set of input symbols which are to be encoded using alphabet of $D$ symbols. It has been shown Feinstein [2] that there is a unique decipherable code with lengths $l_1, l_2, \ldots, l_n$ and satisfying

\[(1.3) \quad \sum_{i=1}^{n} D^{-l_i} \leq 1\]

where $D$ is the size of the code alphabet.

Noiseless coding theorem for Shannon’s entropy with ordinary code mean length

\[(1.4) \quad \sum_{i=1}^{n} l_i p_i\]

under the condition (1.3), has played an important role in ordinary communication theory, (see Shannon [5]).

Khan and Haseen [3], Khan, Autar and Haseen [4], Boekeee et al [1] have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under the condition (1.3) of unique decipherability.

In this paper, we study coding theorems by considering a new function depending on the parameters $\alpha$ and $\nu$. Our motivation for studying this new function is that it generalizes some entropy functions already existing in the literature.

2. Coding theorems

Consider a function

\[(2.1) \quad H(p^\nu, R) = \frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} p_i R^{\nu-1}}{\sum_{i=1}^{n} p_i^\nu} \right)^{\frac{1}{R}} \right] \]

for all $R \in \mathbb{R}_+, (\neq 1), \nu \neq 1, \sum_{i=1}^{n} p_i = 1, i = 1, 2, \ldots, n$

(i) When $\nu = 1$, (2.1) reduces to the R-norm information due to Boekeee et al [1].

(ii) When $R \to 1, \nu = 1$, (2.1) reduces to the measure due to Shannon [5].

Further consider

\[(2.2) \quad L(p^\nu, R) = \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} p_i^\nu D^{-\nu \left( \frac{R-1}{R} \right)}}{\sum_{i=1}^{n} p_i^\nu} \right] \]

where $R \in \mathbb{R}_+, R \neq 1$.

(i) For $\nu = 1$, (2.2) reduces to the mean length due to Boekeee et al [1].

(ii) For $R \to 1, \nu = 1$, (2.2) reduces to the optimal code length defined by Shannon [5].
We now establish a result, that in a sense, gives a characterization of $H(P^\nu, R)$ under the condition

\begin{equation}
\sum_{i=1}^{n} p_i^{\nu-1} D^{-i} \leq \sum_{i=1}^{n} p_i^\nu
\end{equation}

**Remark:** For $\nu = 1$ and $\sum_{i=1}^{n} p_i = 1$, (2.3) is a generalization of (1.3).

**Theorem 1:** For every code whose lengths $l_1, l_2, \ldots, l_n$ satisfies (2.3), the average length satisfies

\begin{equation}
L(P^\nu, R) \geq H(P^\nu, R)
\end{equation}

equality holds if and only if

\begin{equation}
l_i = -\log \frac{\sum_{i=1}^{n} p_i^{R+1}}{\left(\sum_{i=1}^{n} p_i^{\nu+1}\right)^{1/\nu}}
\end{equation}

**Proof:** we use Holder's inequality [6]

\begin{equation}
\sum_{i=1}^{n} x_i y_i \geq \left(\sum_{i=1}^{n} x_i^{\frac{1}{p}}\right)^p \left(\sum_{i=1}^{n} y_i^{\frac{1}{q}}\right)^q
\end{equation}

for all $x_i > 0, y_i > 0, i = 1, 2, \ldots, n, p < 1 (\neq 0)$ and $p^{-1} + q^{-1} = 1$

with equality if and only if there exists a positive number $c$ such that

\begin{equation}
x_i^p = c y_i^q
\end{equation}

Setting

\begin{align*}
x_i &= p_i^{\frac{R}{R-1}} D^{-i} \\
y_i &= p_i^{\frac{R+1}{R-1}}
\end{align*}

$P = \frac{R-1}{R}$ and $q = 1 - R$ in (2.6) and using (2.3), Also if $R > 1$ we get

\begin{equation}
\left[\sum_{i=1}^{n} p_i^\nu D^{-i} \left(\frac{R+1}{R-1}\right)^{1-\nu} \right]^{1-\nu} \geq \left(\sum_{i=1}^{n} p_i^{R+1} \right)^{\nu-1} \sum_{i=1}^{n} p_i^\nu
\end{equation}

Dividing both sides of (2.8) by $\left(\sum_{i=1}^{n} p_i^{\nu} \right)^{R-1}$, we get
\[
\left(\sum_{i=1}^{n} p_i^\nu \frac{D_{i}^{R}}{\sum_{i=1}^{n} p_i^\nu}\right)_{1-R}^{R} \leq \left(\sum_{i=1}^{n} p_i^{R+1} \frac{1}{\sum_{i=1}^{n} p_i^\nu}\right)_{1-R}^{1-R}
\]

Raising both sides to the power \(\frac{1-R}{R}\), \(R \neq 1\) also \(\frac{R}{R-1} > 0\) for \(R > 1\) and after suitable operations, we obtain the result (2.4). For \(0 < R < 1\), the inequality (2.4) can be proved in a similar fashion.

**Theorem 2:** For every code with lengths \(l_1, l_2, ..., l_n\) satisfies (2.3). \(L(P^\nu, R)\) can be made to satisfy the inequality

\[(2.9) \quad L(P^\nu, R) < H_\delta(P^\nu, R)D_{\frac{1-R}{R}}^{\frac{R}{R-1}}\left(1 - D_{\frac{1-R}{R}}^{\frac{R}{R-1}}\right)\]

**Proof:** Let \(l_j\) be the positive integer satisfying the inequality

\[(2.10) \quad -\log \frac{p_i^R}{\left(\sum_{i=1}^{n} p_i^{R+1}\right)\left(\sum_{i=1}^{n} p_i^\nu\right)} \leq l_j < -\log \frac{p_i^R}{\left(\sum_{i=1}^{n} p_i^{R+1}\right)\left(\sum_{i=1}^{n} p_i^\nu\right)} + 1\]

Consider the interval

\[(2.11) \quad \delta_j = -\log \frac{p_i^R}{\left(\sum_{i=1}^{n} p_i^{R+1}\right)\left(\sum_{i=1}^{n} p_i^\nu\right)} , -\log \frac{p_i^R}{\left(\sum_{i=1}^{n} p_i^{R+1}\right)\left(\sum_{i=1}^{n} p_i^\nu\right)} + 1\]

of length 1. In every \(\delta_j\), there lies exactly one positive number \(l_j\) such that

\[(2.12) \quad 0 < -\log \frac{p_i^R}{\left(\sum_{i=1}^{n} p_i^{R+1}\right)\left(\sum_{i=1}^{n} p_i^\nu\right)} \leq l_j < -\log \frac{p_i^R}{\left(\sum_{i=1}^{n} p_i^{R+1}\right)\left(\sum_{i=1}^{n} p_i^\nu\right)} + 1\]

We will first show that sequence \(\{l_1, l_2, ..., l_n\}\), thus defined satisfies (2.3), from (2.12) we have
\[- \log \frac{p_i^R}{\sum_{i=1}^{n} p_i^{R+\nu-1}} \leq l_i \]

\[- \log \frac{p_i^R}{\sum_{i=1}^{n} p_i^{R+\nu-1}} \leq - \log D^{-i} \]

\[
(2.13) \quad \frac{p_i^R}{\sum_{i=1}^{n} p_i^{R+\nu-1}} \geq D^{-i} \]

Multiplying both sides by \(\sum_{i=1}^{n} p_i^{-\nu-1}\) and summing over \(i = 1, 2, \ldots, n\), we get (2.3).

The last inequality in (2.12) gives

\[
l_i < - \log \frac{p_i^R}{\sum_{i=1}^{n} p_i^{R+\nu-1}} + 1
\]

\[
l_i < - \log \frac{p_i^R}{\sum_{i=1}^{n} p_i^{R+\nu-1}} + \log_D D
\]

i.e.,

\[
D^{-l_i} < \frac{p_i^R}{\sum_{i=1}^{n} p_i^{R+\nu-1}} D^{-l}
\]
or

\[ D^{-\left(\frac{R-1}{R}\right)} < \left( \frac{\sum_{i=1}^{n} p^R_i}{\sum_{i=1}^{n} p^{R+\nu-1}_i} \right)^{\frac{R-1}{R}} \frac{1-R}{D} \]

Multiplying both sides by \( \frac{p^\nu_i}{\sum_{i=1}^{n} p^\nu_i} \) and summing over \( i = 1, 2, ..., n \) and simplifying, gives (2.9). For \( 0 < R < 1 \), the proof of the upper bound of \( L(p^\nu, R) \) follows along the similar lines.

As \( D \geq 2 \), we have \( \frac{R}{R-1} \left[ 1 - D^\left(\frac{1-R}{R}\right) \right] > 1 \) from which it follows that the upper bound of \( L(p^\nu, R) \) in (2.9) is greater than unity.

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Signed degree sequences in signed 3-partite graphs

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Abstract. A signed 3-partite graph is a 3-partite graph in which each edge is assigned a positive or a negative sign. Let \( G(U, V, W) \) be a signed 3-partite graph with \( U = \{u_1, u_2, \cdots, u_r\}, V = \{v_1, v_2, \cdots, v_q\} \) and \( W = \{w_1, w_2, \cdots, w_r\} \). Then, signed degree of \( u_i(v_j \text{ and } w_k) \) is \( sdeg(u_i) = d_i = d_i^+ - d_i^- \), \( 1 \leq i \leq r \) (\( sdeg(v_j) = e_j = e_j^+ - e_j^- \), \( 1 \leq j \leq q \) and \( sdeg(w_k) = f_k = f_k^+ - f_k^- \), \( 1 \leq k \leq r \)) where \( d_i^+ (e_j^+ \text{ and } f_k^+) \) is the number of positive edges incident with \( u_i(v_j \text{ and } w_k) \) and \( d_i^- (e_j^- \text{ and } f_k^-) \) is the number of negative edges incident with \( u_i(v_j \text{ and } w_k) \). The sequences \( \alpha = [d_1, d_2, \cdots, d_p], \beta = [e_1, e_2, \cdots, e_q] \) and \( \gamma = [f_1, f_2, \cdots, f_r] \) are called the signed degree sequences of \( G(U, V, W) \). In this paper, we characterize the signed degree sequences of signed 3-partite graphs.

1. Introduction

A signed graph is a graph in which each edge is assigned a positive or a negative sign. The concept of signed graphs is given by Harary [3]. Let \( G \) be a signed graph with vertex set \( V = \{v_1, v_2, \cdots, v_n\} \). Then, signed degree of \( v_i \) is \( sdeg(v_i) = d_i = d_i^+ - d_i^- \), \( 1 \leq i \leq n \) where \( d_i^+ (d_i^-) \) is the number of positive (negative) edges incident with \( v_i \). A signed degree sequence \( \sigma = [d_1, d_2, \cdots, d_n] \) of a signed graph \( G \) is formed by listing the vertex signed degrees in non-increasing order. An integral sequence is s-graphical if it is the signed degree sequence of a signed graph. Also, a non-zero sequence \( \sigma = [d_1, d_2, \cdots, d_n] \) is a standard sequence if \( \sigma \) is non-increasing, \( \sum_{i=1}^{n} d_i \) is even, \( d_1 > 0 \), each \( |d_i| < n \), and \( |d_1| \geq |d_n| \).

The following result, due to Chartrand et al. [1], gives a necessary and sufficient condition for an integral sequence to be s-graphical, which is similar to Hakimi’s result for degree sequences in graphs [2].

Theorem 1.1. A standard integral sequence \( \sigma = [d_1, d_2, \cdots, d_n] \) is s-graphical if and only if

\[
\sigma' = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+s+1} - 1, d_{d_1+s+2}, \cdots, d_{n-s}, d_{n-s+1} + 1, \cdots, d_n + 1]
\]

is s-graphical for some \( s \), \( 0 \leq s \leq \frac{n-1-d_1}{2} \).

The next result [5] provides a good candidate for parameter \( s \) in Theorem 1.1.

Theorem 1.2. A standard integral sequence \( \sigma = [d_1, d_2, \cdots, d_n] \) is s-graphical if and only if

\[
\sigma'_m = [d_2 - 1, d_3 - 1, \cdots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \cdots, d_{n-m}, d_{n-m+1} + 1, \cdots, d_n + 1]
\]

is s-graphical, where \( m \) is the maximum non-negative integer such that \( d_{d_1+m+1} > d_{n-m+1} \).

Some results for signed degree sets in signed graphs are given by Pirzada et al. [4].
A signed 3-partite graph is a 3-partite graph in which each edge is assigned a positive or a negative sign. Let \( G( U, V, W) \) be a signed 3-partite graph with \( U = \{u_1, u_2, \ldots, u_p\} \) and \( V = \{v_1, v_2, \ldots, v_q\} \) and \( W = \{w_1, w_2, \ldots, w_r\} \). Then, signed degree of \( u_i \) is \( sdeg(u_i) = d_i^+ - d_i^- \), degree of \( u_i \) is \( deg(u_i) = d_i = d_i^+ + d_i^- \), where \( 1 \leq i \leq p \) and \( d_i^+ (d_i^-) \) is the number of positive(negative) edges incident with \( u_i \), signed degree of \( v_j \) is \( sdeg(v_j) = e_j = e_j^+ - e_j^- \), degree of \( v_j \) is \( deg(v_j) = e_j = e_j^+ + e_j^- \), where \( 1 \leq j \leq q \) and \( e_j^+ (e_j^-) \) is the number of positive(negative) edges incident with \( v_j \), signed degree of \( w_k \) is \( sdeg(w_k) = f = f_k^+ - f_k^- \), degree of \( w_k \) is \( deg(w_k) = f_k = f_k^+ + f_k^- \), where \( 1 \leq k \leq r \) and \( f_k^+ (f_k^-) \) is the number of positive(negative) edges incident with \( w_k \). Clearly, \( |d_i| \leq q + r, |e_j| \leq r + p \) and \( |f_k| \leq p + q \). Then, the sequences \( \alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q] \) and \( \gamma = [f_1, f_2, \ldots, f_r] \) are called the signed degree sequences of the signed 3-partite graph \( G( U, V, W) \). Three sequences \( \alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q] \) and \( \gamma = [f_1, f_2, \ldots, f_r] \) of integers are \( s \)-graphical if \( \alpha, \beta \) and \( \gamma \) are the signed degree sequences of some signed 3-partite graph. We denote a positive edge xy by \( xy^+ \) and a negative edge xy by \( xy^- \).

2. Characterizations of signed degree sequences in signed 3-partite graphs

First we obtain the following result.

**Theorem 2.1.** Let \( G( U, V, W) \) be a signed 3-partite graph with \( m \) edges. Then,\( g = \sum_{u \in U} sdeg(u) + \sum_{v \in V} sdeg(v) + \sum_{w \in W} sdeg(w) \equiv 2m \mod 4 \) and the number of positive edges and negative edges of \( G( U, V, W) \) are respectively \( \frac{2m+g}{4} \) and \( \frac{2m-g}{4} \).

**Proof.** Let \( G( U, V, W) \) be a signed 3-partite graph with \( U = \{u_1, u_2, \ldots, u_p\}, V = \{v_1, v_2, \ldots, v_q\}, W = \{w_1, w_2, \ldots, w_r\} \). Let \( u_i (1 \leq i \leq p) \) be incident with \( d_i^+ \) positive edges and \( d_i^- \) negative edges, \( v_j (1 \leq j \leq q) \) be incident with \( e_j^+ \) positive edges and \( e_j^- \) negative edges and \( w_k (1 \leq k \leq r) \) be incident with \( f_k^+ \) positive edges and \( f_k^- \) negative edges so that

\[
\begin{align*}
\text{sdeg}(u_i) &= d_i^+ - d_i^- \quad \text{while} \quad \text{deg}(u_i) = d_i^+ + d_i^- \quad \text{for} \quad 1 \leq i \leq p, \\
\text{sdeg}(v_j) &= e_j^+ - e_j^- \quad \text{while} \quad \text{deg}(v_j) = e_j^+ + e_j^- \quad \text{for} \quad 1 \leq j \leq q,
\end{align*}
\]

and \( \text{sdeg}(w_k) = f_k^+ - f_k^- \), while \( \text{deg}(w_k) = f_k^+ + f_k^- \), for \( 1 \leq k \leq r \).

Clearly, \( \sum_{i=1}^{p} \text{deg}(u_i) + \sum_{j=1}^{q} \text{deg}(v_j) + \sum_{k=1}^{r} \text{deg}(w_k) = 2m \).

Let \( G(U, V, W) \) have \( x \) positive edges and \( y \) negative edges.

Then \( m = x + y, \sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+ = 2x \) and \( \sum_{i=1}^{p} d_i^- + \sum_{j=1}^{q} e_j^- + \sum_{k=1}^{r} f_k^- = 2y \).

Further, \( \sum_{i=1}^{p} \text{sdeg}(u_i) + \sum_{j=1}^{q} \text{sdeg}(v_j) + \sum_{k=1}^{r} \text{sdeg}(w_k) = \sum_{i=1}^{p} (d_i^+ - d_i^-) + \sum_{j=1}^{q} (e_j^+ - e_j^-) + \sum_{k=1}^{r} (f_k^+ - f_k^-) = (\sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+) - (\sum_{i=1}^{p} d_i^- + \sum_{j=1}^{q} e_j^- + \sum_{k=1}^{r} f_k^-) = 2x - 2y \).

Hence, \( g = \sum_{i=1}^{p} \text{sdeg}(u_i) + \sum_{j=1}^{q} \text{sdeg}(v_j) + \sum_{k=1}^{r} \text{sdeg}(w_k) = 2x - 2y = 2(m - y) - 2y = 2y = 2m - 4y \),

so that \( g = 2m \mod 4 \). Hence, \( x + y = m \) and \( 2x - 2y = g \), we have \( x = \frac{2m+g}{4} \) and \( y = \frac{2m-g}{4} \).

**Corollary 2.2.** A necessary condition for the sequences \( \alpha = [d_1, d_2, \ldots, d_p], \beta = [e_1, e_2, \ldots, e_q] \) and \( \gamma = [f_1, f_2, \ldots, f_r] \) of integers to be \( s \)-graphical is that \( \sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+ \) is even.
A zero sequence is a finite sequence each term of which is 0. Clearly, every three finite zero sequences are the signed degree sequences of a signed 3-partite graph. If $\delta = [a_1, a_2, \ldots, a_n]$ is a sequences of integers, then the negative of $\delta$ is the sequence $-\delta = [-a_1, -a_2, \ldots, -a_n]$.

The following result follows by interchanging positive edges with negative edges.

**Lemma 2.3.** The sequences $\alpha = \{d_1, d_2, \ldots, d_p\}$, $\beta = \{e_1, e_2, \ldots, e_q\}$ and $\gamma = \{f_1, f_2, \ldots, f_r\}$ are the signed degree sequences of some signed 3-partite graph if and only if

$-\alpha = [-d_1, -d_2, \ldots, -d_p]$, $-\beta = [-e_1, -e_2, \ldots, -e_q]$ and $-\gamma = [-f_1, -f_2, \ldots, -f_r]$

are the signed degree sequences of some signed 3-partite graph.

We may assume without loss of generality, that a non-zero sequence $\delta = [a_1, a_2, \ldots, a_n]$ is non-increasing and $|a_1| \geq |a_n|$, for we may always replace $\delta$ by $-\delta$ if necessary. The sequences of integers $\alpha = \{d_1, d_2, \ldots, d_p\}$, $\beta = \{e_1, e_2, \ldots, e_q\}$ and $\gamma = \{f_1, f_2, \ldots, f_r\}$ are said to be standard sequences if $\alpha$ is non-zero and non-increasing, $\sum_{i=1}^{p} d_i^+ + \sum_{j=1}^{q} e_j^+ + \sum_{k=1}^{r} f_k^+$ is even, $d_i > 0$, each $|d_i| \leq q + r$, each $|e_j| \leq r + p$ each $|f_k| \leq p + q$, $|d_i| \geq |d_p|$, $|d_1| \geq |e_j|$ and $|d_1| \geq |f_k|$ for each $j$ and $k$.

The following result provides a useful recursive test whether the three sequences of integers form the signed degree sequences of some complete signed 3-partite graph.

**Theorem 2.4.** Let $\alpha = \{d_1, d_2, \ldots, d_p\}$, $\beta = \{e_1, e_2, \ldots, e_q\}$ and $\gamma = \{f_1, f_2, \ldots, f_r\}$ be standard sequences and let $g = \frac{d_1 + e_1 + f_1}{2}$. Let $\alpha'$ be obtained from $\alpha$ by deleting $d_1$, and $\beta'$ and $\gamma'$ be obtained from $\beta$ and $\gamma$ by reducing $g$ greatest entries of $\beta$ and $\gamma$ by 1 each and adding remaining entries of $\beta$ and $\gamma$ by 1 each. Then, $\alpha, \beta$ and $\gamma$ are the signed degree sequences of some complete signed 3-partite graph if and if $\alpha'$, $\beta'$ and $\gamma'$ are also.

**Proof.** Let $G'(U', V', W')$ be a complete signed 3-partite graph with signed degree sequences $\alpha'$, $\beta'$ and $\gamma'$. Let $U' = \{u_1, u_2, \ldots, u_p\}$, $V' = \{v_1, v_2, \ldots, v_q\}$, and $W' = \{w_1, w_2, \ldots, w_r\}$. Then, a complete signed 3-partite graph with signed degree sequences $\alpha, \beta$ and $\gamma$ can be obtained by adding a vertex $u_1$ to $U'$ so that there are $g$ positive edges from $u_1$ to those $g$ vertices of $V'$ and $W'$, whose signed degrees were reduced by 1 in going from $\alpha, \beta$ and $\gamma$ to $\alpha', \beta'$ and $\gamma'$, and there are negative edges from $u_1$ to the remaining vertices of $V'$ and $W'$, whose signed degrees were increased by 1 in going from $\alpha, \beta, \gamma$ to $\alpha', \beta'$ and $\gamma'$. Note that the signed degree of $u_1$ is $g = (q + r - g) = 2g - (q + r) = d_1$

Conversely, let $\alpha, \beta$ and $\gamma$ be the signed degree sequences of a complete signed 3-partite graph. Let the vertex sets of the complete signed 3-partite graph be $U' = \{u_1, u_2, \ldots, u_p\}$, $V' = \{v_1, v_2, \ldots, v_q\}$, and $W' = \{w_1, w_2, \ldots, w_r\}$ such that $sdeg(u_i) = d_i$, $1 \leq i \leq p$, $sdeg(v_j) = e_j$, $1 \leq j \leq q$ and $sdeg(w_k) = f_k$, $1 \leq k \leq r$. Among all the complete signed 3-partite graphs with $\alpha, \beta$ and $\gamma$ as the signed degree sequences, let $G(U, V, W)$ be one with the property that the sum $S$ of the signed degrees of the vertices of $V$ and $W$ joined to $u_1$ by positive edges is maximum. Let $d_1^+$ and $d_1^-$ be respectively the number of positive edges and the number of negative edges incident with $u_1$. Then, $sdeg(u_1) = d_1 = d_1^+ - d_1^-$, and $deg(u_1) = d_1^+ + d_1^- = q + r$. Therefore, $d_1^+ = \frac{d_1 + q + r}{2} = g$. Let $X$ be the set of $g$ vertices of $V$ and $W$ with highest signed degrees and let $Y = (V \cup W) - X$. We claim that $u_1$ must be joined by positive edges to the vertices of $X$. If this is not true, then there exist vertices $x \in X$ and $y \in Y$ such that the edge $u_1x$ is negative and the edge $u_1y$ is positive. Since
\( s\text{deg}(x) \geq s\text{deg}(y) \), therefore there exists a vertex \( u_n(\neq u_1) \) of \( U \) such that \( u_nx \) is a positive edge and \( u_ny \) is a negative edge. Now, change the signs of these edges so that \( u_1x \) and \( u_ny \) are positive and \( u_1y \) and \( u_nx \) are negative.

Hence, we obtain a complete signed 3-partite graph with signed degree sequences \( \alpha, \beta \) and \( \gamma \) in which the sum of the signed degrees of the vertices of \( V \) and \( W \) joined to \( u_1 \) by positive edges exceeds \( S \), a contradiction.

So, assume that \( u_1 \) is joined by positive edges to the vertices of \( X \) and by negative edges to the vertices of \( Y \). Therefore \( G(U,V,W) - u_1 \) is a complete signed 3-partite graph with \( \alpha', \beta', \gamma' \) as the signed degree sequences.

Theorem 2.4 provides an algorithm for determining whether or not the standard sequences \( \alpha, \beta \) and \( \gamma \) are the signed degree sequences, and for constructing a corresponding complete signed 3-partite graph. Suppose \( \alpha = [d_1, d_2, \cdots, d_p], \beta = [e_1, e_2, \cdots, e_q] \) and \( \gamma = [f_1, f_2, \cdots, f_r] \) be the standard signed degree sequences of a complete signed 3-partite graph with parts \( U = \{u_1, u_2, \cdots, u_p\} \) and \( V = \{v_1, v_2, \cdots, v_q\} \) and \( W = \{w_1, w_2, \cdots, w_r\} \). Deleting \( d_1 \) and reducing \( g = \frac{d_1 + p + r}{2} \) greatest entries of \( V \) and \( W \) by 1 each and adding remaining entries of \( V \) and \( W \) by 1 each to form \( V' \) and \( W' \). Then, edges are defined by \( u_1v_j^+ (u_1w_k^-) \) if \( e_j(f_k) \) is reduced by 1 and \( u_1v_j^- (u_1w_k^+) \) if \( e_j(f_k) \) is increased by 1. For \( -\alpha, -\beta \) and \( -\gamma \), the edges are defined by \( u_1v_j^- (u_1w_k^+) \) if \( e_j(f_k) \) is reduced by 1 and \( u_1v_j^+ (u_1w_k^-) \) if \( e_j(f_k) \) is increased by 1. If the conditions of the standard sequences do not hold, then we delete \( e_1 \) or \( f_1 \) for which the conditions of the standard sequences get satisfied. If this method is applied recursively, then a complete signed 3-partite graph with signed degree sequences \( \alpha, \beta \) and \( \gamma \) is constructed.

The next result gives a necessary and sufficient conditions for the three sequences of integers to be the signed degree sequences of some signed 3-partite graph.

**Theorem 2.5.** Let \( \alpha = [d_1, d_2, \cdots, d_p], \beta = [e_1, e_2, \cdots, e_q] \) and \( \gamma = [f_1, f_2, \cdots, f_r] \) be standard sequences. Then, \( \alpha, \beta \) and \( \gamma \) are the signed degree sequences of a signed 3-partite graph if and only if there exist integers \( g \) and \( h \) with \( d_1 = g - h \) and \( 0 \leq h \leq \frac{d_1 - d_2}{2} \) such that \( \alpha', \beta' \) and \( \gamma' \) are the signed degree sequences of a signed 3-partite graph, where \( \alpha' \) is obtained from \( \alpha \) by deleting \( d_1 \) and \( \beta' \) and \( \gamma' \) are obtained from \( \beta \) and \( \gamma \) by reducing \( g \) greatest entries of \( \beta \) and \( \gamma \) by 1 each and adding \( h \) least entries of \( \beta \) and \( \gamma \) by 1 each.

**Proof.** Let \( g \) and \( h \) be integers with \( d_1 = g - h \) and \( 0 \leq h \leq \frac{d_1 - d_2}{2} \) such that \( \alpha', \beta' \) and \( \gamma' \) are the signed degree sequences of a signed 3-partite graph \( G' (U', V', W') \). Let \( U' = \{u_1, u_2, \cdots, u_p\}, V' = \{v_1, v_2, \cdots, v_q\} \) and \( W' = \{w_1, w_2, \cdots, w_r\} \). Let \( X \) be the set of \( g \) vertices of \( V' \), and \( W' \) with highest signed degrees; \( Y \) be the set of \( h \) vertices of \( V' \) and \( W' \) with least signed degrees and let \( Z = (V' \cup W') - X - Y \). Then, a signed 3-partite graph with signed degree sequences \( \alpha, \beta \) and \( \gamma \) can be obtained by adding a vertex \( u_1 \) to \( U' \) so that there are \( g \) positive edges from \( u_1 \) to the vertices of \( X \) and \( h \) negative edges from \( u_1 \) to the vertices of \( Y \). Note that the signed degree of \( u_1 \) is \( g - h = d_1 \).

Conversely, let \( \alpha, \beta \) and \( \gamma \) be the signed degree sequences of a signed 3-partite graph. Let the vertex sets of the signed 3-partite graph be \( U = \{u_1, u_2, \cdots, u_p\}, V = \{v_1, v_2, \cdots, v_q\} \)
and $W = \{w_1, w_2, \ldots, w_r\}$ such that $sdeg(u_i) = d_i, 1 \leq i \leq p$, $sdeg(v_j) = e_j, 1 \leq j \leq q$ and $sdeg(w_k) = f_k, 1 \leq k \leq r$. Among all the signed 3-partite graphs with $\alpha, \beta$ and $\gamma$ as the signed degree sequences, let $G(U, V, W)$ be one with the property that the sum $S$ of the signed degrees of the vertices of $V$ and $W$ joined to $u_1$ by positive edges is maximum. Let $d_1^+ = g$ and $d_1^- = h$ be respectively the number of positive edges and the number of negative edges incident with $u_1$. Then, $sdeg(u_1) = d_1 = d_1^+ - d_1^-$, and $sdeg(u_1) = d_1^+ + d_1^- = g + h \leq q + r$, and hence $0 \leq h \leq \frac{q + r - d_1}{2}$. Let $X$ be the set of $g$ vertices of $V$ and $W$ with highest signed degrees and let $Y = (V \cup W) - X$. We claim that $u_1$ must be joined by positive edges to the vertices of $X$. If this is not true, then there exist vertices $x \in X$ and $y \in Y$ such that the edge $u_1y$ is positive and either (i) $u_1x$ is negative or (ii) $u_1$ and $x$ are not adjacent in $G(U, V, W)$. As $sdeg(x) \geq sdeg(y)$, therefore we consider only case (i), while as case (ii) is similar to case (i).

We note that if there exists a vertex $u_n(\neq u_1)$ such that $u_nx$ is a positive edge and $u_ny$ is a negative edge, then change the sign of these edges so that $u_1x$ and $u_ny$ are positive, and $u_1y$ and $u_nx$ are negative. Hence, we obtain a signed 3-partite graph with signed degree sequences $\alpha, \beta$ and $\gamma$ in which the sum of the signed degrees of the vertices of $V$ and $W$ joined to $u_1$ by positive edges exceeds $S$, a contradiction. So, assume that no such vertex $u_n$ exists.

Now, suppose that $x$ is not incident to any positive edges. Since $sdeg(x) \geq sdeg(y)$, then there exists at least two vertices $u_n$ and $u_t$ (both distinct from $u_1$) such that $u_ny$ and $u_1y$ are negative edges and both $u_n$ and $u_t$ are not adjacent to $x$. Then, by changing the edges so that $u_1x$ is a positive edge and $u_1y$ and $xu_n$ and $xu_t$ are negative edges, we again get a contradiction. Hence, $x$ must be incident to at least one positive edge.

We claim that there exists at least one vertex $u_t$ such that $u_mx$ is a positive edge and $u_t$ is not adjacent to $y$. Suppose on contrary that whenever $x$ is joined to a vertex by a positive edge, then $y$ is also joined to this vertex by a positive edge. Since $sdeg(x) \geq sdeg(y)$, then again we have the same situation as above, from which we get a contradiction. Thus, there exists a vertex $u_t$ such that $u_tx$ is a positive edge and $u_t$ is not adjacent to $y$. Similarly, it can be shown that there exists a vertex $u_n$ such that $u_ny$ is a negative edge and $u_n$ is not adjacent to $x$. By changing the edge so that $u_1x$, $yu_1$ are positive edges and $u_1y$, $xu_n$ are negative edges, we again get a contradiction. Thus, $u_1$ must be joined by positive edges to the vertex of $X$.

In a similar way, it can be shown that $u_1$ is joined by negative to the $h$ vertices of $V$ and $W$ with least signed degrees. Hence, $G(U, V, W) - u_1$ is a signed 3-partite graph with $\alpha', \beta'$ and $\gamma'$ as the signed degree sequences.

Theorem 2.5 provides an algorithm of checking whether the standard sequences $\alpha, \beta$ and $\gamma$ are the signed degree sequences, and for constructing a corresponding signed 3-partite graph. Suppose $\alpha = [d_1, d_2, \ldots, d_p]$, $\beta = [e_1, e_2, \ldots, e_q]$ and $\gamma = [f_1, f_2, \ldots, f_r]$ be the standard signed degree sequences of signed 3-partite graph with parts $U = \{u_1, u_2, \ldots, u_p\}$, $V = \{v_1, v_2, \ldots, v_q\}$ and $W = \{w_1, w_2, \ldots, w_r\}$. Let $d_1 = g - h$ and $0 \leq h \leq \frac{q + r - d_1}{2}$. Deleting $d_1$ and reducing $g$ greatest entries of $\beta$ and $\gamma$ by 1 each and adding $h$ least entries of $\beta$ and $\gamma$ by 1 each to form $\beta'$ and $\gamma'$. Then, edges are defined by $u_1v_j^+(u_1w_j^+)$ if $e_j(f_k)$ is reduced by 1; $u_1v_j^-(u_1w_j^-)$ if $e_j(f_k)$ is increased by 1, and $u_1$ and $v_j(u_1$ and $w_k$) are not
adjacent if $e_j(f_k)$ are unchanged. For $-\alpha, -\beta$ and $-\gamma$, edges are defined by $u_1 v_j^-(u_1 w_k^-)$ if $e_j(f_k)$ is not decreased by 1, $u_1 v_j^+(u_1 w_k^+)$ if $e_j(f_k)$ is increased by 1, and $u_1 v_j^-(u_1 w_k^-)$ and $v_j(u_1 w_k^-)$ are not adjacent if $e_j(f_k)$ are unchanged. If the conditions of the standard sequences do not hold, then we delete $e_1$ or $f_1$ for which the conditions of the standard sequences get satisfied. If this method is applied recursively, then a signed 3-partite graph with signed degree sequences $\alpha, \beta$ and $\gamma$ is constructed.

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ON SOME MODELS LEADING TO QUASI-NEGATIVE-BINOMIAL DISTRIBUTION

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Abstract In this paper, we explore some interesting models of the quasi-negative-binomial distribution based on difference differential equations applicable to theory of microorganisms and the situations like that. Some characterizations based on conditional distributions and damage process have been obtained. Further, the distribution of number of accidents as the quasi-negative-binomial distribution in the light of Irwin’s theory of “proneness-liability” model has been derived. Finally, the proposed model (QNBD) has been applied to study the Shunting accidents, home injuries, and strikes in industries.

Keywords: quasi-negative-binomial distribution, models based on difference differential equations, characterization, distribution of number of accidents, chi-square fitting.

1. INTRODUCTION AND MOTIVATION:

Unlike Jain and Consul’s (1973) GPD model, the quasi-negative-binomial distribution also reveals the fact that the probability of success from trial to trial does not remain constant. However, in the real world of living beings the value of the probability changes according to the circumstances. These changes may be due to the inheritance of genes, psychological effects, feelings of social togetherness, previous experience, determination for successor or to face a common danger, adjustments needed for changes in environments, wisdom etc. In such cases, the classical negative binomial distribution does not fit well the data arising from these cases. In fact, these are situations where the probability does remain linearly dependent on the number of successes.
The quasi-negative-binomial distribution is an interesting distribution and has not been studied in detail so far. It was obtained in different forms by Janardan (1975), Nandi and Das (1994) and Sen and Jain (1996).

In this paper, we applied the proposed model to study microorganisms and obtained some models based on difference differential equations leading to the quasi-negative-binomial distribution. This has been shown in Section 2. Section 3 deals with some characterizations based on conditional distribution and damage process. The distribution of number of accidents as quasi-negative-binomial distribution in the light of Irwin’s theory of “proneness-liability” model has been derived in Section 4. Finally, in Section 5, we applied the proposed model to study the Shunting accidents, home injuries, and strikes in industries and obtained some remarkable fit than GPD model.

2. MODELS BASED ON DIFFERENCE DIFFERENTIAL EQUATIONS

In a paper by Tukey (1949), the probability distribution of balls in boxes is a Poisson distribution while assuming that the probability \( II_x \) of finding \( x \) balls in a box is a function of mean number \( \lambda \) of balls in the boxes such that \( II_0 = 1 \) for \( \lambda = 0 \),

\[
\frac{dII_0}{d\lambda} = -II_0 \quad \text{and} \quad \frac{dII_x}{d\lambda} = -II_{x-1} \quad \text{for} \quad x \geq 1 .
\]

This result was generalized by Consul (1988) for generalized Poisson distribution and proved two results for the generalization of Tukey’s result. Consul (1990d) also proved these results for quasi-binomial distribution.

It has been shown by many authors that the classical negative binomial distribution has become increasingly more successful and more flexible alternative than Poisson distribution in accounting the data especially arising in the study of entomology, bacteriology, ecology etc. Taking this fact into consideration and noting that quasi-negative-binomial distribution is a generalization of Jain and Consul’s (1973) generalized Poisson distribution, in Gurland’s (1957) terminology, and that of classical negative-binomial distribution, we had made an attempt here to prove these results for the proposed model.

Let there be an infinite but countable number of available spaces for insects, bacteria, viruses or microbes. Let \( \theta_1 \) be the probability of initial desire of each one of them to get into a particular location. This value of \( \theta_1 \) may increase or decrease by a small quantity \( \theta_2 \) due to some factors like psychological effects, feeling of social togetherness, mutual consultations, communications, determination, prevalent conditions, and the numbers succeeding to get in that location. Let \( M_x(a, \theta_1, \theta_2) \) be the probability of finding exactly \( \cdot \) a number of microbes in that location which will be function of \( a, \theta_1 \) and \( \theta_2 \). By changing each one of the parameters \( \theta_1 \) and \( \theta_2 \) we get the following two theorems.

Theorem 2.1. If the mean \( \mu(a, \theta_1, \theta_2) \) for the probability distribution of microorganisms is increased by changing the parameter \( \theta_1 \) to \( \theta_1 + \Delta \theta_1 \) in such a manner that
\[
\frac{\partial}{\partial \theta_1} M_\theta(a, \theta, \theta_2) = -\frac{a}{(1+\theta_1)} M_\theta(a, \theta, \theta_2) \tag{2.1}
\]
and
\[
\frac{\partial}{\partial \theta_1} M_x(a, \theta, \theta_2) = -\frac{(a+x)}{(1+\theta_1 + x\theta_2)} M_x(a, \theta, \theta_2) + am_{\theta - 1}(a + l, \theta_1 + \theta_2, \theta_2) \tag{2.2}
\]
For all integral values of \( x > 0 \) with the initial condition \( M_\theta(a, 0, \theta_2) = 1 \) and \( M_x(a, 0, \theta_2) = 0 \) for \( x > 0 \), then the probability model \( M_x(a, \theta, \theta_2) \) is the QNBD model \( P_x(a, \theta, \theta_2) \).

**Proof.** Equation (2.1) is a linear differential equation with integrating factor \((1+\theta_1)^x\) and the general solution \( M_x(a, \theta, \theta_2) = C_0 (1+\theta_1)^{-x} \). By making use of the initial condition \( M_\theta(a, 0, \theta_2) = 1 \), the constant \( C_0 = 1 \) and therefore,
\[
M_\theta(a, \theta, \theta_2) = (1+\theta_1)^{-x} = P_\theta(a, \theta, \theta_2)
\]
Taking \( x = 1 \) in (2.2) and using the result above, we get the linear differential equation
\[
\frac{\partial}{\partial \theta_1} M_x(a, \theta, \theta_2) = -\frac{(a + 1)}{(1+\theta_1 + \theta_2)} M_x(a, \theta, \theta_2) + \frac{a}{(1+\theta_1 + \theta_2)^{x+1}}
\]
with the integrating factor \((1+\theta_1 + \theta_2)^{x+1}\) and the general solution
\[
(1+\theta_1 + \theta_2)^{x+1} M_x(a, \theta, \theta_2) = a\theta_1 + C_1
\]
By making use of the initial condition \( M_x(a, 0, \theta_2) = 0 \), the constant \( C_1 = 0 \) and
\[
M_x(a, \theta, \theta_2) = a\theta_1 (1+\theta_1 + \theta_2)^{-x-1} = P_x(a, \theta, \theta_2)
\]
Using the result above in (2.2) for \( x = 2 \), we get
\[
\frac{\partial}{\partial \theta_1} M_x(a, \theta, \theta_2) = -\frac{(a + 2)}{(1+\theta_1 + 2\theta_2)} M_x(a, \theta, \theta_2) + \frac{a(a + 1)}{(1+\theta_1 + 2\theta_2)^{x+2}}
\]
On multiplying the differential equation above with integrating factor \((1+\theta_1 + 2\theta_2)^{x+2}\) and then integrating w.r. to \( \theta_1 \), we get
\[
(1+\theta_1 + 2\theta_2)^{x+2} M_x(a, \theta, \theta_2) = a(a + 1) \left[ \frac{\theta_1^2}{2} + \theta_1 \theta_2 \right] + C_2
\]
By making use of the initial condition for \( x = 2 \), we get \( C_2 = 0 \) and
\[
M_x(a, \theta, \theta_2) = \frac{a(a + 1)}{2!} \frac{\theta_1 (\theta_1 + 2\theta_2)}{(1+\theta_1 + 2\theta_2)^{x+2}} = P_x(a, \theta, \theta_2) \tag{2.3}
\]
Now, for \( x = 3 \), equation (2.2) together with relation (2.3) gives
\[
\frac{\partial}{\partial \theta_1} M_x(a, \theta, \theta_2) = -\frac{(a + 3)}{(1+\theta_1 + 3\theta_2)} M_x(a, \theta, \theta_2) + \frac{a(a + 1)(a + 2)}{2!} \frac{(\theta_1 + \theta_2)(\theta_1 + 3\theta_2)}{(1+\theta_1 + 3\theta_2)^{x+3}}
\]
The general solution of the linear differential equation above is
\[
(1+\theta_1 + 3\theta_2)^{x+3} M_x(a, \theta, \theta_2) = \frac{a(a + 1)(a + 2)}{3!} \left[ \frac{\theta_1^3}{3} + \theta_1 \theta_2 \theta_3 \right] + C_3
\]
Again, by making use of the initial condition \( M_0(a, 0, \theta_2) = 0 \), the constant \( C_3 \) becomes zero and we get

\[
M_y(a \theta_1, \theta_2) = \frac{a(a+1)(a+2)}{3!} \frac{\theta_1(\theta_1 + 3\theta_2)^2}{(1 + \theta_1 + 3\theta_2)^{a+3}} = P_y(a \theta_1, \theta_2)
\]

Thus we have proved that \( M_x(a \theta_1, \theta_2) = P_x(a \theta_1, \theta_2) \) for \( x = 0, 1, 2, 3 \). Now, it can be easily shown by the method of induction that

\[
M_x(a \theta_1, \theta_2) = \frac{(a+x-1)! \theta_1^{x-1}}{(a-1)! x!} \frac{(a+1, \theta_1 + x \theta_2)^{a+x}}{(1 + \theta_1 + x \theta_2)^{a+x}} = P_x(a \theta_1, \theta_2)
\]

for all nonnegative integral values of \( x \).

**Theorem.** 2.2. If the mean \( \mu(a \theta_1, \theta_2) \) for the probability distribution of microorganisms is increased by changing the parameter \( \theta_2 \) to \( \theta_2 + \Delta \theta_2 \) in such a manner that

\[
\frac{\partial}{\partial \theta_2} M_0(a \theta_1, \theta_2) = 0
\]

and

\[
\frac{\partial}{\partial \theta_2} M_x(a \theta_1, \theta_2) = -\frac{x(a+x)}{(1 + \theta_1 + x \theta_2)} M_x(a \theta_1, \theta_2) + \frac{a(x-1)! \theta_1}{(1 + \theta_1 + x \theta_2)^{a+x}} M_{x-1}(a+1, \theta_1 + \theta_2, \theta_2)
\]

for all integral values of \( x > 0 \) with the initial condition

\[
M_0(a \theta_1, 0) = (1 + \theta_1)^{-a} \quad \text{and} \quad M_x(a \theta_1, 0) = \frac{(a+x-1)! \theta_1^x}{(a-1)! x!} \frac{(1 + \theta_1 + x \theta_2)^{a+x}}{(1 + \theta_2)^{a+x}}
\]

for \( x > 0 \), then the probability model \( M_x(a \theta_1, \theta_2) \) is the QNBD model \( P_x(a \theta_1, \theta_2) \).

**Proof.** Integrating equation (2.4) w.r. to \( \theta_2 \) and by making use of the initial condition \( M_0(a \theta_1, 0) = (1 + \theta_1)^{-a} \), the constant \( C_0 = (1 + \theta_1)^{-a} \) and thus

\[
M_x(a \theta_1, \theta_2) = (1 + \theta_1)^{-a} = P_x(a \theta_1, \theta_2)
\]

For \( x = 1 \), the difference differential equation (2.5) becomes

\[
\frac{\partial}{\partial \theta_2} M_y(a \theta_1, \theta_2) = -\frac{(a+1)}{(1 + \theta_1 + \theta_2)} M_y(a \theta_1, \theta_2)
\]

The integrating factor for the above is \( (1 + \theta_1 + \theta_2)^{a-1} \) and the general solution is

\[
M_y(a \theta_1, \theta_2) = (1 + \theta_1 + \theta_2)^{a-1} C_1
\]

By making use of the initial condition \( M_y(a \theta_1, 0) = a \theta_1 (1 + \theta_1)^{-a-1} \) we get \( C_1 = a \theta_1 \) and

\[
M_y(a \theta_1, \theta_2) = a \theta_1 (1 + \theta_1 + \theta_2)^{-a-1} = P_y(a \theta_1, \theta_2)
\]

Taking \( x = 2 \) in (2.5) and making use of the result above, we get

\[
\frac{\partial}{\partial \theta_2} M_2(a \theta_1, \theta_2) = -\frac{2(a+2)}{(1 + \theta_1 + 2 \theta_2)} M_2(a \theta_1, \theta_2) + \frac{a(a+1) \theta_1}{(1 + \theta_1 + 2 \theta_2)^{a+2}}
\]

(2.6)
On multiplying the differential equation above with its integrating factor \((1+\theta_1+2\theta_2)^{n+2}\) and then integrating w.r.t. \(\theta_2\), we get

\[
(1+\theta_1+2\theta_2)^{n+2}M_2(a,\theta_1,\theta_2) = a(a+1)\theta_1\theta_2 + C_2
\]

By making use of the initial condition \(M_2(a,\theta_1,0) = \frac{a(a+1)}{2!} \frac{\theta_1^2}{(1+\theta_1)^{n+2}}\), the constant \(C_2 = \frac{a(a+1)}{2!} \theta_1^2\). Therefore,

\[
M_2(a,\theta_1,\theta_2) = \frac{a(a+1)}{2!} \frac{\theta_1(\theta_1+2\theta_2)}{(1+\theta_1+2\theta_2)^{n+2}} = P_2(a,\theta_1,\theta_2)
\]

Using the result above in (2.5) for \(x=3\), we get

\[
\frac{\partial}{\partial \theta_2} M_3(a,\theta_1,\theta_2) = -\frac{3(a+3)}{(1+\theta_1+3\theta_2)^{n+3}} M_3(a,\theta_1,\theta_2) + \frac{a(a+1)(a+2)\theta_1(\theta_1+3\theta_2)^2}{(1+\theta_1+3\theta_2)^{n+3}}
\]

The integrating factor for the differential equation above is \((1+\theta_1+3\theta_2)^{n+3}\) and the general solution is

\[
(1+\theta_1+3\theta_2)^{n+3} M_3(a,\theta_1,\theta_2) = a(a+1)(a+2)\theta_1\theta_2 \left[ \theta_1 + \frac{3}{2} \theta_2 \right] + C_3
\]

By the initial condition \(M_3(a,\theta_1,0) = \frac{a(a+1)(a+2)}{3!} \frac{\theta_1^3}{(1+\theta_1)^{n+3}}\), the constant \(C_3 = \frac{a(a+1)(a+3)}{3!} \theta_1^3\) and thus

\[
M_3(a,\theta_1,\theta_2) = \frac{a(a+1)(a+2)}{3!} \frac{\theta_1(\theta_1+3\theta_2)^2}{(1+\theta_1+3\theta_2)^{n+3}} = P_3(a,\theta_1,\theta_2)
\]

In a similar manner it can be easily shown by the method of induction that the unknown constants are determined by \(C_x = \frac{a(a+1)\ldots(a+x-1)}{x!} \theta_1^x\) for \(x=1,2,\ldots\)

and that

\[
M_x(a,\theta_1,\theta_2) = \frac{(a+x-1)! \theta_1(\theta_1+x\theta_2)^{x-1}}{(a-1)! x! (1+\theta_1+x\theta_2)^{n+x}} = P_x(a,\theta_1,\theta_2)
\]

for all nonnegative integral values of \(x\).

3. Characterization Based on Conditional Distribution and by Damage Process

Theorem 3.1. If \(X\) and \(Y\) are two independent random variables defined on the set of all non-negative integers such that
\[ p[X = k/X + Y = n] = \binom{n + k - 1}{k} \binom{n + k - 1}{n - k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-1}(\theta_1 + (n - k)\theta_2)^{n-k-1}(1 + \theta_1 + n\theta_2)^{n + n - 1}}{(1 + \theta_1 + k\theta_2)^{k}p_i + (1 + \theta_1 + (n - k)\theta_2)^{n + n - 1}(\theta_1 + n\theta_2)^{n - 1}} \]

for \( k = 0, 1, 2, \ldots, n \) and zero else where

Then show that \( X \) and \( Y \) must have QNBD with parameters \((n_1, \theta_1, \theta_2)\) and \((n_2, \theta_1, \theta_2)\) respectively.

**Proof.** Let \( P(X = x) = f(x) > 0, \sum f(x) = 1 \) and \( P(Y = y) = g(y) > 0, \sum g(y) = 1 \)

By the given condition the random variables \( X \) and \( Y \) are independent, therefore

\[ p[X = k/X + Y = n] = \frac{f(k)g(n-k)}{\sum_{k=0}^{n} f(k)g(n-k)} \]  

(3.2)

By making use of (3.1) in the equation above, for \( n \geq 1 \) and \( 0 \leq k \leq n \), we get functional relation

\[ \frac{f(k)g(n-k)}{f(k-1)g(n-k+1)} = \frac{(n_1 + k - 1)(n + k + 1)}{k(n_2 + n - k)} \frac{(\theta_1 + k\theta_2)^{k-1}(\theta_1 + (n - k)\theta_2)^{n-k-1}}{(\theta_1 + (k - 1)\theta_2)^{k-2}(\theta_1 + (n - k + 1)\theta_2)^{n-k}} \times \frac{(1 + \theta_1 + (n - k + 1)\theta_2)^{n + n - k + 1}}{(1 + \theta_1 + (n - k)\theta_2)^{n + n - k}} \]  

(3.3)

Replacing \( k \) by \( k + 1 \) and \( n \) by \( n + 1 \) in the above, we get

\[ \frac{f(k+1)g(n-k)}{f(k)g(n-k+1)} = \frac{(n_1 + k)(n + k - 1)}{k(n_2 + n - k)} \frac{(\theta_1 + (k+1)\theta_2)^{k}(\theta_1 + (n - k)\theta_2)^{n-k-1}}{(\theta_1 + k\theta_2)^{k-1}(\theta_1 + (n - k + 1)\theta_2)^{n-k}} \times \frac{(1 + \theta_1 + (n - k + 1)\theta_2)^{n + n - k + 1}}{(1 + \theta_1 + (n - k)\theta_2)^{n + n - k}} \]  

(3.4)

On dividing (3.4) by (3.3), we get

\[ \frac{f(k+1)f(k-1)}{[f(k)]^2} = \frac{(n_1 + k)k}{(k+1)(n_1 + k-1)} \frac{(\theta_1 + (k+1)\theta_2)^{k}(\theta_1 + (k-1)\theta_2)^{k-2}}{(\theta_1 + k\theta_2)^{2k+1}(1 + \theta_1 + (k+1)\theta_2)^{n + n - k + 1}} \times \frac{(1 + \theta_1 + (n - k+1)\theta_2)^{n + n - k + 1}(1 + \theta_1 + (k - 1)\theta_2)^{n + n - k}}{(1 + \theta_1 + (n - k)\theta_2)^{n + n - k + 1}(1 + \theta_1 + n\theta_2)^{n + n - k}} \]

Taking \( k = 1, 2, \ldots, n - 1 \) in the equation above and then on multiplying together, we get

\[ \frac{f(n)}{f(n-1)} = \frac{f(1)}{f(0)} \frac{(n_1 + n - 1)}{nn_1} \frac{(\theta_1 + n\theta_2)^{n-1}}{(\theta_1 + (n - 1)\theta_2)^{n-2}} \frac{(1 + \theta_1 + (n - 1)\theta_2)^{n + n - 1}}{(1 + \theta_1 + n\theta_2)^{n + n - 1}} \]

Setting \( \frac{f(1)}{f(0)} = \frac{n\theta_1(1 + \theta_1)^{n-1}}{(1 + \theta_1 + \theta_2)^{n + n - 1}} \) in the equation above, we get

\[ f(n) = \frac{(n_1 + n - 1)}{n} \frac{(\theta_1 + n\theta_2)^{n-1}}{(\theta_1 + (n - 1)\theta_2)^{n-2}} \frac{(1 + \theta_1 + (n - 1)\theta_2)^{n + n - 1}}{(1 + \theta_1 + n\theta_2)^{n + n - 1}} f(n-1) \]

A repeated use of the equation above gives
On Some Models Leading to Quasi-Negative-Binomial Distribution

\[
f(n) = \frac{(n_i+n-1)! \theta_i(n_i+1)(1+\theta_i)^n_i}{(n_i-1)! n_i! (1+\theta_i+n \theta_2)^{n_i+n}} \cdot f(0)
\]

Now, by making use of the fact that \(\sum f(x) = 1\), the recurrence relation above gives \(f(0) = (1+\theta_i)^{-n_i}\) and thus

\[
f(x) = \frac{(n_i+x-1)!}{(n_i-1)! x!} \frac{\theta_i(x+1\theta_i)^n_i}{(1+\theta_i+n \theta_2)^{n_i+n}} \quad x = 0,1,2,\ldots
\]

Hence the random variable \(X\) has a QNBD with parameters \((n_i, \theta_i, \theta_2)\).

On taking \(k = 1\) in (3.3), we get

\[
g(n) = \frac{f(0)}{f(1)} \frac{(n_i+n-1)!}{n_i!} \frac{(\theta_i+n \theta_2)^n_i}{(1+\theta_i+n \theta_2)^{n_i+n}} \frac{(1+\theta_i+(n_i-1) \theta_2)^n_i}{(1+\theta_i)^n_i}
\]

Setting as usual \(\frac{f(1)}{f(0)} = \frac{n \theta_i (1+\theta_i)^n_i}{(1+\theta_i+n \theta_2)^{n_i+n}}\) in the equation above, we get

\[
g(n) = \frac{(n_i+n-1)!}{n_i!} \frac{(\theta_i+n \theta_2)^n_i}{(1+\theta_i+(n_i-1) \theta_2)^n_i} \frac{(1+\theta_i+(n_i-1) \theta_2)^{n_i+n}}{g(n-1)}
\]

A repeated use of the equation above gives

\[
g(n) = \frac{(n_i+n-1)!}{(n_i-1)! n_i!} \frac{\theta_i(n_i+1)(1+\theta_i)^n_i}{(1+\theta_i+n \theta_2)^{n_i+n}} \cdot g(0)
\]

Since \(\sum g(x) = 1\), the recurrence relation above gives \(g(0) = (1+\theta_i)^{-n_2}\) and thus

\[
g(y) = P(Y = y) = \frac{(n_i+y-1)!}{(n_i-1)! y!} \frac{\theta_i(n_i+y \theta_2)^{n_i+n}}{(1+\theta_i+y \theta_2)^{n_i+n}} \quad y = 0,1,2,\ldots
\]

Hence the random variable \(Y\) also possesses QNBD with parameters \((n_i, \theta_i, \theta_2)\).

**Theorem.** 3.2. Let \(X_1\) and \(X_2\) be two independent discrete random variables whose sum \(Y\) is a QNB variate with parameters \((a, \theta_1, \theta_2)\). Then \(X_1\) and \(X_2\) must each be a QNB variate defined over all non-negative integers.

**Proof.** Consul (1974) and Famoye (1994) proved this theorem for GPD and GNB respectively. Since QNB is a generalization, in Gurland's (1957) terminology, of restricted GPD model with parameters \((\theta, a \theta)\) obtained by compounding the GPD model through the values of \(\theta\) with gamma distribution \(g(a, b)\) as mixing distribution. Further, QNB is also generalization of NBD. Therefore, taking these facts into consideration, an attempt has been made here to prove that the theorem under consideration also holds good for the proposed model.

Since the QNB variate \(Y\) has a lattice distribution defined over all non-negative integers. Therefore, using arguments of Raikov (1937), the random variables \(X_1\) and \(X_2\) must have also lattice distribution defined over all non-negative integers.

Let the pgf of the random variable \(X_i\), \(i = 1, 2\) be denoted by

\[
g_i(u) = \sum_{k=0}^{\infty} p_i(x) u^x, \quad |u| < 1
\]
Where \( P(x) = P(X_i = x) \) represents the pdf of \( X_i, i = 1, 2 \). Since the sum \( Y = X_1 + X_2 \) has a QNBD with parameters \((a, \theta_1, \theta_2)\), therefore its pgf is

\[
g(u) = \phi(t) = (1 + \theta_1 - \theta_1 t)^{-a} \quad \text{where} \ t = (1 + \theta_2 - \theta_2 t)^{-a}
\]

and

\[
g_1(u) = g_2(u) = \phi(t) = (1 + \theta_1 - \theta_1 t)^{-a} \quad \text{where} \ t = (1 + \theta_2 - \theta_2 t)^{-a}
\]

Now, using the arguments of Raikov (1937), the pgf of QNBD can only be factorized into pgf's of negative binomial distributions. Thus, the factors \( \phi_i(t) \) and \( \phi_2(t) \) of \( \phi(t) = (1 + \theta_1 - \theta_1 t)^{-a} \) must be given by \( \phi_1(t) = (1 + \theta_1 - \theta_1 t)^{-a_1} \) and \( \phi_2(t) = (1 + \theta_1 - \theta_1 t)^{-a_2} \) where \( a_1 \) is an arbitrary number such that \( 0 < a_1 < a \). Hence pgf’s of \( X_1 \) and \( X_2 \) becomes

\[
g_1(u) = (1 + \theta_1 - \theta_1 t)^{-a_1} \quad \text{and} \quad g_2(u) = (1 + \theta_1 - \theta_1 t)^{-a_2}
\]

Because of the uniqueness property, the pgf’s \( g_1(u) \) and \( g_2(u) \) must represent QNBD models. Thus, \( X_1 \) and \( X_2 \) must be a QNBD variate defined over all non-negative integers with parameters \((a, \theta_1, \theta_2)\) and \((a - a_1, \theta_1, \theta_2)\) respectively, where \( 0 < a_1 < a \).

**Theorem. 3.3.** If a non-negative QNBD variate \( X \) is subdivided into two components \( X_1 \) and \( X_2 \) such that the conditional distribution \[ P\left[ X_1 = k, X_2 = x - k \mid X = x \right] \]

is a hypergeometric-quaizi-negative-binomial distribution

\[
\binom{a+k-1}{a_1+k-1-k} \frac{\theta_1^k \theta_2^{x-k} (1 + \theta_1 + \theta_2)^{x-k}}{(1 + \theta_1 + \theta_2)^{a+k}} \quad \text{with}
\]

\[
\binom{a+k-1}{a_1+k-1-k} \frac{\theta_1^k \theta_2^{x-k} (1 + \theta_1 + \theta_2)^{x-k}}{(1 + \theta_1 + \theta_2)^{a+k}}
\]

Parameters \((a, a_1, k, \theta_1, \theta_2), 0 < a_1 < a \) then the random variables \( X_1 \) and \( X_2 \) are independent and have the QNBD.

**Proof.** Let \( X \) be a QNBD variate with parameters \((a, \theta_1, \theta_2)\) then its probability distribution is

\[
P(X = x) = \binom{a+k-1}{a_1+k-1-k} \frac{\theta_1^k \theta_2^{x-k} (1 + \theta_1 + \theta_2)^{x-k}}{(1 + \theta_1 + \theta_2)^{a+k}} \quad x = 0, 1, 2, ....
\]

The joint probability distribution of random variables \( X_1 \) and \( X_2 \) is given by the conditional distribution

\[
P(X_1 = k, X_2 = x - k) = P\left[ X_1 = k, X_2 = x - k \mid X = x \right] P(X = x)
\]

On substituting the values in the equation above, we get

\[
P(X_1 = k, X_2 = x - k) = \binom{a+k-1}{a_1+k-1-k} \frac{\theta_1^k \theta_2^{x-k} (1 + \theta_1 + \theta_2)^{x-k}}{(1 + \theta_1 + \theta_2)^{a+k}}
\]

Which is a product of two quazi-negative-binomial probabilities corresponding to two random variables \( X_1 \) and \( X_2 \). Thus the two random variables are independent and have QNBD models with parameters \((a, \theta_1, \theta_2)\) and \((a - a_1, \theta_1, \theta_2)\) respectively.
Characterization by damage process.

When an investigator collects a sample of observations produced by nature, according to a certain model, the original distribution may not be produced due to non-observability of some events or the partial destruction of some units. This problem was first pointed out by Rao (1963) when he studied the resultant model after the observations, generated by some probability models, were damaged by other probability models. Subsequently Rao and Rubin (1964), Srivastava and Srivastava (1970) and Consul (1975) also studied the same problem in case of Poisson and GPD models. In this paper, we have made an attempt to obtain some characterization of QNBD model in the light of the same theory.

Let $X$ be a random variable defined on non-negative integers with probability distribution $\{P_n\}$ and let $Z$ be a random variable denoting the undamaged part of the random variable $X$ when it is subject to destruction process such that

$$P(Z=k/X=x) = S[k/X], \quad k=0,1,\ldots,x,$$

then we have the following theorem.

**Theorem 3.4.** If $X$ is a QNB variate with parameters $(n, \theta_1, \theta_2)$ and if the destructive process is hypergeometric-quazi-negative-binomial variate given by

$$S[k/X] = \binom{n+k-1}{k} \binom{n-k}{k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}(\theta_1 + (x-k)\theta_2)^{x-k-l}(1+\theta_1 + x\theta_2)^{n-x}}{(1+\theta_1 + k\theta_2)^{n+k}(1+\theta_1 + (x-k)\theta_2)^{n-k-x-1}(1+\theta_1 + (x-k)\theta_2)^{n-x-k}}$$

$k=0,1,\ldots,x$

Show that

i) $Z$ is a QNB variate with parameters $(n, \theta_1, \theta_2)$.

ii) $P(Z=k) = P\left(Z=k/X\text{ damaged}\right) = P\left(Z=k/X\text{ undamaged}\right)$

**Proof.**

i) $P(Z=k) = \sum_{x=k}^{\infty} S[k/x] P_x (n, \theta_1, \theta_2)$

$$= \sum_{x=k}^{\infty} \binom{n+k-1}{k} \binom{n-k}{x-k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}(\theta_1 + (x-k)\theta_2)^{x-k-l}(1+\theta_1 + x\theta_2)^{n-x}}{(1+\theta_1 + k\theta_2)^{n+k}(1+\theta_1 + (x-k)\theta_2)^{n-k-x-1}(1+\theta_1 + (x-k)\theta_2)^{n-x-k}}$$

$$= \binom{n+k-1}{k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}}{(1+\theta_1 + k\theta_2)^{n+k}} \sum_{r=0}^{\infty} \binom{n-r-1}{r} \frac{(\theta_1 + \theta_2)^{r-l}}{(1+\theta_1 + \theta_2)^{n-1+r}}$$

$$= \binom{n+k-1}{k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}}{(1+\theta_1 + k\theta_2)^{n+k}} = P_k (n, \theta_1, \theta_2)$

Thus the random variable $Z$ is a QNB variate with parameters $(n, \theta_1, \theta_2)$.

ii) $P\left(Z=k/X\text{ damaged}\right) = \sum_{x=k}^{\infty} \sum_{x=k}^{\infty} S[k/x] P_x (n, \theta_1, \theta_2)$

$$= \sum_{k=0}^{\infty} \sum_{x=k}^{\infty} \binom{n+k-1}{k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}}{(1+\theta_1 + k\theta_2)^{n+k}} \sum_{r=0}^{\infty} \binom{n-r-1}{r} \frac{(\theta_1 + \theta_2)^{r-l}}{(1+\theta_1 + \theta_2)^{n-1+r}}$$

$$= \sum_{k=0}^{\infty} \sum_{x=k}^{\infty} \binom{n+k-1}{k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}}{(1+\theta_1 + k\theta_2)^{n+k}} + \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\theta_1(\theta_1 + k\theta_2)^{k-l}}{(1+\theta_1 + k\theta_2)^{n+k}}$$
Similarly it can be shown that

\[
P[Z = k / X \text{undamaged}] = \frac{\sum_{k=0}^{\infty} S[k]{\binom{n}{k}} P(n, \theta_1, \theta_2)}{\sum_{k=0}^{\infty} S[k]{\binom{n}{k}} P(n, \theta_1, \theta_2)} \frac{\binom{n+k-1}{k} \theta_1 (\theta_1 + k \theta_2)^{k-1}}{(1 + \theta_1 + k \theta_2)^n} = P(Z = k)
\]

where \(\sum_{k=0}^{\infty} S[k]{\binom{n}{k}} P(n, \theta_1, \theta_2) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \theta_1 (\theta_1 + k \theta_2)^{k-1} \frac{1}{(1 + \theta_1 + k \theta_2)^n} = 1\)

\[\frac{\binom{n+k-1}{k} \theta_1 (\theta_1 + k \theta_2)^{k-1}}{(1 + \theta_1 + k \theta_2)^n} = P(Z = k)\]

4. **QNBD MODEL AS A DISTRIBUTION OF NUMBER OF ACCIDENTS IN THE LIGHT OF LRWIN’S THEORY OF “PRONENESS-LIABILITY” MODEL**

Traffic accidents remain in concern for every one’s life. Various theories have been developed concerning the interpretation of different situations. A natural model which assumes that the probabilities of having an accident are only result of random factors is that the number of accidents is poisson distribution with parameter \(\lambda\) i.e,

\[P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0,1,2,\ldots\]

Where \(N\) is the random variable which describes the number of accidents of a single person.

Another theory, the “accident proneness” theory which takes into account the individual’s difference in probabilities of having an accident or in their “accident proneness” which remain constant in time. This theory takes into consideration both the factors random as well as non-random where the non-random factors refers to individual’s psychology, explaining in this way, more or less, why the individual’s have unequal accident proneness. Taking these situations into consideration, Greenwood and Woods (1919), obtained that the number of accidents \(N\) has a negative binomial distribution with parameter \(k\) and \(\frac{1}{v}\) i.e.,

\[P(N = n) = \binom{k+n-1}{n} v^n (1 + v)^{n+k}, \quad n = 0,1,2,\ldots\]

Consul (1989) also described the use of GPD model in an accident theory and applied it to a number of data sets pertaining to Shunting accidents, home injuries, and strikes in industries and obtained a better fit as compared to poisson and negative binomial distribution. The GPD model is

\[P(N = n) = \frac{\lambda(\lambda + n \theta)^{n-1} e^{-(\lambda + n \theta)}}{n!}, \quad n = 0,1,2,\ldots\]

He also gave the interpretation of the parameters-the parameter \(\lambda\) represents the “accident proneness” and \(\theta\) represents the “rate of restitution process” of the subject under study.
The Irwin’s theory of “proneness-liability” model which assumes also that the non-random factors can be further split into psychological and external factors provides more explanation as to why some individuals in the population tend to have more accidents than others. In the context of this model, the individual accident proneness does not remain constant, because the population is exposed to a variable risk. In his model, Irwin used the term “accident proneness” to refer to a person’s predisposition to accidents, and the term “accident-liability” to refer to a person’s exposure to external risk of the accident and he derived the univariate generalized Waring distribution as the distribution of number of accidents. Irwin (1975b) applied this model to data on accidents sustained by men in a soap factory, providing an improved fit as compared to the negative binomial.

In fact, Irwin derived his model by compounding the parameter of Poisson distribution with gamma distribution thus resulting in NBD which when compounded with beta- II distribution gives Irwin’s model. This model is one interesting member of the family of mixed Poisson distribution. Making this basis, here, we have made an attempt to derive the distribution of the number of accidents N in the light of Irwin’s theory starting with the restricted Consul and Jain’s (1973) GPD model

$$P(N=n) = \frac{\lambda^n (1+n\alpha)^{n-1} e^{-\lambda(1+n\alpha)}}{n!}, \quad n=0,1,2, \ldots \ldots$$  \hspace{1cm} (4.1)

As per the interpretation of the Consul (1989), the parameter $\lambda$ represents the “accident proneness” and $\theta = \alpha \lambda$ represents the “rate of restitution process” of the subject under study. In the light of Irwin’s theory, the individual accident proneness does not remain constant in time, and thus if we allow the parameter $\lambda$ in the model (4.1) to follow gamma distribution with parameters $k$ and $\frac{\lambda}{\nu}$ i.e. $\lambda$ for given $\nu$ is a random variable with density given by

$$f(\lambda/\nu) = \frac{\nu^k}{\Gamma(k)} \lambda^{k-1} e^{-\lambda/\nu} \quad \lambda \geq 0$$ \hspace{1cm} (4.2)

and then on compounding (4.1) through the values of $\lambda$ by (4.2), we get the distribution of the number of accidents $N$ as the quasi-negative binomial distribution

$$P(N=n) = \binom{k+n-1}{n} \nu^k (1+n\alpha)^{n-1} (1+\nu^{-1}+n\alpha)^{k+n} \quad n=0,1,2, \ldots \ldots$$

Taking $\theta = \alpha \nu$, the distribution becomes

$$P(N=n) = \binom{k+n-1}{n} \nu(\nu+n\theta)^{n-1} (1+\nu+n\theta)^{k+n} \quad n=0,1,2, \ldots \ldots$$  \hspace{1cm} (4.3)

Where $(k, \nu, \theta)$ represents the parameters of the QNBD.

Hence, in the light of Irwin’s theory of “proneness-liability” model, the QNB model explains both the variations in accident-proneness as well as in accident-liabilities of the subject under consideration.

5. APPLICATION OF QNBD MODEL IN SHUNTING ACCIDENTS, HOME INJURIES, AND STRIKES IN INDUSTRIES

In this section, we are more interested in the fitting of the proposed model and its comparison with the GPD model but not in the application of the chi-square test for testing the significance of the discrepancies between the observed and expected
frequencies as the degree of freedom provided by most of the data sets (tables (5.1)-(5.4)) for the proposed model is Zero which makes the chi-square test of significance invalid.

The data sets in tables (5.1) to (5.3) were previously used by Adelstein (1952) for Poisson and negative binomial distributions and concluded that the negative binomial fits well than Poisson distribution. Consul (1989a) used the same data sets for GPD model and reached to a conclusion that GPD model gives best fit than Poisson and negative binomial distributions, for more explanation and details; see GPD Consul (1989), pages 117-121. Here, we applied the proposed model to the same data sets. The parameters of the proposed model have been estimated by ML method with the help of a computer programme in R-soft wear.

**TABLE 5.1**

Comparison of observed frequencies for first-year shunting accidents and for a five year record of experienced men with expected QNBD frequencies for different age groups.

<table>
<thead>
<tr>
<th>No. of accidents</th>
<th>Age 21-25 yr</th>
<th>Age 26-30 yr</th>
<th>Age 31-35 yr</th>
<th>5-yr record for experience</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Obs.</td>
<td>QNBD</td>
<td>Obs.</td>
<td>QNBD</td>
</tr>
<tr>
<td>0</td>
<td>80</td>
<td>76.30</td>
<td>121</td>
<td>123.39</td>
</tr>
<tr>
<td>1</td>
<td>56</td>
<td>65.15</td>
<td>85</td>
<td>80.18</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>23.50</td>
<td>19</td>
<td>20.66</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5.10</td>
<td>1</td>
<td>2.61</td>
</tr>
<tr>
<td>≥4</td>
<td>0</td>
<td>0.16</td>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>Total</td>
<td>170</td>
<td>170</td>
<td>227</td>
<td>227</td>
</tr>
</tbody>
</table>

**ML Estimate**

\[ a = 21.94218080 \]
\[ \theta_1 = 0.037215153 \]
\[ \theta_2 = -0.00369338 \]
\[ a = 38.1442428 \]
\[ \theta_1 = 0.028172153 \]
\[ \theta_2 = -0.00347625 \]
\[ a = 41.94190 \]
\[ \theta_1 = 0.0173925 \]
\[ \theta_2 = -0.00365469 \]
\[ \theta_2 = 0.00046 \]
\[ 3.504626 \]
\[ 0.683468 \]
\[ 0.05219945 \]
\[ 1.48454 \]
TABLE 5.2
Comparison of observed frequencies of accidents of 122 experienced shunting men over 11 years (1937-1947) with expected QNBD frequencies

<table>
<thead>
<tr>
<th>No. of accidents</th>
<th>1937-1942</th>
<th></th>
<th>1943-1947</th>
<th></th>
<th>1937-1947</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Obs.</td>
<td>QNBD</td>
<td>Obs.</td>
<td>QNBD</td>
<td>Obs.</td>
<td>QNBD</td>
</tr>
<tr>
<td>0</td>
<td>40</td>
<td>39.86</td>
<td>50</td>
<td>50.88</td>
<td>21</td>
<td>20.07</td>
</tr>
<tr>
<td>1</td>
<td>39</td>
<td>39.68</td>
<td>43</td>
<td>40.52</td>
<td>31</td>
<td>30.55</td>
</tr>
<tr>
<td>2</td>
<td>26</td>
<td>23.80</td>
<td>17</td>
<td>19.51</td>
<td>26</td>
<td>27.53</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>11.24</td>
<td>9</td>
<td>7.46</td>
<td>19</td>
<td>19.29</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>4.65</td>
<td>2</td>
<td>7</td>
<td>11.66</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3.63</td>
<td>9</td>
<td>6.43</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2.77</td>
<td>1</td>
<td>9</td>
<td>6.47</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
</tr>
</tbody>
</table>

ML Estimate
\[ a = 21.94214414 \]
\[ \theta_1 = 0.05230512 \]
\[ \theta_2 = 0.00419416 \]
\[ \chi^2 = 1.560489 \]

\[ a = 11.209125667 \]
\[ \theta_1 = 0.081133953 \]
\[ \theta_2 = 0.004864953 \]
\[ \chi^2 = 0.917171 \]

\[ a = 31.942233844 \]
\[ \theta_1 = 0.058137631 \]
\[ \theta_2 = 0.004578100 \]
\[ \chi^2 = 4.018035 \]

TABLE 5.3
Comparison of observed frequencies for home injuries of 122 experienced men during 11 years (1937-1947) with the expected QNBD frequencies

<table>
<thead>
<tr>
<th>No. of Injuries</th>
<th>1937-1942</th>
<th></th>
<th>1943-1947</th>
<th></th>
<th>1937-1947</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Obs.</td>
<td>QNBD</td>
<td>Obs.</td>
<td>QNBD</td>
<td>Obs.</td>
<td>QNBD</td>
</tr>
<tr>
<td>0</td>
<td>73</td>
<td>73.23</td>
<td>88</td>
<td>87.92</td>
<td>58</td>
<td>57.07</td>
</tr>
<tr>
<td>1</td>
<td>36</td>
<td>35.34</td>
<td>18</td>
<td>18.78</td>
<td>34</td>
<td>34.68</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10.3</td>
<td>11</td>
<td>9.71</td>
<td>14</td>
<td>16.63</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3.05</td>
<td>4</td>
<td>5.59</td>
<td>8</td>
<td>7.50</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>-</td>
<td>6</td>
<td>0.12</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
</tr>
</tbody>
</table>

ML Estimate
\[ a = 6.929456168 \]
\[ \theta_1 = 0.076422976 \]
\[ \theta_2 = 0.002652596 \]
\[ \chi^2 = 0.0277794 \]

\[ a = 0.2276245 \]
\[ \theta_1 = 3.2170325 \]
\[ \theta_2 = -0.6553422 \]
\[ \chi^2 = 0.2661209 \]

\[ a = 32.942033133 \]
\[ \theta_1 = 0.023331560 \]
\[ \theta_2 = 0.006401901 \]
\[ \chi^2 = 1.055267 \]

After close examination of the data sets in tables (5.1) to (5.3) we conclude that the proposed model fits well than GPD model except that the aggregated data in the last columns of table (5.1) and table (5.2). As explained in section 4 of this paper, the best fit obtained is due to the fact that the proposed model explains both the variation in accident-proneness as well as in accident-liabilities of the subject under study.
Now, we present more data sets in table (5.4) on the number of strikes in 4-week periods in four leading industries in the United Kingdom during (1948-1959) were previously used by Kendall (1961) and concluded that the aggregate data for the four industries agrees with the Poisson law but that it does not hold that well for the individual industries. Consul (1989) used the same data sets for GPD model and observed that the data follows GPD model in Vehicle manufacturing industry, Ship-building industry, and Transport industry but that the pattern in the Coal-mining industry can not be well described by GPD model. We also applied the proposed model to the same data sets and the expected frequencies are shown in table (5.4), the parameters have been estimated by ML method with the help of a computer programme in R-soft wear.

**TABLE 5.4**

Comparison of observed frequencies of the number of outbreaks of strike in four leading industries in the U.K. during (1948-1959) with the expected QNBD frequencies

<table>
<thead>
<tr>
<th>No. of outbreaks</th>
<th>Coal mining</th>
<th>Vehicle manufacture</th>
<th>Ship building</th>
<th>Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Obs. QNBD</td>
<td>Obs. QNBD</td>
<td>Obs. QNBD</td>
<td>Obs. QNBD</td>
</tr>
<tr>
<td>0</td>
<td>46 50.22</td>
<td>110 109.79</td>
<td>117 116.73</td>
<td>114 114.84</td>
</tr>
<tr>
<td>1</td>
<td>76 65.42</td>
<td>33 33.45</td>
<td>29 30.27</td>
<td>35 31.90</td>
</tr>
<tr>
<td>2</td>
<td>24 32.29</td>
<td>9 9.19</td>
<td>9 6.94</td>
<td>4 7.23</td>
</tr>
<tr>
<td>3</td>
<td>9 8.07</td>
<td>3 3.57</td>
<td>0 2.06</td>
<td>2 2.03</td>
</tr>
<tr>
<td>≥ 4</td>
<td>156 156</td>
<td>156 156</td>
<td>156 156</td>
<td>156 156</td>
</tr>
</tbody>
</table>

**ML Estimate**

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a = 32.94235746)</td>
<td>(a = 37.94234634)</td>
<td>(a = 38.942347767)</td>
<td>(a = 32.09423493)</td>
</tr>
<tr>
<td></td>
<td>(\theta_1 = 0.035003245)</td>
<td>(\theta_1 = 0.009301366)</td>
<td>(\theta_1 = 0.007475730)</td>
<td>(\theta_1 = 0.009588379)</td>
</tr>
<tr>
<td></td>
<td>(\theta_2 = -0.00475275)</td>
<td>(\theta_2 = 0.003574957)</td>
<td>(\theta_2 = 0.00273226)</td>
<td>(\theta_2 = 0.00283426)</td>
</tr>
<tr>
<td></td>
<td>4.655564</td>
<td>0.06217639</td>
<td>1.210815</td>
<td>2.213897</td>
</tr>
</tbody>
</table>

After comparing the chi-square values of the proposed model with the GPD model we did not find any improvement in fitting by the proposed model for the table (5.4) and observed almost equal fit.

**REFERENCE**


TEACHING APPLIED MATHEMATICS FOR ENGINEERS – A NEW TEACHING PARADIGM BASED ON INDUSTRIAL MATHEMATICS

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veli-matti.taavitsainen@evtek.fi

INTRODUCTION

My interest in pedagogical aspects of engineering mathematics arose when I was working as a scientist in a Finnish chemical company called Kemira. My role was to help the engineers to solve problems of mathematical nature in chemical and biotechnical engineering. In the early years of my career, I was astonished to notice the lack of understanding, even in some very elementary mathematical concepts, and the inability to describe problems in mathematical terms. Astonishingly enough, some engineers seemed to be almost proud of not having needed mathematics in their works. Yet, all the engineers I worked with had studied a huge amount of mathematics, at least in terms of time spent for it, typically 12 years in school and 2 to 3 years at university. Why were the results so poor, is mathematics really so difficult, or are we not teaching it in the right way? Could we do something differently? One of the objectives of this lecture is to discuss these questions.

Another objective of this lecture is to discuss whether the present engineering mathematics courses meet with the challenges emerging from applied industrial mathematics, and if not, what kind of changes are needed. Especially, we shall discuss the role of numerical mathematics, multivariate methods and applied statistics.

The ideas in this lecture are based on 10 years experience in teaching engineering mathematics at EVTEK University of Applied Sciences, mostly to students whose major subject is chemical or biotechnical engineering. However, my firm belief, partly based on discussions with teachers and students of other fields of engineering, is that many of these ideas and conclusions can be generalized into these other fields as well. The generality of the conclusions is naturally somewhat limited by the fact that my experience is mostly based on the European educational systems.

OBJECTIVES OF ENGINEERING MATHEMATICS

Before discussing pedagogical aspects or contents of engineering mathematics we have to be aware of the objectives of studying mathematics. We also have to know what a typical engineer expects to learn, what he is really interested in. For a mathematically oriented person, or a becoming mathematician, mathematics is interesting in itself. He or she is fascinated by the beauty, generality and logic of mathematics. An engineer thinks differently: he or she wants to solve engineering problems and mathematics is just a tool among others. In order to gain permanent results, one has to be able to motivate the engineering students that mathematics is an essential tool in solving real problems.

It is certainly true that many real engineering problems of industrial origin lead to mathematics that is beyond the mathematical abilities and skills of an average engineer. In
such cases, a project team of engineers needs to consult a professional applied mathematician for a successful completion of their project. However, such cooperation is bound to fail if the engineers do not know in what kind of sub-problems mathematics can help, or if they are not able to communicate with the mathematician due to the ignorance of mathematical terms or methods. Of course, there is a similar demand for the mathematician of having some understanding of the physical, chemical or biological and engineering aspects of the problem in question.

**THE PROBLEMS IN THE TRADITIONAL WAY OF TEACHING MATHEMATICS FOR ENGINEERS**

Still, in most universities, mathematics and some other subjects, e.g. physics, are taught almost solely during the first two years of engineering studies. After that, it is assumed that the student has achieved such a mastery of mathematical skills that he or she is able to apply them in the engineering subjects of the following years. This sounds good, and it would be ideal if it worked - but it doesn't. And the less there are practical examples within the math courses, linking the theory to real problems - or, to be more accurate, simplified real life problems, the less this kind of pedagogy works. Why is that so? The main reason is very simple: only a very small minority of people is able to model problems in mathematical terms without practicing. Just as a simple example, consider that of a student, who just has learned the definition of the derivative. It is very unlikely that he is able to apply it e.g. in physics without practicing it by examples of increasing complexity. But there are other reasons as well. Probably the one of the greatest importance is the question of motivation. A typical engineering student is not well motivated to study abstract mathematics, rather he or she would like to be convinced that what he or she is doing is useful in engineering problems - and very few people are convinced only for the reason that the mathematics teacher says that this will be useful in the future. A good reason is also the fact that many practically oriented students, who may become very good engineers, understand mathematical concepts best by practical examples.

Are there alternative ways of teaching mathematics for engineers? This will be discussed within the question of a need to change also the contents of mathematics courses.

**THE NATURE OF INDUSTRIAL ENGINEERING MATHEMATICS**

In the introduction we discussed how the recent development in computer technology has influence on the relevance of different branches of industrial and engineering mathematics. Let us look closer at some important aspects that also have influence on how to teach math.

The principle of parsimony

The main goal in solving industrial engineering problems is to get a solution, as quickly as possible and at as small expenses as possible. Of course, before any decisions, also an estimate of reliability of the solution is needed. This leads to the conclusion that the problem should be solved by the simplest possible method. The problem is that the academic world does not emphasize this kind of thinking. Academic research is looking forward to new methods, which usually are more complicated than the old ones. Naturally, the professors want to make publications in which the novelty of methods is a typical requirement. As a consequence, most graduate and especially PhD theses involve the newest most, and often complicated methodology. Exaggerating slightly it can be said that, if a simple method solves
a problem quickly and cost effectively, it does not have academic value. Of course, academic research has to produce new methodology, but it is important make it clear to the engineering student what is important in industry. However, one must differentiate between the simplicity of methods and simplicity in thinking: a simple method combined with simple thinking leads to erroneous conclusions, but insightful thinking, i.e. mathematical understanding, helps to find the simplest method.

To be able to show this to the students, exercises allowing solutions of different complexity levels are very important in teaching. The following examples illustrate the importance of simple mathematical methods combined with mathematical understanding.

Example 1: Quite often an engineer is looking for correlations in experimental data. Figure 1 depicts a typical case where the dependent variable is $y$ and the explanatory variables are $x_1$ and $x_2$. The question is: can we conclude that $y$ tends to increase along with both $x_1$ and $x_2$?

![Figure 1](image.png)

If the engineer hasn’t studied any statistics or statistical design of experiments the probable answer is affirmative, which in this case is wrong. This would probably be the case even if he or she would look at the original data table. Actually, these data have been calculated using a model $y = 0.25 - 0.75x_1 + 1.75x_2 + E$, where $E$ is a normally distributed random number, and thus $y$ decreases along with $x_1$. An engineering with some knowledge of basic statistics would immediately realize that the reason is the correlation between $x_1$ and $x_2$, commonly exhibited in real process data. The true dependencies could of course be easily detected by simple multiple linear regression, or by some more sophisticated graphical tools.

Example 2: Consider a case where 5 absorbances at different wavelengths have been measured from 21 food oil samples. The task is to find out whether any of them is a fake that would cause exceptional ratios in absorbances. The table of arbitrarily scaled absorbances is given below.
Now, it is quite obvious that the task is not easy just by looking at the numbers, neither by elementary graphical tools, e.g. simple scatter plots. Actually, the problem is really difficult unless one is familiar with elementary multivariate methods. On the other hand, it is very easy if one just knows the simplest multivariate method, the principal component analysis (PCA). Figure 2 shows the so-called score plot of the second and third principal components.
From this figure it is obvious that the oil number 9 is different from all the other and consequently possibly a fake oil. Such tasks of finding outliers, or groups in general, in multivariate data, also in process data, are quite common in industrial problems. In many cases they can be solved with such simple methods as the one above.

The importance of modeling

Modeling or mathematical formulation is the first step in all mathematical problems of industrial origin. In spite of this obvious fact, modeling has a minor role in the elementary courses of engineering mathematics. As a consequence, the students typically have difficulties in separating the tasks of formulating a problem and solving the problem. Yet, to solve any complicated enough problem, it is essential first to formulate the problem, i.e. to get a model and after that, to analyze and classify the model. After classifying the model, the choice of efficient tools for solving the problem is usually quite straightforward. The following simple optimization example illustrates these aspects.

Example 3: This is an exercise that is given to the first chemical and biotechnology engineering students. To formulate the problem, only elementary mathematics is needed, but to solve the problem accurately, some basic knowledge of optimization is required. However, an approximate solution can be easily found e.g. by a grid search. The problem is given in the following form.

The product ($B$) of a chemical factory is produced in a batch reactor at constant temperature. The raw material ($A$) is fed into the reactor and it is let to react a given time ($t$). After this, the reactor is emptied and the product is purified. The reaction is assumed to be a first order chemical reaction, i.e. $A(t) = A_0 e^{-\lambda t}$ and $B(t) = A_0 - A(t)$, where $A(t)$ and $B(t)$ are the concentrations at time $t$ and $A_0$ is the initial concentration of the raw material. The reaction rate constant $\lambda$ depends on the temperature $T$ according to the Arrhenius equation $k = A_f e^{R/2.310^3 T}$, where $A_f$ is the frequency factor $E_a$ is the activation energy and $R$ is the gas constant.

The income from the product is $T_p$ $\epsilon$ per produced amount of $B$ ($m_B$). The costs are $K_0$ $\epsilon$ per batch and the purification costs are $K_p$ $\epsilon$ per the amount of $A$ ($m_A$) left after reaction time $t$. The costs due to the heating of the reactor are $K_T T^2$ $\epsilon$ and the time dependent costs during the reaction are $K_r t$ $\epsilon$. The time between two consecutive batches depends linearly on $m_A$, i.e. on $t_0 + k m_A$. The volume of the reactor is $V$.

The task is to optimize the yearly profit (= income - costs) with respect to the temperature ($T$) and the reaction time ($t$). The batches can be produced without breaks assuming 330 operating days per year.

The values for the parameters are (the units of $A_f$, $E_a$ and $R$ are compatible, when time is given in hours):

$$A_f = 10^{12} \quad E_a = 90000 \quad R = 8.3 \quad A_0 = 0.1 \text{ kg/l}$$

$$V = 1000 \text{ l} \quad T_p = 100 \text{ \epsilon/kg} \quad K_p = 10 \text{ \epsilon/kg} \quad K_0 = 2000 \text{ \epsilon}$$

$$K_T = 0.2 \text{ \epsilon/}^\circ \text{C}^2 \quad K_r = 100 \text{ \epsilon/h} \quad t_0 = 2 \text{ h} \quad k_r = 0.5 \text{ h/kg}$$

For a mathematician, approaching the problem in the right way is not difficult, i.e. by first forming a composed function of $T$ and $t$ for both income and costs, and then classifying and solving the optimization task. However, for a beginning engineering student it is not easy. It is very typical that he or she is mixed up with solution and formulation. The idea of first building a model of the problem is something that has to be learned by practicing and purely mathematical problems, unrelated to real problems, or practising only mechanical calculation, simply is not enough.
Such examples should also be an essential ingredient of elementary engineering mathematics courses. As a consequence, the elementary engineering mathematics courses cannot be faculty independent. Mass lectures of general mathematics will not serve purpose. It is also very challenging for the mathematics teacher: in order to make up interesting and motivating examples, he or she should be familiar with the kind of engineering the students are educated for.

Numerical vs. symbolic math

Due to the fast development of both computer hardware and mathematics software, the role of applied mathematics in solving industrial problems has increased. Industrial problems typically lead to complicated, large-scale problems, which can be solved only numerically by computers, emphasizing the role of numerical methods and matrix algebra. It is quite typical in engineering curricula that numerical mathematics is taught in separate courses. A more efficient way is to include numerical methods as an essential part of the elementary engineering mathematics courses. For example, numerical differentiation should be taught together with teaching the concept of a derivative and differentiation rules.

The importance of multivariate methods

Contemporary automatic process monitoring systems and modern analytical instruments produce inherently multivariate data. In interpreting such data, both conventional statistical and multivariate statistical methods have a key role. In biotechnical engineering, the emergence of bioinformatics has increased the need of understanding multivariate methods. Also process data analysis, one of the most common problems in industrial mathematics, is essentially an application of multivariate methods. Many of the speech and image processing techniques are based on multivariate methods as well. Here again, the principle of parsimony is good to remember. The emphasis should be in the simplest, yet powerful methods, such as the principal component analysis (PCA). The following industrial example shows the power of PCA in solving a quality problem.

Example 4: A factory produces a granulated product. The problem was that the particle size distribution of the product varied too much. However, in spite of a lot of effort, the reason of the variation remained unclear. Finally, it was decided to look at the size distributions using PCA score plots. Each size distribution of the daily average of the product was treated as an 8-dimensional vector, and each such vector was projected onto a plane of the first two principal components. This plot is shown in figure 3.

Figure 3
The score plot showed clear grouping into two almost distinct groups. A detailed analysis showed that every 4 to 5 consecutive points, i.e. projections of distributions, belonged to either of the groups. At first, nobody in the factory could explain the peculiar grouping, but finally a production engineer remembered that there were actually two identical granulation units in the process line and that the unit had to be cleaned after 4 to 5 days of operation. Now the reason for the unwanted variation was clear: although the units should have been identical and they were controlled identically, they produced a product of different size distributions. After this, the solution was obvious; the other granulation unit just had to be controlled differently.

Naturally, one might claim that it should have been obvious to suspect the cause of the problem to be in the granulation units. However, the periodicity in the size distributions did not show up in the mean particle size, nor in the standard deviation or in any other univariate measure. In addition, the similarity of the units was taken for granted. Thus, the problem might not have been solved without using multivariate methods.

For an engineer to be able to understand and apply multivariate methods, it is important to introduce the basics of multivariate mathematics at an early stage of studies as possible. This is possible in a natural way in connection with vector algebra and systems of linear equations. The student should find a 10 or 1000 dimensional vector as natural as an ordinary 2 or 3 dimensional vector. Again, it is a challenge for the mathematics teacher to provide motivating examples of high dimensional vectors related with real engineering problems.

The importance of statistical methods

Most industrial engineering problems involve use and analysis of measurement data. Without proper understanding of measurement uncertainty and statistical nature of such data, truly meaningful conclusions can hardly be drawn. Statistical methods are essential also in designing new processes and products of high quality. Nowadays, most big companies use statistically based quality policies, such as Six Sigma. Consequently, a course in statistical design of experiments (DOE) should be included in all fields of engineering. The same holds also for statistical process control (SPC), an area where new interesting multivariate techniques have been developed, e.g. multivariate SPC. However, all that is impossible without a good knowledge of the basics: probability distributions, confidence intervals, estimation, the logic of statistical testing, regression analysis etc. But again, efficient and motivating teaching of the basics is impossible without simple examples that are related to real engineering problems.

**ENHANCING MOTIVATION**

In the previous chapters, we have mainly discussed the kind of mathematics that is needed in order to be able to solve typical engineering problems of modern industrial processes. It has also been stated that a typical engineer, or engineering student, is not much interested in mathematics in itself, rather he or she sees it as a tool to serve quite practical purposes. However, to be able to apply mathematics a good understanding of the basic mathematical concepts and the ‘language’ of mathematics are necessary. To obtain such understanding and knowledge requires hard work and a lot of effort. This, in turn, requires motivation and a firm belief that the effort is worthwhile. The three basic ingredients of enhancing motivation are: 1) good examples in the field of the main subjects of the students, 2) integration of mathematics studies with physics, chemistry, biology, computer science and engineering subjects and 3) use of mathematical software already in the early stages of studies. Let us have a closer look at these topics.

Making up good examples is far from being easy. The minimum requirements of a good example are: it illustrates the mathematical concept that is being taught, it is interesting, it is
simple, but still it is related to a real problem. At least part of the examples should lead to numerical methods, to statistical aspects and to questions of the correct degree of simplification.

Integration of different studies is even more difficult. However, it is possible in small steps, if just the professors and lecturers are open-minded and willing to do it. Key ingredients of any success are interdisciplinary conversations and good timing. Especially important is that not all mathematics is taught in the first two years. An ideal way would be that if an engineering subject requires some special mathematics, a suitable mathematics course is given concurrently. In many cases this may be impossible in practice, but it is a goal worth aiming at. Some mathematical subjects are especially suitable for the 1 or 2 last years of the studies. Such is, for example, statistical design of experiments, whose proper understanding requires a lot of experience in experimentation and measurements. The famous applied statistician George Box has once stated that nobody should study statistics before having worked for at least 5 years after studies. This might be exaggerating the matter, but it contains the essential idea.

The use of mathematical software, properly done, allows the student to focus on understanding instead of training skills that are not any more important. However, the right balance is difficult to achieve. It is not always obvious what kind of mechanical training actually is helpful for the understanding. Let us take, for example matrix multiplication. If the student has understood the dot product of vectors and its geometric interpretation, it is possible to apply matrix multiplication in several fields and there is not much sense to practice matrix multiplication by hand. Similarly, inverting a matrix by hand does not help in understanding what the matrix inverse is. For that purpose the analogy between ordinary linear equations and systems of linear equations is much better. Using computers allows taking up more realistic examples already during the first mathematics courses. The use of computers also makes it easier to give good examples of how the problems are divided into the steps of modeling, solving and assessment of the results.

**Enhancing Understanding**

Finally, we shall discuss what kind of mathematical understanding and knowledge form the basics on which applications can be built on. In a way, learning to apply mathematics is to learn a new language and, as in learning any new language, one has to learn both the vocabulary and the syntax of forming meaningful sentences. The very basics of the language of mathematics are algebra and geometry. Learning any more advanced mathematical concepts or techniques is impossible without fluent mastery of algebra. This fact is often neglected, and the engineering students entering a university are typically assumed to know more than they actually do. Learning algebra resembles learning a language in the sense that it requires a lot of repetition. The required fluency is a product of a massive amount of practicing which does not end in the school.

An equally important matter is to find a balance between learning mechanical mathematical skills and understanding the key concepts. The former is especially important in learning solution techniques and the latter in problem formulation (modeling). Let us take an example, the derivative. If a beginning engineering student is asked to explain what a derivative of a function is, typical answers are usually related to differentiation rules, i.e. to the skills and not to the concept. Yet, to be able to apply the derivative, understanding the concept is what counts, not being able to differentiate elementary functions. Of course the latter is an important skill in solving problems but useless in applications without understanding what the derivative is. It is important to realize that numerical methods can be useful, not only in solving problems where analytical methods fail, but also in gaining
understanding. For example, simple numerical differentiation based on the difference quotient is good practice for understanding what the derivative really is (of course emphasizing that it is not applicable for differentiation using measured values of a function). Naturally, geometrical interpretations are good for this purpose as well. Another good example is the concept of a function. The students typically have great difficulties in learning to form (program) their own functions in any mathematical software. The problem is in the abstractness and the generality of the concepts. Therefore, teaching such concepts should not be separated from programming or numerical methods and these should be taught simultaneously.

Finally, one should not forget that an important part of the mathematical understanding is to realize that intuition can be very misleading, especially in probability and in multivariate dependencies. Examples serving also this purpose, such as example 1 here, should be included whenever applicable.

One of the consequences of teaching mathematics that is not related to real problems containing measured quantities is that students start to think of mathematical models as absolute truths. Though it is important to understand the rigorous nature of mathematical reasoning, it is equally important to understand that mathematical models are only approximate descriptions of real phenomena. Again, the best way to achieve this understanding is to show good examples, preferably such that the students can conduct the experiments themselves for comparing the model with the experimental data.

Example 5: The students are asked to model the cooling of a beer can in cold water using the Newton's law for cooling. After finding the correct ordinary differential equation, they have to solve it. After this, they have to conduct an experiment and to estimate the thermal resistance using the measured temperature profile. Finally, they are asked to discuss the goodness of the model, and to give explanations for the differences between the modeled and observed temperature in the can.

This is a typical example that combines modeling, experimentation, statistical analysis and consequences of the simplifications made in the model.

**SUMMARY**

What is the "new paradigm"? It is impossible express it in one or two words, but if one had to; the closest might be the "holistic approach". The expression can be justified by the fact that the conclusions above lead to a greater intermixing of mathematics with engineering and natural sciences subjects, typically expressed in the form of examples of simplified real problems. They also lead to a greater intermixing of subjects within mathematics so that the courses should have less separation e.g. between symbolic and numerical mathematics.

The conclusions also lead to the spreading the mathematics courses throughout all study years, not just the first two years. Of course, this should be done with great care in order to guarantee studies that are logically linked together.

The new paradigm also means that the needs arising from industrial mathematics must be taken into account in the contents of engineering mathematics courses. Such topics are e.g. multivariate methods, statistics and use of mathematical software.

What are we expected to gain from the paradigm shift? The primary benefit should be in obtaining more productive engineers equipped with a better degree of mathematical preparedness for engineering problems. But in addition, it should also promote more intensive use of applied mathematics and easier communication with professional mathematicians, often needed in complicated industrial problems.
Finally, it can be noted that the new paradigm is in harmony with the basic ideas of the CDIO (Conceive – Design – Implement – Operate) initiative for producing the next generation of engineers [1]. New ideas for engineering education can be found also in the homepage of SEFI (European Society for Engineering Education) [2].

1. http://www.cdio.org/

FINE SEGMENTATION USING GEOMETRIC
ATTRACTION-DRIVEN FLOW AND EDGE-REGIONS

JOOYOUNG HAHN AND CHANG-OCK LEE

ABSTRACT. A fine segmentation algorithm is proposed for extracting objects in an image, which have both weak boundaries and highly non-convex shapes. The image has simple background colors or simple object colors. Two concepts, geometric attraction-driven flow (GADF) and edge-regions are combined to detect boundaries of objects in a sub-pixel resolution. The main strategy to segment the boundaries is to construct initial curves close to objects by using edge-regions and then to make a curve evolution in GADF. Since the initial curves are close to objects regardless of shapes, highly non-convex shapes are easily detected and dependence on initial curves in boundary-based segmentation algorithms is naturally removed. Weak boundaries are also detected because the orientation of GADF is obtained regardless of the strength of boundaries. For a fine segmentation, we additionally propose a local region competition algorithm to detect perceptible boundaries which are used for the extraction of objects without visual loss of detailed shapes. We have successfully accomplished the fine segmentation of objects from images taken in the studio and aphids from images of soybean leaves.

1. INTRODUCTION

In the segmentation problems to extract objects from an image to make, for examples, 3D VR (virtual reality) contents or to estimate sizes of objects, a key issue is fine segmentation which means that the objects can be extracted without visual loss of detailed shapes. Our research is motivated by making 3D VR contents of commercial products. It makes an e-catalog that customers can browse a product in three dimensional virtual space on internet markets. A common way of making a 3D VR content starts from taking hundreds of photographs of a product with different view angles in

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a photo studio. The most difficult step is to extract the product from a background without visual loss of detailed shapes. The images taken in the studio have well-known difficulties in segmentation problems even though they usually have simple background colors and small amount of noises such as JPEG artifacts. The difficulties mainly come from lighting conditions in the studio and complex shapes of products. Most of lighting conditions make shadows which cause weak boundaries between dark objects and the background. More serious weak boundaries are produced by a reflection on some parts of an object due to bright lighting conditions and properties of materials of the object. It changes colors of objects into almost white which is normally used as a background color. Note that other simple colors on a background except white are not usually used because of color bleeding effect. In addition, there is another difficulty; shapes of objects can be highly non-convex.

There have been many boundary-based segmentation algorithms. The snake model in [1] has been a foundation of curve evolution based on the minimization of an energy. After a curve evolution was formulated by the level set method in [2], geodesic active contour model was introduced in [3] as the minimization of a weighted length. Although the model has many advantages over the classical snake, it has drawbacks such as dependence on positions of initial curves, incapacity for capturing weak boundaries changed smoothly from strong boundaries, and slow convergence in non-convex boundaries. Numerous modifications of the snake model and the geodesic active contour model have been developed to address these drawbacks. In [4], the gradient vector flow was proposed for a fast convergence to non-convex boundaries. In [5], a curvature vector flow was introduced to overcome a limitation of [4] for capturing highly non-convex shapes. In [6], the region-aided geometric snake was proposed for more robust detection of weak edges. If an object in an image has both weak boundaries and highly non-convex boundaries, most of boundary-based segmentation algorithms suffer from capturing such boundaries all around the object. Even though they may capture the boundaries, it is not enough to be a fine segmentation for extracting the detailed object from an image.

In this paper, we propose a fine segmentation algorithm for extracting objects in an image without visual loss of detailed shapes, which have both weak boundaries and highly non-convex shapes. There are two concepts, geometric attraction-driven flow (GADF) and edge-regions, which are combined to capture boundaries of objects in a sub-pixel resolution. Since an image is a two dimensional manifold, we obtain GADF by comparing two lengths of curves along the direction of the largest change in the manifold. Edge-regions contain most of edges. They are obtained by computing inward fluxes in the gradient field of a strength of edges. The main strategy to segment boundaries of objects is to construct initial curves close to objects by using edge-regions and then make a curve evolution in GADF. Both problems of dependence on positions of initial curves and slow convergence in non-convex boundaries are naturally solved because the initial curves are already close to objects regardless of shapes. Moreover, weak boundaries are captured because the orientation of GADF near boundaries
of objects points to edges from each side of the boundaries regardless of strength of boundaries. According to the purpose of segmentation, for examples, fine extraction of objects or measurement of sizes of objects, we additionally propose a local region competition algorithm to obtain perceptible boundaries which are used for extraction of objects without visual loss of detailed shapes. We have successfully accomplished the fine segmentation of objects from images taken in the studio. Our algorithm can be applied to other kinds of segmentation problems by taking the appropriate strategy for selecting the edge-regions. An example is to extract aphids from images of soybean leaves. We may count the number of aphids that live on the sampled leaves and obtain an exact size of each aphid. Those information gives the appropriate time to dust powder in a huge farmland.

2. ALGORITHMS

The proposed algorithm consists of five steps to extract objects from an image without visual loss of detailed shapes even though there are weak edges and highly non-convex shapes. We derive GADF in Step 1. Regions which contain most of edges, which we call edge-regions, are detected in Step 2. Initial curves close to boundaries of objects are obtained in Step 3 from edge-regions. In Step 4, we segment objects by using a curve evolution in GADF with the initial curves from Step 3. For a fine segmentation, a post processing is needed in Step 5. The detailed explanation and derivation for each step is appeared in the paper.

Step 1: Derivation of GADF. We derive a vector flow whose orientation near boundaries of objects points to edges from each side of the boundaries regardless of the strength of edges. We call the vector flow as geometric attraction-driven flow (GADF). GADF is obtained by a geometric analysis of eigenspace in a tensor field on a color image as a two-dimensional manifold. Note that the attraction term in the well-known segmentation algorithms [3, 4, 6] does not help to segment weak boundaries changed smoothly from strong boundaries because it is based on the gradient of the strength of boundaries. To the contrary, since the orientation of GADF is regardless of the strength of boundaries, GADF gives a possibility of capturing such weak boundaries in Step 4 when initial curves are close to objects. So, we will focus on finding such curves in Step 2 and Step 3.

Step 2: Detection of edge-regions. Edge-regions are roughly defined as a union of regions that include most of edges in objects. They are obtained by computing inward fluxes in the gradient field of a strength of edges. We use two steps to detect the edge-regions. The first step is to select candidates of edge-regions. It uses global constants applied to all images without considering an individual characteristic in each image. So, the candidates of edge-regions contain useful information in common all through similar images and also contain some points that are unnecessary or even harmful in finding initial curves close to objects. These points usually come from a lack of careful
consideration for an individual characteristic in each image. Therefore, we focus on deleting bad candidates of edge-regions in the second step. The edge-regions will give important information in Step 3 to construct initial curves for the segmentation process.

**Step 3: Construction of Initial Curves for Segmentation.** The main goal in this step is to obtain initial curves for evolution of curves, which is the segmentation process in Step 4. Generally, initial curves close to objects in boundary-based segmentation algorithms [3, 4] solve both problems of dependence on initial curves and slow convergence in non-convex shapes. We will obtain such initial curves by connecting edge-regions along boundaries of objects. In [7], line connection algorithms for contour completion were proposed by using an anisotropic diffusion operator. With the algorithms, the edge-regions become thick due to the diffusion process and it is hard to decide how large areas to be regarded as connected edge-regions. Instead, we propose a Hamilton-Jacobi equation for a curve evolution.

**Step 4: Segmentation using GADF and edge-regions.** In Step 1, GADF was derived, whose orientation near boundaries of objects points to edges from each side of the boundaries, and initial curves close to the boundaries were obtained in Step 3 by connecting edge-regions in Step 2. In this step, we solve a simple advection equation in order to segment objects:

\[
\frac{\partial \phi(x,t)}{\partial t} + \vec{F}(x) \cdot \nabla \phi(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T),
\]

\[
\phi(x,0) = \psi(x) \quad \text{in} \quad \Omega,
\]

where \( \vec{F} \) is the GADF and \( \psi(x) \) is a signed distance function which has the zero level set as the curves obtained in the Step 3. It is the simplest equation which really works for segmenting both highly non-convex shapes and weak edges in images.

**Step 5: Post processing.** In the previous step, we may consider the result as a final segmentation. However, the result is not fine enough for extracting objects from images without visual loss of detailed shapes. People usually recognize a little bit outside of the boundaries as borders of objects because human vision perceives objects without missing any part of the objects. We call such borders as perceptible boundaries of objects. The perceptible boundaries make a big difference for extracting objects from images where colors of objects are changed gradually near boundaries. We obtain perceptible boundaries of objects by using a local region competition algorithm based on comparison of local probability density functions; see [8]. As the region competition algorithm is applied locally, we deduce a Hamilton-Jacobi equation which has a force term based on the difference between local probability density functions.

3. **Examples and numerical aspects**

We illustrate a whole procedure of the proposed algorithm in Figure 1. It has strong edges mostly and weak edges on the bottom due to shadow. The image (a) is original.
Figure 1. A procedure of a fine segmentation algorithm using GADF and edge-regions: (a) is an original image. The black regions in (b) are edge-regions. The curves in (c) are a result of step 3. In (d), initial curves for a segmentation process are shown. The curves in (e) are perceptible boundaries. The image in (f) is an extracted object on white background. The size of image is 940 by 544.

Edge-regions are shown in (b) where original image is overlaid with edge-regions. The curves in (c) are a result of step 3 which connects edge-regions. The curves in (d) are the initial curves for the segmentation process in Step 4. Note that they are close to boundaries of objects. From Step 5, we solve a PDE to obtain final curves in (e). We use an explicit Euler scheme for time discretization. For space discretization a simple upwind scheme is used in Step 3 and a nonoscillatory upwind scheme is used in Step 4 and 5; see [9] for details of numerical schemes. Every computation related to level
sets is done by using fast local level set method [10]. A stopping criterion is given by measuring an error in a small band [11].

4. Conclusions

We introduced a fine segmentation algorithm for extracting objects in an image, which have both weak boundaries and highly non-convex shapes. The image has simple background colors or simple object colors with small amount of noises. The main strategy to segment the boundaries is to construct initial curves close to objects by using edge-regions and then to make curve evolution in GADF. Since the initial curves are close to objects regardless of shapes, highly non-convex shapes are naturally detected and dependence on initial curves in boundary-based segmentation algorithms is removed. Moreover, weak boundaries are captured because the orientation of GADF is obtained regardless of strength of boundaries. For a fine segmentation, we additionally propose a local region competition algorithm to detect perceptible boundaries which are used for extraction of objects without visual loss of detailed shapes. The proposed whole algorithm consists of five steps. In Step 1, we compute GADF and edge-regions are obtained in Step 2. In Step 3, we connect edge-regions in order to find initial curves close to objects for segmentation. From the initial curves, we obtain the boundaries of objects in Step 4. Based on results in Step 3 and 4, we finally obtain the perceptible boundaries of objects in Step 5. The proposed whole algorithm is able to extract objects from an image without visual loss of detailed shapes even though there are weak edges and highly non-convex shapes.

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DIRECTIONAL FILTER BANK-BASED FINGERPRINT IMAGE ENHANCEMENT USING RIDGE CURVATURE CLASSIFICATION

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Abstract: In fingerprints, singular regions including core or delta points have different directional characteristics from non-singular regions. Generally, the ridges of singular regions change more abruptly than those of non-singular areas, thus in order to effectively enhance fingerprint images regardless of region, local ridge curvature information needs to be used. In this paper, we present an improved Directional Filter Bank (DFB)-based fingerprint image enhancement method that effectively takes advantage of such ridge curvature information. The proposed method first decomposes a fingerprint image into 8 directional subbands using the DFB and then classifies the image into background, low curvature, and high curvature regions using the directional energy estimates calculated from the subbands. Thereafter, the weight values for directional subband processing are determined using classification information and directional energy estimates. Finally, the enhanced image is obtained by synthesizing the processed subbands. The experimental results show that the proposed approach is effective in enhancing both singular and non-singular regions.

1. INTRODUCTION

Owing to developments in fingerprint scanning technologies [1], recently fingerprint based person identification or verification could be deployed successfully in many forensic or civilian applications. However, even fingerprint images acquired by modern fingerprint sensors include various kinds of noise components, thus fingerprint image enhancement is still considered as a crucial step for reliable fingerprint feature extraction.

The aim of fingerprint image enhancement is not to produce a good visual appearance of a fingerprint image but to facilitate the subsequent ridge extraction. Therefore, not only suppressing noise but avoiding undesired side effects is very important in fingerprint image enhancement [2]. To accomplish this aim of image enhancement, many approaches have been suggested. O’Gorman et al. proposed a contextual filter whose parameters are automatically determined from image features such as ridge
width [3]. Sherlock et al. suggested a fingerprint image enhancement method using directional Fourier filtering [4]. This method suppresses both directional and frequency noises effectively in the Fourier transform domain, but it has the disadvantage that it requires Fourier transform and inverse Fourier transform operations as many as the number of predefined directional band pass filters. Hong et al. attempted to remove directional and frequency noises in the spatial domain by using a set of Gabor filters, which have oriented and frequency selective characteristics [5]. In this method, they assume that ridge-valley structures have a sinusoidal shape along the direction normal to ridges, thus non-singular regions satisfying this assumption relatively well can be successfully enhanced while singular regions generally forming a high curvature are prone to be distorted. In addition, due to blockwise operation of the Gabor filter based methods, some blocking artifacts appear especially in borders between the blocks with high differences in directions or qualities.

In order to effectively remove directional noises of fingerprints, several directional filter bank (DFB) [6][7] -based methods have been proposed [8]. The method in [8] decomposes a fingerprint image into 8 directional subbands using the DFB, and next is calculated a weight value for each directional subband block using the directional energy ratios and then each directional subband block is multiplied by a weight and synthesized each subband is processed. Finally, the enhanced image is obtained by synthesizing the processed directional subbands. Though it can remove directional noise quite a lot, this method could be unreliable because it is not easy to distinguish the regions of normal fingerprint and undesired noise features. Additionally, since the DFB-based methods do not use local ridge curvature information, it has a poor performance in enhancing singular (high curvature) areas though it enhances well non-singular (low curvature) regions. If the method by [7] is used for fingerprint enhancement, it has an advantage of multiresolution structure which consists of a combination of the Laplacian pyramid and the DFB, but it is not maximally decimated.

Therefore, based on DFB of [8] due to its simple structure and efficient processing time, we propose an improved fingerprint image enhancement method that effectively exploits local ridge curvature information. The proposed method gets such ridge curvature information using the DFB and the local directional subbands are processed differently according to local ridge curvatures. The enhanced image is obtained by synthesizing the processed directional subbands. In the next section, the proposed method is described in detail, and the experimental results and conclusions are given in section 3 and 4, respectively.

2. Fingerprint Image Enhancement

The proposed methods are composed of directional decomposition, subband processing, and synthesis stages. In the analysis stage, an input image is decomposed into 8 directional subband outputs using the 8-band DFB, and segmentation is performed to differentiate foreground regions from background ones and high curvature regions from low curvature ones. As a result of segmentation, a segmentation array is generated. In the subband processing stage, the decomposed directional subband images are processed using a segmentation array and directional estimates of each block. In
the synthesis stage, the processed subbands are synthesized and the enhanced image is obtained.

2.1 Directional Decomposition Using DFB

Directional decomposition of a fingerprint image is performed by the DFB that decomposes an image efficiently and accurately into several directional subbands. Fig. 1 shows the frequency partition map of the 8-band DFB and an example of directional decomposition by the DFB. The size of each directional subband image is illustrated in Fig. 1(c). For an \( N \times N \) image, the size of each subband becomes \( N/4 \times N/2 \) or \( N/2 \times N/4 \) according to its direction. The down sized subbands result from quincunx down samplers and the rectangular shapes of the subbands are due to post sampling for removing the frequency scrambling by down sampling. Though there are a lot of methods that perform directional filtering such as Gabor filters and directional Fourier filters, the DFB has several advantages over the other methods in that the directional separation is accurate and the procedures are efficient. The more detail description about the DFB used in this paper can be found in [6].

![Directional Decomposition](image)

Fig. 1. An example of directional decomposition by 8-band DFB. (a) Partition map of the 8-band DFB, (b) original image, and (c) decomposed directional subband images of (b).

2.2 Classification

The proposed method separates foreground regions from background ones by finding blocks where sum of the directional energy estimates is more than a certain threshold \( (Th) \). Let \( s_{i,j}^{\theta}(x, y) \) denote the coefficient at position \( (x, y) \) of subband \( \theta \) corresponding to an image block \( B_{i,j} \). The directional energy estimate of the image block \( B_{i,j} \) associated with subband \( \theta \) is defined as

\[
e_{i,j}^{\theta} = \sum_{x, y \in B_{i,j}} |s_{i,j}^{\theta}(x, y) - \mu_{i,j}^{\theta}| \tag{1}
\]

where \( \theta = \{0, 1, 2, ..., 7\} \) and \( \mu_{i,j}^{\theta} \) is the mean value of the corresponding subband block coefficients. Once the directional energy estimates of a block are calculated, we determine whether the block belongs to foreground regions or not as follows:
\[ A_s(i, j) = \begin{cases} 
1, & \text{if } \sum_{k=m}^{m+n} e_{i,j}^\theta > Th, \\
0, & \text{otherwise}, 
\end{cases} \quad (2) \]

where \( A_s(i, j) \) is a segmentation array that shows the characteristics of a block \( B_{ij} \). If \( A_s(i, j) = 0 \), the block \( B_{ij} \) belongs to background regions, while if \( A_s(i, j) = 1 \), it belongs to foreground regions. In our experiment, we set the block size for segmentation to 16x16.

Thereafter, to divide the foreground regions into low curvature areas and high curvature ones, the proposed method calculates a block orientation image \( o_{ij} \) by finding the direction with the maximum energy from each block. Since the directional energy values are usually affected by noise components, in order to get a more reliable orientation of the block the extended neighborhood regions are considered for calculating the block orientation. Let \( o_{ij} \) denote the orientation of a block \( B_{ij} \), then the block orientation is given as follows:

\[ o_{i,j} = \theta, \quad \text{if } E_{i,j}^\theta = \max \{ E_{i,j}^\theta' \}, \quad 0 \leq \theta, \theta' \leq 7, \quad (3) \]

\[ E_{i,j}^\theta = \frac{1}{(2m+1)^2} \sum_{k=-m}^{m} \sum_{l=-m}^{m} e_{i+k, j+l}^\theta \quad (4) \]

where \( E_{i,j}^\theta \) is the average directional energy estimate of the neighborhood region of the block \( B_{ij} \). An example of orientation images is given in Fig. 2. We can see that the orientation image calculated from average directional energy estimates is more reliable than the one by the directional energy estimates.

After obtaining an orientation image on a 8x8 block basis, the proposed method differentiates high curvature areas from low curvature ones by checking the directional difference (\( dd_{k,l} \)) between two adjacent 8x8 blocks as follows:

\[ A_s\left( \frac{i}{2}, \frac{j}{2} \right), A_s\left( \frac{i}{2} + k, \frac{j}{2} + l \right) = \begin{cases} 
2, & \text{if } 45^\circ \leq dd_{k,l} \leq 135^\circ \text{ and } d_s < d_{sh}, \\
1, & \text{otherwise}, 
\end{cases} \quad (5) \]

where \( dd_{k,l} = |\theta_{i,j} - \theta_{i+k,j+l}|, (k,l) = (-1,0), (0,-1), (0,1), (1,0), \) and \( d_s \) is a distance from the nearest singular point. According to the above equations, every 16x16 block including at least one high curvature 8x8 block becomes a high curvature area. In case that a local orientation field is affected by noise, the region could be a high curvature area even though it belongs to low curvature areas. For that reason, the proposed method defines high curvature areas as the regions where adjacent two block regions show a high curvature and they are within a certain distance \( d_{sh} \) from the nearest singular point. Limiting the high curvature area in a singular point area enables the proposed method to effectively differentiate the singular regions from noisy ones unless the singular point region is affected severely by noise. Since an orientation image is already obtained, detection of singular points can be performed simply by calculating a Poincare index for each local region [9]. As a result of classification, a segmentation array is generated and each element has a value between 0 and 2.
2.3 Processing and Synthesizing Directional Subbands

In general, in a block of low curvature areas directional energy is concentrated on about one direction, whereas in high curvature areas directional energy is concentrated on more than one direction. To enhance high curvature regions effectively, the proposed adjusts the number of directional subbands to be used for local region reconstruction according to ridge curvatures. In case that block size is fixed, high curvature regions can be represented well by using more number of directional subbands for local subband reconstruction than that for low curvature regions. Let $w_{ij}^\theta$ denote the weight value for $\theta$-directional subband ($s_{ij}^\theta$) corresponding to a block $B_{ij}$, then the weight values for a local region are determined as follows:

$$w_{ij}^\theta = \begin{cases} 
1, & \text{if } \theta \in D_{ij}, \\
0.5, & \text{else if } \{(\theta \pm 1) + n_0\} \mod n_0 \in D_{ij} \text{ and } \theta \notin D_{ij}, \\
0, & \text{otherwise},
\end{cases}$$

(6)

$$D_{ij} = \{ k \mid E_{ij}^k \geq \hat{E}_{ij} \}, \quad 0 \leq \theta, k \leq 7,$$

(7)

$$n = \begin{cases} 
n_t, & \text{if } A_j(i, j) = 1, \\
n_h, & \text{else if } A_j(i, j) = 2,
\end{cases}$$

(8)

where $D_{ij}$ is a set of indices indicating the orientations with dominant energy in each block and the directions within the top $n$ energies are the elements of $D_{ij}$, $n_0$ is the number of orientations, and $\{\hat{E}_0, ..., \hat{E}_7\}$ is a down sorted version of $\{E_0, ..., E_7\}$. In the experiments, we set $n_t$ and $n_h$ to 1 and 2, respectively. Since if only dominant directional subbands are used for reconstruction some artifact could occur, linear interpolation between the dominant orientation and its adjacent orientation is performed to reduce such artifacts as in Eq. (6) [8]. In the proposed method the block size is fixed as 16x16 and the number of directions with dominant energy varies from
as 16x16 and the number of directions with dominant energy varies from 1 to 2 according to a segmentation array value. All weight values for background regions ($A_r(i,j)=0$) are set to 0. The decomposed directional subbands are multiplied by the calculated weight values and the enhanced image is obtained by synthesizing the processed subbands. The block diagram for the proposed fingerprint image enhancement method is given in Fig. 3.

Since the directional filtering procedure sets the coefficients of directional subband blocks that do not belong to dominant directions to 0, we can see some artifacts in the directionally filtered image. In order to reduce such artifacts, the proposed method performs smoothing using a 3x3 mean filter after directional filtering.

**Fig. 3.** Block diagram for the proposed method.

### 3. EXPERIMENTAL RESULTS

The proposed method was first evaluated using the 5 fingerprint images sampled from FVC2000 2a database. The aim of fingerprint image enhancement is to facilitate the following fingerprint feature extraction, so we not only visually observed the enhanced images but investigated how the proposed method affects accurate fingerprint feature extraction. Since the fingerprint features we want to extract here are fingerprint minutiae such as bifurcations and ending points, we evaluate the proposed method analyzing the accuracy of the extracted minutiae after applying the proposed fingerprint image enhancement method to fingerprints.

Once the enhanced image is obtained, it is binarized and thinned. From the thinned image, minutiae can be detected simply by analyzing neighborhood pixels. In our experiments, we compared the proposed methods with the Gabor filter bank-based method [5], and the results are shown in Table 1. As shown in Table 1, the two methods had a similar error rate. To be more specific, the Gabor filter bank-based method showed a good performance in enhancing non-singular and oily regions, whereas severe blocking artifacts or ridge distortion appeared frequently in singular or high curvature regions. For the proposed method, it showed a better performance in enhancing singular areas than the Gabor filter bank-based method. In non-singular (or
low curvature) regions, it showed a reasonably good performance though it was a bit inferior to the Gabor filter bank-based method especially in enhancing the oily regions. In the case that two dominant directions are used for local subband reconstruction, the singular regions are enhanced better than the case that one dominant direction is used, but the broken ridges are not connected well. In Fig. 4, we can see that the proposed method using local ridge curvature information has a better enhancing performance than the previous methods.

**Table 1.** Error rate of each method. M: Number of missing minutiae, S: Number of spurious minutiae. DFB-based (I): DFB-based enhancement using one dominant direction [6], DFB-based (II): DFB-based enhancement using two dominant directions, DFB-based (adaptive): Proposed method

<table>
<thead>
<tr>
<th>Method</th>
<th>1 M</th>
<th>1 S</th>
<th>2 M</th>
<th>2 S</th>
<th>3 M</th>
<th>3 S</th>
<th>4 M</th>
<th>4 S</th>
<th>5 M</th>
<th>5 S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gabor filter-based [5]</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>10</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>DFB-based (I)</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>9</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>DFB-based (II)</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>11</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>DFB-based (adaptive)</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>10</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

4. **CONCLUSIONS AND FUTURE WORK**

We have proposed an improved DFB-based fingerprint image enhancement method that is effective in enhancing both singular and non-singular regions. The proposed method segments the foreground region of a fingerprint image into low curvature and high curvature regions, then processes local directional subbands differently according to where the local region belongs. Experimental results show that the proposed approach suppresses the major noise components of fingerprints effectively regardless of region. In order to improve the enhancing performance of the proposed method, further study on segmenting the singular and non-singular regions more reasonably is required.
Fig. 4. Comparison of the enhancement methods. (a) Original image, (b) enhanced image using one dominant direction, (c) binarized image of (b), (d) enhanced image using two dominant direction, (e) binarized image of (d), (f) enhanced image by the proposed method, (g) binarized image of (f), (h) enhanced image by the Gabor filter bank-based method, and (i) binarized image of (h).

Fig. 4. (Continued).

ACKNOWLEDGEMENTS

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THE PROCESS OF THE DEVELOPMENT OF HYPOXIA IN AN ABNORMAL BLOOD FLOW II

MINKYU KWAK AND JAEGWI GO

ABSTRACT. The oxygen distribution at steady state is analyzed mathematically in a hexagonal cylinder. The domain is penetrated by parallel cylindrical capillaries of different oxygen squirt. Asymptotic solution is used to determine the effect of axial diffusion. Oxygen concentration profiles are displayed at some positions of capillary-beds. At the venous end some tissue areas suffer from a shortage of oxygen.

Keywords: Hypoxia; Oxygen diffusion; Oxyhemoglobin;

1. INTRODUCTION

Measurements of oxygen concentration on a tissue domain is a fundamental problem in the study of oxygen transport. The basic single capillary model was introduced by Krogh[1]. The Krogh-Erlang equation formulated based on Krogh’s model was used to express oxygen tension in the highly regular capillary beds of skeletal muscle, and has been the foundation of most physiological estimates for the last 70 years. The major developments in quantitative and qualitative comprehension of oxygen transport to tissue lie in the studies of hemoglobin-oxygen kinetics, the role of hemoglobin and myoglobin in promoting oxygen diffusion, and the role of morphological hemodynamic heterogeneities.

Salathe and coworker[4] investigated oxygen distribution in steady state in a Krogh cylinder by considering axial diffusion in the tissue and capillary, and an arbitrary oxy-hemoglobin dissociation relationship. The effect of heterogeneity in skeletal muscle was studied by Popel and Charny[2], and Ronald and Chang[7]. Hoofd[6] extended Krogh circle to arbitrary shapes with nonhomogeneous multi-source including myoglobin role to
facilitate oxygen diffusion. The geometrical influences such as capillary network anastomoses and tortuosity were presented in the paper[3]. But, in spite of previous remarkable achievements, their works were limited to constant solute out.

A mathematical analysis of steady state oxygen distribution in a regular hexagonal cylinder is presented in this paper (see Fig. 1). In our mathematical equations the oxyhemoglobin dissociation relationship and axial diffusion in capillary are included. The exact solution for the oxygen concentration in the capillary is obtained using perturbation analysis.

Fig 1. A multi-capillary hexagonal cylinder domain.

2. MATHEMATICAL MODELLING AND PERTURBATION ANALYSIS

2.1. Mathematical Modelling

We consider a regular hexagonal cylinder of length $L$ for tissue domain. The cylinder is penetrated by parallel cylindrical $N$ capillaries arranged in uneven location and diffusion strength (Fig. 1.). Let $(\bar{\rho}_j, \varphi_j)$ be local cylindrical coordinates centered at each capillary and $\rho_{jr}$ be the radius of the $j$th capillary. The oxygen flux $\bar{q}_j$ per unit volume of the $j$th capillary is defined by

$$\bar{q}_j = -D_{ij}\rho_{jr} \int_0^{2\pi} \frac{\partial \bar{C}_i}{\partial \bar{\rho}_j}|_{\bar{\rho} = \rho_{jr}, d\varphi_j} d\varphi_j,$$

(1)
where \( \overline{C}_t \) is the oxygen concentration in the tissue and \( D_{\tau} \) is the radial oxygen diffusivity. The oxygen concentration in the blood of the \( j \)th capillary, \( C_{bj} \), varies with both radial and axial location within the capillary. However, the convective mixing within the bolus of fluid between each red blood cell results in a fairly uniform distribution of oxygen. We may thus neglect radial variation of concentration in the capillary. If \( F_j \) denotes the volume blood flow rate of the \( j \)th capillary and \( \overline{z} \) denotes distance measured along the capillary, then oxygen concentration in the blood is

\[
F_j \frac{\partial}{\partial \overline{z}} \{ C_{bj}(\overline{z}) + \overline{M} S[C_{bj}(\overline{z})] \} = -\overline{q}_j + \epsilon D_b \frac{\partial^2 C_{bj}}{\partial \overline{z}^2}.
\]

(2)

The \( \overline{M} \) denotes the oxygen capacity of blood, \( S[C_{bj}] \) denotes the oxyhemoglobin dissociation relationship, and the \( D_b \) denotes the diffusivity of blood. The \( \epsilon \) is related to the diffusivity in the direction \( \overline{z} [4] \), but we regard it as a small positive number, that is, \( 0 < \epsilon \ll 1 \). The function \( S[C_{bj}] \) may be approximated by the empirical formula

\[
S[C_{bj}] = \frac{\alpha C_{bj}^n}{1 + \alpha C_{bj}^n}
\]

(3)

for suitable choice of the constants \( \alpha \) and \( n \). At the arterial end, the oxygen concentration of the \( j \)th capillary is set:

\[
C_{bj}(0) = \overline{q}_j(0).
\]

(4)

Let \( \kappa \) be the constant consumption per volume of tissue. The longitudinal diffusion of solute may be neglected in the body because the length of the blood vessel cylinder is about 100 times cylinder diameter. At a given \( \overline{z} = \overline{z}_0 \) the oxygen distribution chart of each cross section of the hexagonal cylinder is investigated. The oxygen concentration in the tissue, \( \overline{C}_t(\overline{x}, \overline{y}, \overline{z}_0) \), at a fixed \( \overline{z} = \overline{z}_0 \) (\( 0 \leq \overline{z}_0 \leq L \)), satisfies the equation and boundary condition

\[
\frac{\partial}{\partial \overline{x}} D_{\overline{x}} \frac{\partial \overline{C}_t}{\partial \overline{x}} + \frac{\partial}{\partial \overline{y}} D_{\overline{y}} \frac{\partial \overline{C}_t}{\partial \overline{y}} = \kappa,
\]

(5)
\[
\frac{\partial C_i}{\partial n} = 0, \quad \text{at} \quad \bar{z} = \bar{z}_0. \quad (6)
\]

The \(D_\bar{x}\) and \(D_\bar{y}\) are the oxygen diffusivities of directions \(x\) and \(y\), respectively, and we set that \(D = D_\bar{x} = D_\bar{y}\) is a constant.

Let us suppose that, at the arterial end (at \(\bar{z} = 0\)), everywhere the consumption is \(\kappa\) per volume. Then the balance of mass flux on \(\bar{q}_j\) is

\[
\sum_{j=1}^{N} \bar{q}_j(0) = \kappa R, \quad (7)
\]

where \(R\) is the area of the regular hexagon (the cross section of the hexagonal cylinder).

In terms of non-dimension variables: \(C_{Bj} = C_{bj}/\kappa R\), \(x = \bar{x}/K\), \(y = \bar{y}/K\), \(z = \bar{z}/L\), \(q_j = \bar{q}_j/\kappa R\), \(C = C_iD/\kappa K^2\), \(M = \bar{M}/\kappa R\), \(S[C_{Bj}] = \bar{S}[C_{Bj}\kappa R]\), we obtain equations:

\[
\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} = 1 \quad (8)
\]

\[
\beta_j \frac{\partial}{\partial z} \{C_{Bj}(z) + MS[C_{Bj}(z)]\} = -q_j + \epsilon \tau \frac{\partial^2 C_{Bj}}{\partial z^2} \quad (9)
\]

\[
\frac{\partial C}{\partial n} = 0, \quad \text{at} \quad z = z_0, \quad 0 \leq z_0 \leq 1 \quad (10)
\]

\[
\sum_{j=1}^{N} q_j = 1, \quad \text{at} \quad z = 0 \quad (11)
\]

\[
C_{Bj} = q_j(0), \quad \text{at} \quad z = 0. \quad (12)
\]

Here \(\beta_j = \bar{F}_j/L\), and \(\tau = D_b/L^2\).

2.2. Perturbation Analysis

We assume that \(q_j, 1 \leq j \leq N\), are independent of \(z\) and let us expand \(C_{Bj}\) in the form of an asymptotic series

\[
C_{Bj} \sim C_{j0} + \epsilon C_{j1} + \cdots. \quad (13)
\]
The expansion $S[C_{ Bj}] = S[C_{ j0}] + CS'[C_{ j0}]C_{ j1}$ then yields the consecutive differential equations

$$\beta_j \frac{\partial}{\partial z} \{C_{ j0} + MS[C_{ j0}]\} = -q_j$$  \hspace{1cm} (14)

$$\beta_j \frac{\partial}{\partial z} \{C_{ j1} + MS'[C_{ j0}]C_{ j1}\} = \tau \frac{\partial^2 C_{ j0}}{\partial z^2}.$$  \hspace{1cm} (15)

The solution of Eq (14) satisfying (12) is

$$C_{ j0}(z) + MS[C_{ j0}(z)] = \frac{-q_j}{\beta_j} z + q_j + MS[q_j]$$  \hspace{1cm} (16)

Using the Eq (16) the solution of Eq (15) is

$$C_{ j1}(z)\{1 + MS'[C_{ j0}(z)]\} = -\frac{\tau q_j}{\beta_j^2(1 + MS'[C_{ j0}(z)])} + \frac{\tau q_j}{\beta_j^2(1 + MS'[q_j])}.$$  \hspace{1cm} (17)

We assume that the value of each parameter used in the asymptotic solution is independent of capillary, and the data are shown in table 1. When we combine the equations (16) and (17) for the oxygen concentration in the blood, $\varepsilon = 10^{-2}$ are employed for 2-capillary. The changes of the oxygen concentration in the blood along the capillaries are depicted in figure 2.

<table>
<thead>
<tr>
<th>$F$</th>
<th>$L$</th>
<th>$D_b$</th>
<th>$\alpha$</th>
<th>$n$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.13 \times 10^{-3}$</td>
<td>$6 \times 10^{-4}$</td>
<td>$1.7 \times 10^{-1}$</td>
<td>$8.55 \times 10^5$</td>
<td>$2.0$</td>
<td>$0.204$</td>
</tr>
</tbody>
</table>

![Graph showing the changes of oxygen concentration along capillaries](attachment:graph.png)
Fig 2. The variation of oxygen concentration in blood.

3. NUMERICAL RESULTS AND DISCUSSION

The level curves of oxygen concentration are found using MATHEMATICA 4.1. The term "HYPOXIA" is used in our figures to indicate the tissue area that the oxygen concentration $C$ is less than 0. We consider a hexagonal cylinder which is penetrated by two parallel cylindrical capillaries. The oxygen concentrations in the blood obtained using the perturbation analysis are diffused numerically based on the finite difference method[5]. The locations and strengths of capillaries at $z = 0$ are shown below in table 2 and the amount decreases as $z$ increases.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$q_k$</th>
<th>$x_k$</th>
<th>$y_k$</th>
</tr>
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<tr>
<td>1</td>
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<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
</tr>
</tbody>
</table>

The oxygen concentration profiles along the axis $z$ are displayed in figures 3-(a) $\sim$ 3-(f). As shown in figures 3-(a) $\sim$ 3-(d) the low lever concentration curves show an inclination that more oxygen disperse to the right upper corner of each hexagon. Hypoxic area appears at $z = 0.8$ (see Fig 3-(e)) and develops at the end of capillary bed (see Fig-(f)). The tissue area near the side 1 and the left upper corner of each hexagon are most susceptible to hypoxia. Moreover, Fig. 3-(f) shows that the upper part of side 2 is susceptible to hypoxia even though the distance to capillary is not far away.
Fig 3-(a). 2-capillary domain at z = 0.0. Fig 3-(b). 2-capillary domain at z = 0.2.

Fig 3-(c). 2-capillary domain at z = 0.4. Fig 3-(d). 2-capillary domain at z = 0.6.

Fig 3-(e). 2-capillary domain at z = 0.8. Fig 3-(f). 2-capillary domain at z = 1.0.

ACKNOWLEDGEMENT
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A STUDY OF BRAMBLE-HILBERT LEMMA AND ITS RELATION TO POINCARÉ’S INEQUALITY

SEUNG-WOO KUK

Abstract. This paper is concerned with the proof of so-called Bramble-Hilbert Lemma. We present that Poincaré’s inequality in [3] implies one of results of Morrey which is crucial in the proof. In this point of view, we recognize that removing the average term in Poincaré’s inequality fulfills crucial role in the proof of Bramble-Hilbert Lemma. It is accomplished by adding some polynomial of degree one less than the degree of the Sobolev space in the outset. So, the condition annihilating the set of polynomials $P_{k-1}$ of degree $k-1$ is required necessarily in Bramble-Hilbert Lemma.

Key words.

AMS subject classifications.

1. Introduction. In the paper [1] of so-called Bramble-Hilbert Lemma, the authors gave estimates for a certain class of linear functionals on Sobolev spaces. These functionals have the property that they annihilate the set of polynomials $P_{k-1}$ of degree $k-1$. The bounds were given in terms of the $L_p$ norms of all $k$-th order partial derivatives. In this paper, one of the results of Morrey, Lemma 3.3 is deduced from Poincaré’s inequality in [3]

$$\|u - (u)_U\|_{L^p(U)} \leq Cr\|Du\|_{L^p(U)},$$

(1.1)

where $(u)_U$ is the average of $u$ over $U$, and $r$ is a diameter of $U$. In other words, generalization of Poincaré’s inequality in [3] implies exactly Lemma 3.3. Another result of Morrey, Lemma 3.2 carry out a role removing the term $(u)_U$ in Poincaré’s inequality. Lemma 3.2 fulfills the role by adding some polynomial of degree $k-1$ with $u$. So, Bramble-Hilbert Lemma need the condition annihilating the set of polynomials $P_{k-1}$ of degree $k-1$. The form removed the term $(u)_U$ in Poincaré’s inequality

$$\|u\|_{L^p(U)} \leq Cr\|Du\|_{L^p(U)}$$

(1.2)

is essentially crucial in the proof of Bramble and Hilbert. If we obtain the form of (1.2), then the conclusions of Bramble-Hilbert Lemma is derived naturally. For example, in $W_0^{k,p}$, the Sobolev space vanishing on the boundary, we get (1.2) by so-call Poincaré-Friedrichs inequality, and so the condition annihilating the set of polynomials $P_{k-1}$ is unnecessary in applying Bramble-Hilbert Lemma in $W_0^{k,p}$.

2. Notation and Preliminaries. Let $R$ with boundary $\partial R$ be a bounded domain in Euclidean $n$-space, $\mathbb{R}^n$. Let $\rho$ be the diameter of $R$. We shall assume that $R$ satisfies a strong cone property; that is, there exists a finite open covering $\{O_i\}, i = 1, \ldots, N$ of $\partial R$ and corresponding cones $\{C_i\}$ with vertices at the origin such that $x + C_i$ is contained in $R$ for any $x \in R \cap O_i$ (Figure 2.1). We shall consider complex-valued functions defined on $R$. As usual we define $L_p(R)$ as the set of all functions $u$ such that

$$\|f\|_{p,R} = \left( \frac{1}{\rho^n} \int_R |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

(2.1)

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exist and is finite, where $dx$ denotes Lebesgue measure. The above norm is equivalent to the ordinary usual norm $\|f\|_{L^p(R)} = \left( \int_R |f(x)|^p dx \right)^{\frac{1}{p}}$ since $\rho$ is finite and $R$ is bounded. We shall need the following seminorms:

$$|u|_{p,k,R} = \sum_{|\alpha| = k} \|D^\alpha u\|_{p,R}$$  \hspace{1cm} (2.2)

and

$$|u|_{\infty,k,R} = \sum_{|\alpha| = k} |D^\alpha u|_{\infty,R},$$  \hspace{1cm} (2.3)

where $|u|_{\infty,R} = \text{ess.sup}_{x \in R} |u(x)|$. The above semi norm (2.2) is a little different from the usual semi norm $\left( \sum_{|\alpha| = k} \int_R |D^\alpha u|^p dx \right)^{\frac{1}{p}}$, but is equivalent to the usual semi norm since

$$C \sum_{|\alpha| = k} \left( \int_R |D^\alpha u|^p dx \right)^{\frac{1}{p}} \leq \left( \sum_{|\alpha| = k} \int_R |D^\alpha u|^p dx \right)^{\frac{1}{p}} \leq \sum_{|\alpha| = k} \left( \int_R |D^\alpha u|^p dx \right)^{\frac{1}{p}},$$  \hspace{1cm} (2.4)

where $N_k$ is the number of multi-index $\alpha$ with $|\alpha| = k$, and $C$ is some constant less than or equal to $N_k^{\frac{1}{p} - 1}$. In (2.4), the first inequality is induced by Jensen's inequality, and the second inequality is easy to be shown by Theorem $4.1$. In (2.2) and the sequel, $\alpha$ is a multi-index:

$$\alpha = (\alpha_1, \ldots, \alpha_n) \text{ and } |\alpha| = \sum_{i=1}^n \alpha_i, \quad D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$  

Now we shall consider to introduce the Sobolev space and the weak derivative in [3]. Assume that $U \subset \mathbb{R}^n$ is open. Fix $1 \leq p \leq \infty$ and let $k$ be a nonnegative integer.

**Notation.**

(i) $C^k(U) = \{ u : U \to \mathbb{C} \mid u \text{ is } k\text{-times continuously differentiable } \}$

(ii) $C^\infty(U) = \{ u : U \to \mathbb{C} \mid u \text{ is infinitely differentiable } \} = \bigcap_{k=0}^{\infty} C^k(U)$
(iii) $C_c(U), C_c^k(U)$, etc. denote these functions in $C(U), C^k(U)$, etc. with compact support.
(iv) $L^p(U) = \{ u : U \to \mathbb{C} | u \text{ is Lebesgue measurable, } \|u\|_{L^p(U)} < \infty \} \quad (1 \leq p < \infty)$
(v) $L_{loc}^p(U) = \{ u : U \to \mathbb{C} | u \in L^p(V) \text{ for each } V \subset U \}$, where $V \subset U$ denotes that $V$ is compactly embedded in $U$.

We define the Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{C}$, that is, $u \in L^1_{loc}(U)$ such that for each multi index $\alpha$ with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

In this paper we take the norm on $W^{m,p}(R)$ to be

$$
\|u\|_{p,m,R}^p = \sum_{k=0}^{m} \rho^{kp}|u|_{p,k,R}^p. \quad (2.5)
$$

It is trivial that this is equivalent to the usual norm for $W^{m,p}(R)$.

We shall also consider the space of functions which have continuous derivatives of order up to and including $m$ in $R$; this space will be denoted by $C^m(R)$. For the purpose of this paper we take the norm on $C^m(R)$ to be:

$$
\|u\|_{\infty,m,R} = \sum_{k=0}^{m} \rho^k|u|_{\infty,k,R}. \quad (2.6)
$$

Again, the usual norm on $C^m(R)$ is equivalent to (2.6). We shall denote by $P_k$ the set of polynomials of degree less than to equal to $k$, restricted to $R$. Throughout this paper we shall use $C$ to denote a generic constant not necessarily the same in any two places.

3. Estimation of linear functionals. Let us consider $B$ a Banach space with norm $\| \cdot \|_B$ and let $B_1$ be a closed linear subspace of $B$. We define $Q$ by the quotient or factor space of $B$ with respect to $B_1$, denoted by $B/B_1$. The elements of $Q$ are equivalence classes $[u]$, where $[u]$ is the class containing $u$. The equivalence relation is given by $\sim$ where for $u, v \in B$, $u \sim v$ if and only if $u - v \in B_1$. The usual norm on $Q$ is given by $\| [u] \|_Q = \inf_{v \in B_1} \| u + v \|_B$. Under the assumptions we have made for $B$ and $B_1$, it is well known that $Q$ is a Banach space with norm $\| \cdot \|_Q$.

Now consider the (closed) finite-dimensional subspace of $W^{k,p}_p(R)$ given by $P_{k-1}$. (We know that a finite-dimensional subspace of a normed space is closed.) Here, $p(x) \in P_{k-1}$ if and only if $p(x) = \sum_{|\gamma| \leq k-1} a_\gamma x^\gamma$ for $x \in R$, where $a_\gamma$ are complex numbers and $\gamma$ is a multi-index.

**Theorem 3.1.** Let $Q = W^{k,p}(R)/P_{k-1}$. Then $\| u \|_{k,p,R}$ is a norm on $Q$ equivalent to $\| [u] \|_Q$. Further, there exist $C$ independent of $\rho$ and $u$ such that for any $u \in W^{k,p}(R)$

$$
\rho^k |u|_{k,p,R} \leq \| [u] \|_Q \leq C \rho^k |u|_{k,p,R}. \quad (3.1)
$$

We shall make use of two lemmas which can be found in Morrey [4], p.85. In this paper, we give each lemma its proof which is not cited from Morrey.

**Lemma 3.2.** For any $u \in W^{k,p}(R)$ there is a unique polynomial $p$ of degree less than or equal to $k-1$ (or 0 ) such that

$$
\int_R D^\alpha (u + p) = 0 \quad (3.2)
$$
for all \( \alpha \) with \( 0 \leq |\alpha| \leq k - 1 \).

Proof. of Lemma 3.2. Let \( p \) be an element in \( P_{k-1} \). Then \( p(x) \) is written by 
\[
\sum_{|\gamma| \leq k-1} a_{\gamma} x^{\gamma}
\]
for all \( x \in R \), where \( a_{\gamma} \) are complex numbers and \( \gamma \) is a multi-index.

Now, we shall show that \( p(x) \) satisfying (3.2) is unique in \( P_{k-1} \). Since \( u \in W^k_p(R) \), weak derivatives of order less than or equal to \( k \) of \( u \) satisfies 
\[
\int_R |D^\alpha u|^p < \infty
\]
for all \( \alpha \) with \( |\alpha| \leq k \). we note that each \( \int_R D^\alpha u \) is bounded since
\[
\int_R D^\alpha u \leq \int_R |D^\alpha u| \leq \int_R |D^\alpha u|^p.
\]
The above latter inequality is easily shown by Jensen inequality with using convex function \( x^p \) and bounded domain. Even though the above inequalities is not considered, weak derivative may be defined conventionally in a set of locally summable functions. We see that (3.2) means
\[
\int_R D^\alpha (p(x)) = -\int_R D^\alpha u,
\]
for all \( \alpha \) with \( 0 \leq |\alpha| \leq k - 1 \). Here, the left term of (3.3) stands for
\[
\int_R D^\alpha (p(x)) = \int_R D^\alpha \left( \sum_{|\gamma| \leq k-1} a_{\gamma} x^{\gamma} \right) = \sum_{|\gamma| \leq k-1} a_{\gamma} \int_R D^\alpha (x^{\gamma}).
\]
Hence, (3.3) is represented with the system of linear equations
\[
\begin{cases}
  a_{\gamma_1} \int_R D^{\alpha_1} (x^{\gamma_1}) + a_{\gamma_2} \int_R D^{\alpha_2} (x^{\gamma_2}) + \cdots + a_{\gamma_N} \int_R D^{\alpha_N} (x^{\gamma_N}) = -\int_R D^{\alpha_1} u \\
  a_{\gamma_1} \int_R D^{\alpha_2} (x^{\gamma_1}) + a_{\gamma_2} \int_R D^{\alpha_2} (x^{\gamma_2}) + \cdots + a_{\gamma_N} \int_R D^{\alpha_N} (x^{\gamma_N}) = -\int_R D^{\alpha_2} u \\
  \vdots \\
  a_{\gamma_1} \int_R D^{\alpha_N} (x^{\gamma_1}) + a_{\gamma_2} \int_R D^{\alpha_N} (x^{\gamma_2}) + \cdots + a_{\gamma_N} \int_R D^{\alpha_1} (x^{\gamma_N}) = -\int_R D^{\alpha_N} u
\end{cases}
\]
where \( \alpha_i \) and \( \gamma_i \) are attached to index \( i \) for \( \alpha \) and \( \gamma \), respectively, and \( N \) is the number of all multi indices \( \alpha \) with \( |\alpha| \leq k - 1 \). Here, we note that the number of all multi indices \( \alpha \) with \( |\alpha| \leq k - 1 \) is the same with the number of all multi indices \( \gamma \) with \( |\gamma| \leq k - 1 \), since \( \alpha \) and \( \gamma \) have the same dimension. We may assume that \( \alpha \) and \( \gamma \) are the same each other, and that \( \{\alpha_1, \alpha_2, \ldots, \alpha_N\} \) is arranged in the order that satisfies some entry of \( \alpha_i \) is larger than or equal to all entries of \( \alpha_{i-1} \), (i.e. there exist an entry \( \alpha_{i,j} \) in \( \alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n}) \) such that \( \alpha_{i,j} \geq \alpha_{i-1,k} \), \( \forall 1 \leq k \leq n \)).

Example of the case of dimension \( n = 2 \) and \( k - 1 = 2 \) illustrates
\[
\{\alpha_1 = (0,0), \alpha_2 = (1,0), \alpha_3 = (0,1), \alpha_4 = (1,1), \alpha_5 = (2,0), \alpha_6 = (0,2)\}.
\]
Then, we observe that
\[
D^{\alpha_i} (x^{\gamma_j}) = \begin{cases} 
\text{is constant,} & \text{if } i = j, \\
0, & \text{if } i > j,
\end{cases}
\]
because \( D^{\alpha_i} (x^{\gamma_j}) = D^{(\alpha_i - \alpha_j)} (D^{\alpha_j} x^{\gamma_j}) \), where \( \alpha_i - \alpha_j = (\alpha_{i,1} - \alpha_{j,1}, \alpha_{i,2} - \alpha_{j,2}, \ldots, \alpha_{i,n} - \alpha_{j,n}) \), and obviously \( \alpha_i - \alpha_j \) has some positive entry. Consequently, the coefficient matrix of (3.4) is
where C denotes a constant not necessary the same each other. (3.6) is in echelon form whose diagonal entries is not vanishing. Thus, (3.6) is invertible. In conclusion, (3.4) has the only solution. Thus, a polynomial p is uniquely determined in P_{k-1}. □

**Lemma 3.3.** Let R satisfy a strong cone condition. Then (since R is contained in a sphere of radius ρ)

\[ |u|_{j,p,R} \leq C \rho^{k-j} |u|_{k,p,R} \]  

(3.7)

for \(0 \leq j \leq k-1\) for all \(u \in W^{k,p}(R)\) such that the average over R of each \(\partial^\alpha u\) is 0 for \(0 \leq |\alpha| \leq k-1\), where C is a constant independent of ρ and u.

**Note.** Morrey assumes that his domain is strongly Lipschitz, but the proof is exactly the same if the domain satisfies a strong cone condition. Hence, we may assume that R satisfies strongly Lipschitz. Before proof of Lemma 3.3, we shall state Poincaré's inequality which is introduced in Evans [3], pp.275.

**Notation.**

(i) \((u)_U = \frac{1}{\text{meas}(U)} \int_U u. \) (i.e. \((u)_U\) means the average of u over U.)

(ii) \(Du\) denotes the gradient of u; that is, \(Du = \left[ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right]\)

**Theorem 3.4 (Poincaré's inequality).** Let U be a bounded, connected, open subset of \(\mathbb{R}^n\), with a \(C^1\) boundary \(\partial U\). Assume \(1 \leq p \leq \infty\). Then there exists a constant C, depending only on n, p and U, such that

\[ \|u - (u)_U\|_{L^p(U)} \leq C \|Du\|_{L^p(U)} \]  

(3.8)

for each function \(u \in W^{1,p}(U)\). In particular, there exists a constant C, depending only on n and p, such that

\[ \|u - (u)_U\|_{L^p(U)} \leq C \tau \|Du\|_{L^p(U)} \]  

(3.9)

for each function \(u \in W^{1,p}(B^0(x,r))\), where \(B^0(x,r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}\).

**Proposition 3.5.** Under the assumptions of Lemma 3.3 and Theorem 3.4, the above inequality (3.9) implies just the same as (3.7) in Lemma 3.3 for \(j = 0, k = 1\), in particular, if \(u \in W^{2,p}(U)\) then (3.9) implies just the same as (3.7) in Lemma 3.3 for \(j = 0, 1, k = 2\).
Proof. We observe at first that \( \|Du\|_{L^p(U)} \) is less than \( |u|_{p,1,U} \) because

\[
\|Du\|_{L^p(U)} = \left\| D\left( \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \right) \right\|_{L^p(U)} \leq \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|_{L^p(U)} \quad (\text{by Theorem 4.1 in the following section.})
\]

\[
= \sum_{|\alpha|=1} |D^{\alpha}u|_{L^p(U)} \quad (\text{by the notation of partial derivative } D^{\alpha}.)
\]

\[
\leq \sum_{|\alpha|=1} \|D^{\alpha}u\|_{L^p(U)} = |u|_{p,1,U} \quad (\text{by (2.2)})
\]

Thus, In Theorem 3.4, (3.9) implies

\[
\|u - (u)_U\|_{L^p(U)} \leq Cr|u|_{p,1,U}. \tag{3.10}
\]

Since \( \| \cdot \|_{L^p(U)} \) is equivalent to \( | \cdot |_{p,0,U} \) by (2.1) and (2.2), it follows that

\[
|u - (u)_U|_{p,0,U} \leq Cr|u|_{p,1,U}. \tag{3.11}
\]

Thus, the above inequality (3.11) is just the same as (3.7) in Lemma 3.3 for \( j = 0, k = 1 \) since \( (u)_U = 0 \) from assumption of Lemma 3.3.

Now, let us consider to suppose \( u \in W^{2,p}(U) \). Then \( D^{\alpha}u \in W^{1,p}(U) \) for some \( \alpha \) with \( |\alpha| = 1 \) ([3], p.247). We get from (3.10)

\[
\|D^{\alpha}u - (D^{\alpha}u)_U\|_{L^p(U)} \leq Cr|D^{\alpha}u|_{p,1,U}, \tag{3.12}
\]

and from assumption of Lemma 3.3 so that

\[
\|D^{\alpha}u\|_{L^p(U)} \leq Cr|D^{\alpha}u|_{p,1,U}. \tag{3.13}
\]

Taking a summation over \( |\alpha| = 1 \) for (3.13),

\[
\sum_{|\alpha|=1} \|D^{\alpha}u\|_{L^p(U)} \leq Cr \sum_{|\alpha|=1} |D^{\alpha}u|_{p,1,U}. \tag{3.14}
\]

The left side of the above (3.14) is equivalent to \( |u|_{p,1,U} \) by (2.1) and (2.2), and a part of the right side of (3.14) is

\[
\sum_{|\alpha|=1} |D^{\alpha}u|_{p,1,U} = \sum_{|\alpha|=1} \left( \sum_{|\beta|=1} \|D^{\beta}(D^{\alpha}u)\|_{p,U} \right) = \sum_{|\gamma|=2} \|D^{\gamma}u\|_{p,U} = |u|_{p,2,U}. \tag{3.15}
\]

Hence, we obtain that for \( u \in W^{2,p}(U) \) satisfying assumptions of Lemma 3.3 and Theorem 3.4,

\[
|u|_{p,1,U} \leq Cr|u|_{p,2,U}. \tag{3.16}
\]

With considering (3.10) and multiplying \( Cr \) in both sides of (3.16), consequently

\[
|u|_{p,2,U} \leq Cr|u|_{p,1,U} \leq C^2 r^2 |u|_{p,2,U}. \tag{3.17}
\]

This completes the proof. \( \Box \)
Thus we are motivated to verify that a generalization of Theorem 3.4 (Poincaré's inequality) implies exactly the Lemma 3.3. It is shown below.

Proof. of lemma 3.3. It is trivial that strongly Lipschitz condition of \( R \) satisfies \( C^1 \) boundary condition. Since \( u \in W^{k,p}(R) \), \( D^\alpha u \in W^{1,p}(R) \) for some \( \alpha \) with \( |\alpha| = j \). Hence we can apply (3.9) in Theorem 3.4 (Poincaré's inequality) since \( R \) is contained in a sphere of radius \( \rho \). From (3.10) and (2.1) and the assumption that the average over \( R \) of each \( D^\alpha u \) is 0,

\[
\|D^\alpha u\|_{p,R} \leq C \rho \|D^\alpha u\|_{p,1,U}. \tag{3.18}
\]

Taking a summation over all \( \alpha \) with \( |\alpha| = j \) for the above inequality,

\[
\sum_{|\alpha|=j} \|D^\alpha u\|_{p,R} \leq C \rho \sum_{|\alpha|=j} \|D^\alpha u\|_{p,1,U}. \tag{3.19}
\]

The left side of (3.19) is the sum \( |u|_{p,j,R} \) by (2.2), and a part of right side of (3.19) is

\[
\sum_{|\alpha|=j} \|D^\alpha u\|_{p,1,U} = \sum_{|\alpha|=j} \left( \sum_{|\beta|=1} \|D^\beta (D^\alpha u)\|_{p,U} \right) = \sum_{|\gamma|=j+1} \|D^\gamma u\|_{p,U} = |u|_{p,j+1,U}.
\]

Thus, we get

\[
|u|_{p,j,R} \leq C \rho |u|_{p,j+1,R}. \tag{3.20}
\]

Similarly, since \( D^\alpha u \in W^{n,1}(R) \) for each of \( \alpha \) with \( |\alpha| = j+1, \ldots, k-1 \) ([3], p.247), we obtain

\[
|u|_{p,j+1,R} \leq C \rho |u|_{p,j+2,R},
\]

\[
|u|_{p,j+2,R} \leq C \rho |u|_{p,j+3,R},
\]

\[
\vdots
\]

\[
|u|_{p,k-1,R} \leq C \rho |u|_{p,k,R}.
\]

Thus, in all,

\[
|u|_{p,j,R} \leq C \rho |u|_{p,j+1,R} \leq C^2 \rho^2 |u|_{p,j+2,R} \leq \ldots \leq C^{(k-j)} \rho^{(k-j)} |u|_{p,k,R}.
\]

This completes the proof. \( \square \)

Proof. of Theorem 3.1. we shall now prove the right hand inequality of (3.1) in Theorem 3.1. By Lemma 1 we can choose \( \bar{p} \in P_{k-1} \) such that \( \int_R D^\gamma (u + \bar{p}) = 0 \) for \( |\gamma| \leq k-1 \). Hence using Lemma 2 it follows that

\[
\|u + \bar{p}\|_{k,p,R} = \sum_{i=0}^k \rho^i |u + \bar{p}|_{p,i,R} \leq C \rho^k |u + \bar{p}|_{k,p,R} = C \rho^k |u|_{k,p,R}. \tag{3.22}
\]

In the above, the first inequality is derived by our notation, and the second inequality is shown by using Lemma 2, and the last equality holds since \( D^\alpha p = 0 \) for each \( |\alpha| = k \). However, since \( \bar{p} \in P_{k-1} \) we have that \( ||[u]|_Q \leq \|u + \bar{p}\|_{k,p,R} \). Hence \( ||[u]|_Q \leq C \rho^k |u|_{k,p,R} \) for each \( u \in W^{p,k}(R) \). The other inequality of (3.1) is easily
seen from the observation that $\rho^k|u+p|_{k,p,R} = \rho^k|u|_{k,p,R}$ for any $p \in P_{k-1}$ from which we immediately obtain
\[ \rho^k|u|_{k,p,R} \leq \inf_{p \in P_{k-1}} \|u + p\|_{k,p,R} = \|[u]\|_Q. \]
\[ \square \]

Now, the main result of this section is the following theorem.

**Theorem 3.6 (Bramble-Hilbert Lemma).** Let $F$ be a linear functional on $W^{p,k}(R)$ which satisfies

(i) $|F(u)| \leq C\|u\|_{k,p,R}$ for all $u \in W^{p,k}(R)$ with $C$ independent of $\rho$ and $u$

(ii) $F(p) = 0$ for all $p \in P_{k-1}$.

Then $|F(u)| \leq C_1 \rho^k|u|_{k,p,R}$ for any $u \in W^{p,k}(R)$ with $C_1$ independent of $\rho$ and $u$.

*Proof.* Since $F$ is linear and satisfies condition (ii),
\[ |F(u)| = |F(u + p)| \quad \text{for all } p \in P_{k-1}. \] (3.23)

By condition (i) and (3.23) we have
\[ |F(u)| \leq C\|u + p\|_{k,p,R}. \] (3.24)

Taking the infimum over $P_{k-1}$ in (3.24) we have
\[ |F(u)| \leq C\|[u]\|_Q. \] (3.25)

The result now follows from Theorem 3.1. \[ \square \]

4. Consideration on the preceding section. In the preceding section, we used the following theorem to prove the latter inequality about the equivalence of norm in (2.4). It seems to be analogous to Jensen’s inequality, but is quite different from the inequality. It is introduced as an exercise of [6], p.15.

**Notation** \( \mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\} \)

**Theorem 4.1.** Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a strictly convex function with $f(0) \leq 0$. Assume that $a_1, \ldots, a_n \geq 0$ and at least two $a_i$ are non-zero. Then
\[ \sum_{i=1}^{n} f(a_i) < f(\sum_{i=1}^{n} a_i). \]

*Proof.* Since $f$ is a strictly convex function on $\mathbb{R}^+$, $f$ is satisfied with the requirement that
\[ \frac{f(t) - f(s)}{t - s} < \frac{f(u) - f(t)}{u - t}, \] (4.1)
whenever $0 \leq s < t < u < \infty$ ([7], p.61). Furthermore, it follows from (4.1) that
\[ \frac{f(t) - f(s)}{t - s} < \frac{f(v) - f(u)}{v - u}, \] (4.2)
whenever $0 \leq s < t < u < v < \infty$.

It is sufficient to prove the statement for the just any two numbers in \( \{a_1, \ldots, a_n\} \). At first, it is trivial that for some non-zero $a$,
\[ f(a) + f(0) \leq f(a). \]
Next, we consider for two non-zero $a$ and $b$ in $\mathbb{R}^+$. By applying (4.2),
\[
\frac{f(a) - f(0)}{a} < \frac{f(a + b) - f(b)}{a}.
\]
(4.3)
We get
\[
f(a) - f(0) < f(a + b) - f(b).
\]
(4.4)
Since $f(0) \leq 0$,
\[
f(a) \leq f(a) - f(0) < f(a + b) - f(b).
\]
Thus, we obtain
\[
f(a) + f(b) < f(a + b).
\]
(4.5)
(We note that strictly inequality incurs here.) Since the preceding method is similarly used for $n$ elements $a_1, \ldots, a_n$, this completes the proof. \qed

We need to consider the following corollary to apply the case of concave function as like that $x^p$, $p > 1$.

**Corollary 4.2.** Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a strictly concave function with $f(0) \geq 0$. Assume that $a_1, \ldots, a_n \geq 0$ and at least two $a_i$ are non-zero. Then
\[
\sum_{i=1}^{n} f(a_i) > f\left(\sum_{i=1}^{n} a_i\right).
\]
\]
In the preceding section, let us observe the process of method of proof in from (3.18) to (3.22). Then we recognize that (3.18) is essentially important, that is, if we assume (3.18), then the conclusion to (3.22) is deduced naturally. So, we get the following corollary from this point of observation.

**Corollary 4.3.** Suppose that
\[
\|D^\alpha u\|_{p,U} \leq C\rho\|D^\alpha u\|_{p,1,U},
\]
(4.6)
for all $D^\alpha u \in W^{1,p}(U)$ with $|\alpha| = j$. Then
\[
|u|_{p,j,U} \leq C\rho|u|_{p,j+1,U}.
\]
In particular, supposing (4.6) for $j = 0, \ldots, k - 1$,
\[
|u|_{p,0,U} \leq C\rho|u|_{p,1,U} \leq C^2\rho^2|u|_{p,2,U} \leq \cdots \leq C^k\rho^k|u|_{p,k,U},
\]
Moreover,
\[
\|u\|_{p,k,U} \leq C\rho^k|u|_{p,k,U},
\]
So, $|\cdot|_{p,k,U}$ is norm-equivalent to $\|\cdot\|_{p,k,U}$. \qed

The following theorem is introduced as Poincaré-Friedrichs inequality in Braess [5], p.30. In this paper, however, its proof is not cited in [5], but almost analogous to the proof of Poincaré’s inequality in [3], p.275.
Notation \( W^{1,p}_0(U) = \{ u \in W^{1,p}(U) \mid u = 0 \text{ on } \partial U \} \)

**Theorem 4.4** (Poincaré-Friedrichs inequality). Let \( U \) be a bounded, connected, open subset of \( \mathbb{R}^n \), with a \( C^1 \) boundary \( \partial U \). Assume \( 1 \leq p \leq \infty \). Then there exists a constant \( C \), depending only on \( n, p \) and \( U \), such that
\[
\| u \|_{L^p(U)} \leq C \| Du \|_{L^p(U)}
\]
(4.7)
for each function \( u \in W^{1,p}_0(U) \). In particular, there exists a constant \( C \), depending only on \( n \) and \( p \), such that
\[
\| u \|_{L^p(B(x,r))} \leq C r \| Du \|_{L^p(B(x,r))}
\]
(4.8)
for each function \( u \in W^{1,p}(B^0(x,r)) \), where \( B^0(x,r) = \{ y \in \mathbb{R}^n \mid |x - y| < r \} \).

The proof of Theorem 4.4 only requires zero boundary conditions on a part of the boundary. It suffices that the function vanishes on a part of a set \( V \), where \( V \) is a set with positive measure (Braess [5], p.30). The following corollary is a generalization of Corollary 4.4 from this point of view.

**Notation** \( W^{1,p}_{0,0}(U) = \{ u \in W^{1,p}(U) \mid u = 0 \text{ on } V \} \), where \( V \) is a subset of \( U \) with \( m(V) > 0 \).

**Corollary 4.5** (Poincaré-Friedrichs inequality). Let \( U \) be a bounded, connected, open subset of \( \mathbb{R}^n \), with a \( C^1 \) boundary \( \partial U \). Assume \( 1 \leq p \leq \infty \). Then there exists a constant \( C \), depending only on \( n, p \) and \( U \), such that
\[
\| u \|_{L^p(U)} \leq C \| Du \|_{L^p(U)}
\]
(4.9)
for each function \( u \in W^{1,p}_{0,0}(U) \).

**Remark.** In \( W^{p,k}_{0,0}(R) \), the condition (ii) that \( F(p) = 0 \) for all \( p \in P_{k-1} \) of Theorem 3.6 (Bramble-Hilbert Lemma) becomes unnecessary. Thus we get the following theorem.

**Theorem 4.6.** Let \( F \) be a linear functional on \( W^{p,k}_{0,0}(R) \) which satisfies \( |F(u)| \leq C \| u \|_{k,p,R} \) for all \( u \in W^{p,k}_{0,0}(R) \) with \( C \) independent of \( \rho \) and \( u \). Then \( |F(u)| \leq C_1 \rho^k |u|_{k,p,R} \) for any \( u \in W^{p,k}_{0,0}(R) \) with \( C_1 \) independent of \( \rho \) and \( u \).

**Proof.** By Corollary 4.5, since \( D^\alpha u \in W^{1,p}_{0,0}(U) \) for all \( \alpha \) with \( |\alpha| = 0, \ldots, k - 1 \), we obtain
\[
\| D^\alpha u \|_{p,U} \leq C \| D^\alpha u \|_{p,1,U},
\]
from (3.10) and (2.1). This is just the assumption of Corollary 4.3. So, we obtain the conclusion of the corollary. This completes the proof.

**5. Conclusion and Discussion.** If we discuss Poincaré inequality in a Sobolev space nonvanishing on a set of measure of nonzero, the term of average and the degree of the Sobolev space are looked upon. In just the above space but vanishing on a set of measure of nonzero, it is not easy to remove the term of average and to raise the degree of the Sobolev space. Here, if the term of average is removed, then to raise the degree of the Sobolev space is obtained naturally. Otherwise, it is very difficult or impossible on the point of my observation without changing in itself. Bramble and Hilbert used the result of Morrey to remove the term of average in Poincaré inequality.
The result of Morrey is to make the term of average be zero by adding a polynomial of degree of less one than the Sobolev space in the outset. It is deduced originally from linearly independence of each term of a polynomial. If we consider linearly independence, we may discover a little application with Bramble-Hilbert Lemma, for example, to deal with trigonometric polynomial.

REFERENCES