A PRIORI ERROR ESTIMATES OF A DISCONTINUOUS GALERKIN METHOD FOR LINEAR SOBOLEV EQUATIONS

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ABSTRACT. A discontinuous Galerkin method with interior penalty terms is presented for linear Sobolev equation. On appropriate finite element spaces, we apply a symmetric interior penalty Galerkin method to formulate semidiscrete approximate solutions. To deal with a damping term \(\nabla \cdot (\nabla u_t)\) included in Sobolev equations, which is the distinct character compared to parabolic differential equations, we choose special test functions. A priori error estimate for the semidiscrete time scheme is analyzed and an optimal \(L^\infty(L^2)\) error estimation is derived.

1. INTRODUCTION

Discontinuous Galerkin methods using interior penalties have been used very widely for solving various types of differential equations, including computational fluid problems. By virtue of the potential of error control and mesh adaptation and the local mass conservation, DG methods are preferred over the standard Galerkin method.

Since Baker [4] firstly introduced the interior penalty method with nonconforming elements for elliptic equations, discontinuous Galerkin methods with interior penalties for elliptic and parabolic equations have been developed by several authors [1, 5, 14]. They generalized the Nitsche method in [6] which treated the Dirichlet boundary condition by introducing the penalty terms on the boundary.

New applications of discontinuous Galerkin methods with interior penalties to nonlinear parabolic equations are considered in [9, 10, 11]. The authors in [9, 10, 11] developed

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elementwise conservative DG methods and derived a priori and a posteriori error estimates in higher dimensions.

The purpose of this paper is to consider the discontinuous Galerkin approximations of Sobolev differential equations with one time derivative appearing in the highest space derivative term. Sobolev equations are used to study the consolidation of clay, heat conduction, homogeneous fluid flow in fissured material, shear in second order fluids and other physical models.

In [12, 13], the authors constructed semidiscrete DG approximations and fully discrete DG approximations and obtained the optimal $L^\infty(H^1)$ error estimates for the nonlinear Sobolev equations. In this paper we construct semidiscrete DG approximations and analyze a priori optimal $L^\infty(L^2)$ error estimates for the linear Sobolev equations. In section 2 we introduce a model problem and some assumptions. In section 3 several notations and preliminaries are described and discontinuous Galerkin semidiscrete scheme is formulated. Finally an optimal a priori $L^\infty(L^2)$ error estimate is analyzed in section 4.

2. Model problems and assumptions

Consider the following linear Sobolev equation

$$u_t - \nabla \cdot (\nabla u + \nabla u_t) = f(x, u) \quad \text{in } \Omega \times (0, T],$$

with the boundary condition

$$(\nabla u + \nabla u_t) \cdot n = 0 \quad \text{on } \partial \Omega \times (0, T],$$

and the initial condition

$$u(x, 0) = u_0(x) \quad \text{in } \Omega,$$

where $\Omega$ is a bounded convex domain in $\mathbb{R}^d$, $d = 2, 3$ and $n$ is the unit outward normal vector to $\partial \Omega$.

We assume that the following conditions are satisfied.

1. $f$ is uniformly Lipschitz continuous with respect to its second variable.
2. The model problem has a unique solution satisfying the following regularity conditions:

$$u \in L^\infty((0, T), H^s(\Omega)), \quad u_t \in L^2((0, T), H^s(\Omega))$$

for $s \geq 2$. 
3. Notations and discontinuous Galerkin approximations

Let $\mathcal{E}_h = \{ E_1, E_2, \cdots, E_{N_h} \}$ be a regular quasi-uniform subdivision of $\Omega$, where $E_j$ is a triangle or a quadrilateral if $d = 2$ and $E_j$ is a 3-simplex or 3-rectangle if $d = 3$. Let $h_j = \text{diam}(E_j)$ be the diameter of $E_j$ and $h = \max_{1 \leq j \leq N_h} h_j$. The regularity means that there exists a constant $\rho > 0$ such that each $E_j$ contains a ball of radius $\rho h_j$. The quasi-uniformity requirement is that there is a constant $\gamma > 0$ such that

$$\frac{h}{h_j} \leq \gamma, \quad j = 1, \cdots, N_h$$

These quasi-uniformity and regularity assumptions are required for driving error estimates in terms of the degree of polynomials.

We denote the set of all edges of the elements by $\{ e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h} \}$ where $e_k \subset \Omega$, for $1 \leq k \leq P_h$, $e_k \subset \partial \Omega$ for $P_h + 1 \leq k \leq M_h$. $n_k$ is the unit outward normal vector to $E_j$ if $e_k = \partial E_i \cap \partial E_j$ for $i < j$ and $1 \leq k \leq P_h$ and $n_k = n$, $P_h + 1 \leq k \leq M_h$.

For an $s \geq 0$ and a domain $E \subset \mathbb{R}^d$, the usual norm of Sobolev space $H^s(E)$ is denoted by $\| \cdot \|_{s,E}$, and the usual seminorm is denoted by $| \cdot |_{s,E}$. If $E = \Omega$ we write $\| \cdot \|_{s}, | \cdot |_{s}$ instead of $\| \cdot \|_{s,\Omega}, | \cdot |_{s,\Omega}$ and if $s = 0$ we use $\| \cdot \|$ instead of $\| \cdot \|_0$.

For $s \geq 0$ and a given subdivision $\mathcal{E}_h$, we define the following space

$$H^s(\mathcal{E}_h) = \{ v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1, \cdots, N_h \}.$$  

Now, for $\phi \in H^s(\mathcal{E}_h)$, $s > \frac{1}{2}$, we define the following average function $\{ \phi \}$ and jump function $[\phi]$,

$$\{ \phi \} = \frac{1}{2} (\phi|_{E_i})|_{e_k} + \frac{1}{2} (\phi|_{E_j})|_{e_k}, \quad \forall x \in e_k, \quad 1 \leq k \leq P_h$$

$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \quad \forall x \in e_k, \quad 1 \leq k \leq P_h,$$

where $e_k = \partial E_i \cap \partial E_j$, $i < j$.

The usual $L^2$ inner product, for the functions $\phi, \psi \in L^2(E)$, is denoted by $(\phi, \psi)_E$. If $E = \Omega$ we use $(\phi, \psi)$ instead of $(\cdot, \cdot)_{\Omega}$.

We define the following broken norms associated with $H^s(\mathcal{E}_h)$ for $s \geq 2$,

$$\| \phi \|^2_0 = \sum_{j=1}^{N_h} \| \phi \|^2_{0, E_j}$$

$$\| \phi \|^2_1 = \sum_{j=1}^{N_h} (\| \phi \|^2_{1, E_j} + h_j^2 \| \phi \|^2_{2, E_j}) + J''(\phi, \phi)$$

$$\| \phi \|^2_2 = \sum_{j=1}^{N_h} \| \phi \|^2_{2, E_j}$$
where \( J^\sigma(\phi, \psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \int_{e_k} [\phi][\psi] ds \) is an interior penalty term and \( \sigma \) is a discrete positive function that takes the constant value \( \sigma_k \) on the edge \( e_k \) and is bounded below by \( \sigma_0 > 0 \) and above by \( \sigma^* \).

Let \( r \) be a positive integer. The finite element space is taken by

\[
D_r(E_h) = \prod_{j=1}^{N_h} P_r(E_j)
\]

where \( P_r(E_j) \) denotes the set of all polynomials of total degree not greater than \( r \) on \( E_j \).

Throughout this paper the symbol \( C \) indicates a generic positive constant independent of \( h \) and is not necessarily the same in any two places. The following \( hp \) approximation properties are proved in [2, 3].

**Lemma 3.1.** Let \( E_j \in E_h \), and \( u \in H^s(E_j) \). There are a constant \( C \) independent of \( u \), \( r \) and \( h \), and \( \hat{u} \in P_r(E_j) \) such that for any \( 0 \leq q \leq s \),

\[
\|u - \hat{u}\|_{q, E_j} \leq C h^{\mu - q} \gamma_j^{q} \|u\|_{s, E_j} \quad s \geq 0
\]

\[
\|u - \hat{u}\|_{0, e_j} \leq C h^{\mu - 1/2} \gamma_j^{1/2} \|u\|_{s, E_j} \quad s > \frac{1}{2}
\]

\[
\|u - \hat{u}\|_{1, e_j} \leq C h^{\mu - 3/2} \gamma_j^{3/2} \|u\|_{s, E_j} \quad s > \frac{3}{2}
\]

where \( \mu = \min(r + 1, s) \), and \( e_j \) is an edge or a face of \( E_j \).

The following Lemma states the trace inequalities whose proofs are given in [1].

**Lemma 3.2.** For each \( E_j \in E_h \), there exists a positive constant \( C \) depending only on \( \gamma \) and \( \rho \) such that the following two trace inequalities hold:

\[
\|\phi\|_{0, e_j} \leq C \left( \frac{1}{h_j} |\phi|_{0, E_j}^2 + h_j |\phi|_{1, E_j}^2 \right), \forall \phi \in H^1(E_j),
\]

\[
\|\nabla \phi \cdot n_j\|_{0, e_j} \leq C \left( \frac{1}{h_j} |\phi|_{1, E_j}^2 + h_j |\phi|_{2, E_j}^2 \right), \forall \phi \in H^2(E_j),
\]

where \( e_j \) is an edge or a face of \( E_j \) and \( n_j \) is the unit outward normal vector to \( e_j \).

We define a bilinear functional \( A \) on \( H^2(E_h) \times H^2(E_h) \) by

\[
A(\phi, \psi) = \sum_{k=1}^{N_h} (\nabla \phi, \nabla \psi)_{E_k} - \sum_{k=1}^{P_h} \int_{e_k} \{\nabla \phi \cdot n_k\}[\psi] ds - \sum_{k=1}^{P_h} \int_{e_k} \{\nabla \psi \cdot n_k\}[\phi] ds + J^\sigma(\phi, \psi).
\]
From (2.1), $u$ satisfies the following weak formulation

$$
(u_t, v) + A(u, v) + A(u_t, v) = (f(u), v), \forall v \in H^s(\mathcal{E}_h).
$$

(3.1)

Now we formulate a semidiscrete DG approximation to (3.1) as follows: Find $U(\cdot, t) \in D_r(\mathcal{E}_h)$ satisfying

$$
\begin{aligned}
(U_0, v) + A(U, v) + A(U_1, v) &= (f(U), v), \forall D_r(\mathcal{E}_h), \\
U(\cdot, 0) &= U_0
\end{aligned}
$$

(3.2)

where $U_0$ is an appropriate projection of the initial condition $u_0(x)$ onto $D_r(\mathcal{E}_h)$. For example, we can choose $U_0$ as $\tilde{u}(x, 0)$ to be defined later.

Define $A_{\lambda}(\phi, \psi) = A(\phi, \psi) + \lambda(\phi, \psi)$, with $\lambda > 0$. Then we obtain the following lemmas which can be proved easily by using Lemma 3.2 and the definition of $\| \cdot \|_1$.

**Lemma 3.3.** For $\lambda > 0$, there exists a constant $C$ independent of $h$ satisfying

$$
|A_{\lambda}(\phi, \psi)| \leq C\|\phi\|_1\|\psi\|_1, \forall \phi, \psi \in H^2(\mathcal{E}_h).
$$

**Lemma 3.4.** For a sufficiently large $\sigma$ and $\lambda > 0$, there exists a positive constant $\beta$ such that

$$
A_{\lambda}(v, v) \geq \beta\|v\|_1^2, \forall v \in D_r(\mathcal{E}_h).
$$

**Proof.** For an arbitrary small constant $\delta > 0$, we have, by Lemma 3.2

$$
A_{\lambda}(v, v) = \sum_{j=1}^{N_h} \langle \nabla v, \nabla v \rangle_{E_j} - 2 \sum_{k=1}^{P_h} \int_{e_k} \{\nabla v \cdot n_k\} [v] + J^{\sigma}(v, v) + \lambda(v, v)
$$

\[
\geq \|\nabla v\|_0^2 - \delta \sum_{k=1}^{P_h} [e_k]\|\{\nabla v\}\|_{E_j}^2 \|\nabla v\|_{E_j} + \delta^{-1} \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \|\{v\}\|_{E_j}^2 \|\nabla v\|_{E_j} + J^{\sigma}(v, v) + \lambda\|v\|^2 \\
\geq \|\nabla v\|_0^2 - C\delta \sum_{j=1}^{N_h} h_j(h_j^{-1}\|\nabla v\|_{E_j}^2 + h_j\|\nabla v\|_{E_j}^2 - \frac{\delta^{-1}}{\sigma_0}) + \left(1 - \frac{\delta^{-1}}{\sigma_0}\right) J^{\sigma}(v, v) \\
+ \lambda\|v\|^2 \\
\geq \|\nabla v\|_0^2 - C\delta\|\nabla v\|_0^2 + \left(1 - \frac{\delta^{-1}}{\sigma_0}\right) J^{\sigma}(v, v) + \lambda\|v\|^2 \\
= \left(\frac{1}{2} - C\delta\right)\|\nabla v\|_0^2 + \frac{1}{2}\|\nabla v\|_0^2 + \left(1 - \frac{c\delta^{-1}}{\sigma_0}\right) J^{\sigma}(v, v) + \lambda\|v\|^2 \\
\geq \left(\frac{1}{2} - C\delta\right)\|\nabla v\|_0^2 + C\sum_{j=1}^{N_h} h_j^2\|\nabla v\|_{E_j}^2 + \left(1 - \frac{C\delta^{-1}}{\sigma_0}\right) J^{\sigma}(v, v) + \lambda\|v\| \\
\geq \beta\|v\|_1^2.
\]
By Lemma 3.3 and Lemma 3.4, if $\lambda > 0$ there exists $\tilde{u} \in D_r(E_h)$ satisfying
$$A_\lambda(u - \tilde{u}, \chi) = 0,$$
$\forall \chi \in D_r(E_h)$.

Now we state the following Lemma which is essential for the proof of the optimal convergence of the semidiscrete approximation in the norm $L^\infty(L^2)$. The proof can be found in [8].

**Lemma 3.5.** For $\lambda \geq 0$, we let $t \in [0, T]$ be fixed and suppose that there exists $\phi \in H^2(E_h)$ satisfying
$$A_\lambda(\phi, v) = F(v), \quad \forall v \in D_r(E_h),$$
where $F : H^2(E_h) \to \mathbb{R}$ is a linear map. If there exist $M_1, M_2 > 0$ satisfying
$$|F(\psi)| \leq M_1\|\psi\|_1, \quad \psi \in H^2(E_h)$$
$$|F(\psi)| \leq M_2\|\psi\|_2, \quad \psi \in H^2(\Omega) \cap H^1_0(\Omega),$$
then we have the following estimation
$$\|\phi\| \leq C(\|\phi\|_1 + M_1)h + M_2.$$

**Proof.** For $\phi \in L^2(\Omega)$, let $\psi \in H^2(\Omega) \cap H^1_0(\Omega)$ be the solution of an elliptic problem
$$-\Delta \psi + \lambda \psi = \phi. \tag{3.3}$$
From the regularity property of the elliptic problem, then we have
$$\|\psi\|_2 \leq C\|\phi\|_1.$$
Let $\psi_I$ be the interpolant of $\psi$ such that $\|\psi - \psi_I\|_1 \leq Ch\|\psi\|_2$. Then from (3.3) and the assumptions we get the following inequalities
$$\|\phi\|^2 = (\phi, \phi) = (\phi, -\Delta \psi + \lambda \psi) = A_\lambda(\phi, \psi)$$
$$= A_\lambda(\phi, \psi - \psi_I) + A_\lambda(\phi, \psi_I)$$
$$\leq C\|\phi\|_1\|\psi - \psi_I\|_1 + F(\psi_I)$$
$$\leq C\|\phi\|_2\|\psi\|_2 + F(\psi) - F(\psi - \psi_I)$$
$$\leq Ch\|\phi\|_1\|\psi\|_2 + M_2\|\psi\|_2 + M_1\|\psi - \psi_I\|_1$$
$$\leq Ch\|\phi\|_1\|\psi\|_2 + M_2\|\psi\|_2 + ChM_1\|\psi\|_2$$
$$\leq C(h\|\phi\|_1\|\phi\| + M_2\|\phi\| + M_1h\|\phi\|).$$
Therefore we get
$$\|\phi\| \leq C[(\|\phi\|_1 + M_1)h + M_2].$$
4. Optimal $L^\infty (L^2)$ Error Estimate

Now, to prove the $L^\infty (L^2)$ optimal convergence of $u - U$, we denote

$$\eta = u - \tilde{u}, \quad \theta = \tilde{u} - \hat{u}, \quad \xi = \tilde{u} - U, \quad e = u - U.$$  \hspace{1cm} (4.1)

**Theorem 4.1.** For $r, s \geq 2$, there exists a constant $C$ independent of $h$ satisfying the following statements:

(i) $\|u - \tilde{u}\|_1 \leq C \frac{h^\mu}{r^{s-2}} \|u\|_s$

(ii) $\|u - \tilde{u}\| \leq C \frac{h^\mu}{r^{s-2}} \|u\|_s$

(iii) $\|u_t - \tilde{u}_t\|_1 \leq C \frac{h^{\mu-1}}{r^{s-2}} \|u_t\|_s$

(iv) $\|u_t - \tilde{u}_t\| \leq C \frac{h^{\mu}}{r^{s-2}} \|u_t\|_s$

**Proof.** From Lemma 3.3 and Lemma 3.4, we get

\[
\|\theta\|_1^2 \leq C A_\lambda(\theta, \theta) = C A_\lambda(u - \tilde{u}, \theta) \leq C \|u - \tilde{u}\|_1 \|\theta\|_1
\]

from which we get

\[
\|\theta\|_1 \leq C \|u - \tilde{u}\|_1. \hspace{1cm} (4.2)
\]

By the definition of $\| \cdot \|_1$, Lemma 3.1 and Lemma 3.2, we have

\[
\|u - \hat{u}\|_1^2 = \sum_{j=1}^{N_h} \left( \|u - \hat{u}\|_{1,E_j}^2 + h^2 \|u - \hat{u}\|_{2,E_j}^2 \right) + J^\sigma(u - \hat{u}, u - \tilde{u})
\]

\[
\leq \sum_{j=1}^{N_h} C \left( \frac{h^{2(\mu-1)}}{r^{2(s-1)}} \|u\|_{s,E_j}^2 + \frac{h^2}{r^{2(s-2)}} \|u\|_{s,E_j}^2 \right) + \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \int_{e_k} [u - \hat{u}]^2 \, ds
\]

\[
\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 + \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \|u - \hat{u}\|_{0,e_k}^2
\]

\[
\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 + C \sum_{j=1}^{N_h} h_j^{-1} \left( h_j^{-1} \|u - \hat{u}\|_{0,E_j}^2 + h_j \|\nabla(u - \tilde{u})\|_{0,E_j}^2 \right)
\]

\[
\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 + C \sum_{j=1}^{N_h} h_j^{-2} \left( \frac{h_j^{2\mu}}{r^{2s}} + \frac{h_j^2}{r^{2(s-1)}} \right) \|u\|_{s,E_j}^2
\]

\[
\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2,
\]
which implies
\[ \| u - \hat{u} \|_1 \leq C \frac{h^{\mu - 1}}{r^{s-2}} \| u \|_s. \]

From the triangle inequality, (4.2) and (4.3), we obtain
\[ \| u - \hat{u} \|_1 \leq \| u - \hat{u} \|_1 + \| \hat{u} - \tilde{u} \|_1 \leq C \| u - \hat{u} \|_1 \leq C \frac{h^{\mu - 1}}{r^{s-2}} \| u \|_s, \]
which proves (i).

By applying the result of Lemma 3.5 with \( M_1 = M_2 = 0 \), we get the statement (ii) as follows
\[ \| u - \hat{u} \| \leq C \| u - \hat{u} \|_1 h \leq C \frac{h^{\mu}}{r^{s-2}} \| u \|_s. \]

Differentiating \( A_\lambda (\eta, v) = 0 \) with respect to \( t \), we get
\[ \sum_{j=1}^{N_h} (\nabla \eta_t, \nabla v)_E_j - \sum_{k=1}^{P_h} \int_{e_k} \{ \nabla \eta_t \cdot n_k \} [v] - \sum_{k=1}^{P_h} \int_{e_k} \{ \nabla v \cdot n_k \} [\eta_t] + J^\sigma (\eta_t, v) + \lambda(\eta_t, v) = 0 \]
which implies
\[ A_\lambda (\eta_t, v) = 0. \]

By applying Lemma 3.5 with \( M_1 = M_2 = 0 \), we get
\[ \| \eta_t \| \leq ch \| \eta_t \|_1. \]

From the definition of \( \eta \) and \( \theta \), we separate \( \eta_t \) into
\[ \| \eta_t \|_1 \leq \| \theta_t \|_1 + \| u_t - \hat{u}_t \|_1. \]

The results of Lemma 3.3 and Lemma 3.4 imply that
\[ \| \theta_t \|_1 \leq CA_\lambda (\theta, \theta_t) = CA_\lambda (u_t - \hat{u}_t, \theta_t) - CA_\lambda (\eta_t, \theta_t) \]
\[ = CA_\lambda (u_t - \hat{u}_t, \theta_t) \leq C \| u_t - \hat{u}_t \|_1 \| \theta_t \|_1. \]

Therefore we get
\[ \| \theta_t \|_1 \leq C \| u_t - \hat{u}_t \|_1, \]
\[ \| \eta_t \|_1 \leq C \| u_t - \hat{u}_t \|_1. \]

By applying Lemma 3.1 and Lemma 3.2, we get
\[ \| u_t - \hat{u}_t \|_1^2 \leq C \sum_{j=1}^{N_h} \left( \frac{h^{2(\mu - 1)}}{r^{2(s-1)}} \| u_t \|_{s,E_j}^2 + h_j^2 \frac{h^{2(\mu - 2)}}{r^{2(s-2)}} \| u_t \|_{s,E_j}^2 \right) + C \sum_{k=1}^{P_h} |c_k|^{-1} \| u_t - \hat{u}_t \|_{0,e_k}^2 \]
\[ \leq C \sum_{j=1}^{N_h} \frac{h^{2(\mu - 1)}}{r^{2(s-2)}} \| u_t \|_{s,E_j}^2 + C \sum_{j=1}^{N_h} h_j^2 \left( \| u_t - \hat{u}_t \|_{0,E_j}^2 + h_j^2 \| \nabla (u_t - \hat{u}_t) \|_{0,E_j}^2 \right) \]
\[ \leq C \sum_{j=1}^{N_h} \frac{h^{2(\mu - 1)}}{r^{2(s-2)}} \| u_t \|_{s,E_j}^2, \]
which implies
\[
\| u_t - \hat{u}_t \|_1 \leq C h^{\mu-1} s^{-2} \| u_t \|_s
\]
and
\[
\| \eta_t \|_1 \leq C h^{\mu-1} s^{-2} \| u_t \|_s.
\]

**Theorem 4.2.** If \( \lambda > 0 \) is sufficiently small, then there exists a constant \( C \) independent of \( h \) satisfying the followings:

1. \( \| u - U \|_{L^\infty(L^2)} \leq C h^{\mu} \left( \| u \|_{L^\infty(H^s)} + \| u_t \|_{L^2(H^s)} \right) \)
2. \( \| u - U \|_{L^2(H^s)} \leq C h^{\mu-1} \left( \| u \|_{L^2(H^s)} + \| u_t \|_{L^2(H^s)} \right) \)
3. \( \| u_t - U_t \|_{L^2(L^2)} \leq C h^{\mu} \left( \| u \|_{L^2(H^s)} + \| u_t \|_{L^2(H^s)} \right) \)
4. \( \| u_t - U_t \|_{L^2(\mathcal{E}_h)} \leq C h^{\mu-1} \left( \| u \|_{L^2(H^s)} + \| u_t \|_{L^2(H^s)} \right) \)

**Proof.** From the notation (4.1), we have \( e = \eta + \xi \). By subtracting (3.2) from (3.1), we have
\[
(e_t, v) + A(e, v) + A(e_t, v) = (f(u) - f(U), v), \quad \forall v \in D_r(\mathcal{E}_h).
\]

By applying the definition of \( A_\lambda \), we get
\[
(e_t, v) + A_\lambda(e, v) + A_\lambda(e_t, v) = (f(u) - f(U), v) + \lambda(e, v) + \lambda(e_t, v), \quad \forall v \in D_r(\mathcal{E}_h).
\]

From the equation above, we can deduce
\[
(\xi_t, v) + A_\lambda(\xi, v) + A_\lambda(\xi_t, v) = - (\eta_t, v) - A_\lambda(\eta, v) - A_\lambda(\eta_t, v) + (f(u) - f(U), v) + \lambda(u - U, v)
+ \lambda(u_t - U_t, v)
= - (\eta_t, v) + (f(u) - f(U), v) + \lambda(u - U, v) + \lambda(u_t - U_t, v)
= - (\eta_t, v) + (f(u) - f(U), v) + \lambda(\xi, v) + \lambda(\eta, v) + \lambda(\xi_t, v) + \lambda(\eta_t, v) + \lambda(\xi_t, v).
\]

Now we choose \( v = \xi + \xi_t \) in (4.4), to get
\[
\| \xi_t \|^2 + \frac{1}{2} \frac{d}{dt} (\xi, \xi) + A_\lambda(\xi, \xi) + 2 A_\lambda(\xi, \xi_t) + A_\lambda(\xi_t, \xi_t)
= - (\eta_t, \xi) + (f(u) - f(U), \xi + \xi_t) + \lambda(\xi, \xi + \xi_t) + \lambda(\eta, \xi + \xi_t)
+ \lambda(\xi_t, \xi + \xi_t) + \lambda(\eta_t, \xi + \xi_t),
\]
which yields the following inequality
\[
\|\xi_t\|^2 + \frac{1}{2} \frac{d}{dt} (\xi, \xi) + A_\lambda(\xi, \xi) + A_\lambda(\xi_t, \xi_t) + 2 \left[ \sum_{k=1}^{N_h} (\nabla \xi, \nabla \xi_t) E_k \right] \\
- \sum_{k=1}^{P_h} \int_{e_k} \{ \nabla \xi \cdot n_k \} [\xi_t] - \sum_{k=1}^{P_h} \int_{e_k} \{ \nabla \xi_t \cdot n_k \} [\xi] + J^\sigma(\xi, \xi_t) + \lambda(\xi, \xi_t)
\]
\[
\leq (1 + \lambda)\|\eta\|\|\xi\| + (1 + \lambda)\|\eta\|\|\xi_t\| + K(\|\eta\| + \|\xi\|)(\|\xi\| + \|\xi_t\|) + \lambda\|\xi\|^2 \\
+ 2\lambda(\xi, \xi_t) + \lambda\|\eta\|\|\xi\| + \lambda|\xi_t|^2 + \lambda\|\eta\|\|\xi_t\|.
\]

If \( \lambda \) is sufficiently small, we obtain the following inequality
\[
\|\xi_t\|^2 + A_\lambda(\xi, \xi) + A_\lambda(\xi_t, \xi_t) + \frac{1}{2} \frac{d}{dt} \left[ ((1 + \lambda)\xi, \xi) + 2 \sum_{k=1}^{N_h} (\nabla \xi, \nabla \xi) E_k + 2J^\sigma(\xi, \xi) \right] \\
\leq C \left( \|\eta\|^2 + \|\eta\|^2 + \|\xi\|^2 \right) + \varepsilon \|\xi_t\|^2 + 2 \left( \sum_{k=1}^{P_h} \int_{e_k} \{ \nabla \xi \cdot n_k \} [\xi] + \sum_{k=1}^{P_h} \int_{e_k} \{ \nabla \xi_t \cdot n_k \} [\xi] \right).
\]

By the definition of \( J \) and Lemma 3.2, we can find that
\[
\|\xi_t\|^2 + A_\lambda(\xi, \xi) + A_\lambda(\xi_t, \xi_t) + \frac{1}{2} \frac{d}{dt} \left[ (\xi, \xi) + 2 \sum_{k=1}^{N_h} (\nabla \xi, \nabla \xi) E_k + 2J^\sigma(\xi, \xi) \right] \\
\leq C \left( \|\eta\|^2 + \|\eta\|^2 + \|\xi\|^2 \right) + C\|\nabla \xi\|_{0}^2 + \varepsilon J^\sigma(\xi_t, \xi_t) + C J^\sigma(\xi, \xi) + \varepsilon \|\nabla \xi\|_{0}^2,
\]
which implies that
\[
\|\xi_t\|^2 + \|\xi\|^2 + \|\xi_t\|^2 + \frac{1}{2} \frac{d}{dt} \left[ \|\xi\|^2 + \|\nabla \xi\|_{0}^2 + J^\sigma(\xi, \xi) \right] \\
\leq C \left( \|\eta\|^2 + \|\eta\|^2 + \|\xi\|^2 + \|\nabla \xi\|_{0}^2 + J^\sigma(\xi, \xi) \right). \tag{4.5}
\]

By integrating (4.5) from \( t = 0 \) to \( t = \tau \), we have
\[
\|\xi\|^2(\tau) + \|\nabla \xi\|_{0}^2(\tau) + J^\sigma(\xi, \xi)(\tau) + \int_{0}^{\tau} \left( \|\xi_t\|^2 + \|\xi\|^2 + \|\xi_t\|^2 \right) dt
\]
\[
\leq \|\xi\|^2(0) + \|\nabla \xi\|^2(0) + J^\sigma(\xi, \xi)(0) + C \int_{0}^{\tau} \left( \|\xi\|^2 + \|\nabla \xi\|_{0}^2 + J^\sigma(\xi, \xi) \right) dt \\
+ C \int_{0}^{\tau} (\|\eta\|^2 + \|\eta\|^2) dt.
\]
Gronwall’s Lemma and the approximation results from Theorem 4.1 imply that

\[
\|\xi\|^2(\tau) + \|\nabla \xi\|^2_0(\tau) + J^s(\xi, \xi)(\tau) + \int_0^\tau \|\xi_t\|^2 + \|\xi_t\|_1^2 + \|\xi_t\|_2^2) dt
\]

\[
\leq C \int_0^\tau (\|\eta_t\|^2 + \|\eta\|^2) dt
\]

\[
\leq C \frac{h_{2\mu}}{r^{2(s-2)}} (\|\eta\|_{L^2(H^s)}^2 + \|\eta_t\|_{L^2(H^s)}^2).
\]

From (4.6), we get the following approximations

\[
\|\xi\|_{L^\infty(L^2)} \leq C \frac{h_{\mu}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)})
\]

\[
\|\xi\|_{L^2(\|\cdot\|_1)} \leq C \frac{h_{\mu}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)}).
\]

Using the inequality (4.6) again, we have

\[
\|\xi_t\|_{L^2(L^2)} \leq C \frac{h_{\mu}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)})
\]

\[
\|\xi_t\|_{L^2(\|\cdot\|_1)} \leq C \frac{h_{\mu}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)}).
\]

Therefore by the triangle inequality and Theorem 4.1, we obtain the statements (i) and (ii) as follows:

\[
\|e\|_{L^\infty(L^2)} \leq \|\eta\|_{L^\infty(L^2)} + \|\xi\|_{L^\infty(L^2)}
\]

\[
\leq C \frac{h_{\mu}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)})
\]

and

\[
\|e\|_{L^2(\|\cdot\|_1)} \leq \|\eta\|_{L^2(\|\cdot\|_1)} + \|\xi\|_{L^2(\|\cdot\|_1)}
\]

\[
\leq C \frac{h_{\mu-1}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)}).
\]

Again applying the triangle inequality and Theorem 4.1, we prove the statements (iii) and (iv) as follows

\[
\|e_t\|_{L^2(L^2)} \leq \|\eta_t\|_{L^2(L^2)} + \|\xi_t\|_{L^2(L^2)}
\]

\[
\leq C \frac{h_{\mu}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)})
\]

and

\[
\|e_t\|_{L^2(\|\cdot\|_1)} \leq \|\eta_t\|_{L^2(\|\cdot\|_1)} + \|\xi_t\|_{L^2(\|\cdot\|_1)}
\]

\[
\leq C \frac{h_{\mu-1}}{r^{s-2}} (\|\eta\|_{L^2(H^s)} + \|\eta_t\|_{L^2(H^s)}).
\]
REFERENCES


EPIDEMIOLOGICAL APPROACH TO THE SOUTH KOREAN BEEF PROTESTS WITH HIDDEN AGENDA

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\textbf{Abstract.} Hundreds of thousands of South Korean protesters staged candlelight vigils and demonstrations against US beef imports in 2008. The problems, however, went far beyond that of beef imports. The political party veterans, who lost the presidential election, exploited labor unions that were discontent with the economy and ideological student groups to weaken the majority party. In this study, an epidemiological model is constructed with a system of three nonlinear differential equations. The model seeks to examine the dynamics of the system through stability analysis. Two threshold conditions that spread the protests are identified and a sensitivity analysis on the conditions is performed to isolate the parameters to which the system is most responsive. The results are also explored by deterministic simulations. This model can be easily modified to apply to other protests that may occur in various circumstances.

1. INTRODUCTION

South Korea ("Korea") banned most U.S. beef imports in 2003 due to fears of mad cow disease (BSE) \cite{1} after two BSE-infected cows were identified, one born in U.S. and one in Canada \cite{2}. U.S. lawmakers had pressed the Korean government to lift the restrictions on beef imports before approving a free trade agreement between two nations.

The new Korean president who was inaugurated in February 2008 agreed to relax the restrictions on beef imports from the U.S. in April. The Japanese government soon followed Korea on relaxing regulations on U.S. beef imports and Koreans realized that their deal was not as effective as the Japanese one. For example, the Korean government allowed the imports of beef from U.S. cattle 30 months old or younger, but the Japanese government limited the age of cattle to 20 months or younger, which is thought to be potentially less at risk from the disease. Extensive media reporting on BSE and American beef imports of which some were exaggerated and not even supported by science created paranoia surrounding infected U.S. beef \cite{3}.
The first candlelight vigil concerning relaxed regulations was on May 2 and thousands of protesters rallied almost nightly for more than a month. Demonstrations continued even with scientific facts and international assurance of safety of U.S. beef. As protesters became aggressive and violent, it was clear that the problems went far beyond that of beef imports and the Korean police began to arrest some protesters.

The new president had been elected with the biggest margin of victory in decades. Although the citizens in Korea had high expectations from the new president, soaring gas prices in April and the global credit crisis began affecting Korea resulting in significant job losses. Many Koreans viewed the deal of relaxed regulations on beef imports as an embarrassing concession to the US and thought that the new president compromised public health standards to improve Korea - U.S. relations. The beef deal triggered public discontent with the worsening economy and political turmoil and provided outlet to express doubts on the governing style of the new administration. The political party veterans, who lost the presidential election, exploited labor unions that were discontent with the economy and ideological student groups to weaken the majority party. Hundreds of thousands of Koreans participated in candlelight vigils and demonstrations; it was an epidemic.

In this study, we take a modeling approach to the situation and construct an epidemiological model with ordinary differential equations to examine the dynamics of the protests. [4] and [5] modeled social phenomena using contact interactions and the theory of random nets, respectively. We assume that during the events of all of the beef protesting, individuals are encouraged and persuaded to participate in candlelight vigils and demonstrations through peer pressure and the media similar to epidemics in a population. Through the stability analysis we identify threshold conditions that describe the different scenarios for the protests. We perform a sensitivity analysis on the threshold conditions to isolate the parameters to which our system is most responsive. Our system produces a backward bifurcation. We explore our results through deterministic simulations.

2. Model

Our model consists of three classes, susceptible (S), demonstrating (D), and leading (L). Individuals in D participate in candlelight vigils and demonstrations, but they do not lead a protest, whereas individuals in L organize protests and recruit L members from the D class in addition to participating in protests. The total population is \( N = S + D + L \). Since the system is considered for only one hundred days, we assume that there is no birth or death coming into or leaving the system. Individuals move among the classes by a number of different processes. The parameter \( \alpha \) is the peer driven recruitment rate of S into D by individuals in D and L. Individuals in L are \( \xi \) times as effective in recruiting susceptible as individuals in D. The linear term \( \phi \) represents the role of the secondary source that moves individuals in D to S such as reports on the safety of American beef from international news programs, no more time to protest, or being arrested among others. The primary contact in the transition of demonstrators to the susceptible class is via the \( \delta \) term. Demonstrators transit to the L class at rate \( \gamma DL/N \). \( \eta \) is the rate at which individuals in L moving back to S. Individuals in L may have distinct
reasons to stage protests or distinct goals to achieve by organizing protests. Hence, some may move back to the S class when they obtained what they desired, and some may get arrested.

For sociological reasons, we assume that \( \delta < \alpha \) and \( \phi > \eta \). Our unit time is a day. These are summarized in schematic form in Figure 1.

\[
\begin{align*}
S & \xrightarrow{\alpha S/N (D + \xi L)} D \\
& \xleftarrow{\phi D + \delta D/N S} \\
D & \xrightarrow{\gamma D/L} L \\
& \xleftarrow{\eta L}
\end{align*}
\]

**Figure 1.** A schematic diagram of the model

Under these assumptions the governing compartmental model is the following system.

\[
\begin{align*}
\frac{dS}{dt} &= -\alpha \frac{S}{N} (D + \xi L) + \phi D + \delta \frac{D}{N} S + \eta L \\
\frac{dD}{dt} &= \alpha \frac{S}{N} (D + \xi L) - \phi D - \delta \frac{D}{N} S - \gamma \frac{D}{N} L \\
\frac{dL}{dt} &= \gamma \frac{D}{N} L - \eta L \\
N &= S + N + L
\end{align*}
\]  

(2.1)

3. Analysis

3.1. Rescaling the System of Equations. We define our variables as proportions of the entire population by letting \( s = S/N \), \( d = D/N \), and \( l = L/N \). Then the last equation of the system (1) provides \( d = 1 - s - l \) and we obtain a rescaled system of ordinary differential equations.

\[
\begin{align*}
s' &= (1 - s - l)(-\alpha s + \phi + \delta s) + l(-\alpha \xi s + \eta) \\
l' &= l\gamma(1 - s - l) - \eta l
\end{align*}
\]  

(3.1)

(3.2)

3.2. Equilibria and Stability. We set equations (2) and (3) equal zero and solve for equilibria. There are four possible equilibria: the susceptible-only equilibrium (SOE), the leader-free equilibrium (LFE), and two endemic equilibria (EE). In order to analyze the stability of the system at each equilibrium we linearize the system. By computing partial first derivatives with respect to each of the variables \( s \) and \( l \), we obtain the Jacobian for the system

\[
J = \begin{bmatrix}
(\alpha - \delta)s - \phi + (1 - s - l)(\delta - \alpha) - \alpha \xi l & s(\alpha - \delta) - \phi - \alpha \xi s + \eta \\
-\gamma l & \gamma(1 - s) - 2\gamma l - \eta
\end{bmatrix}.
\]
Applying $J$ to the SOE (1,0), we find that the eigenvalues are negative if \( \frac{\alpha - \delta}{\phi} < 1 \). Define

$$R_1 = \frac{\alpha - \delta}{\phi}$$  \hspace{1cm} (3.3)

Then the SOE is locally asymptotically stable if \( R_1 < 1 \). \( R_1 \) describes an average number of susceptibles that an individual in D would convert if dropped in a homogeneous population of susceptibles. It is the multiplication of \( \alpha - \delta \), the net pressure on susceptibles by individuals in \( D \), and \( 1/\phi \), the average time spent in the \( D \) class.

The LFE \((1/R_1, 0)\) exists when \( R_1 > 1 \). Define

$$R_2 = \frac{\gamma}{\eta}(1 - \frac{1}{R_1}).$$  \hspace{1cm} (3.4)

The Jacobian \( J \) evaluated at the LFE has negative eigenvalues if \( R_2 < 1 \). Hence, the LFE exists and is locally asymptotically stable if \( R_1 > 1 \) and \( R_2 < 1 \). \( R_2 \) is the multiplication of the average time spent in \( L \), the proportion of the population \( D \) in \( N \), and the rate of conversion from \( D \) to \( L \). Hence, \( R_2 \) is interpreted as the average number of demonstrators that an individual in \( L \) would convert. \( R_2 \) measures how serious the effect of \( L \) to \( D \) is.

In order to discuss two endemic equilibria, we first obtain \( l^* = 1 - s^* - \frac{\eta}{\gamma} \) from setting equation (3.2) equal zero and substitute this in equation (3.1), \( s^t = 0 \), to solve for \( s^* \). Define the following:

\[
A = \frac{\alpha}{\gamma} \\
B = \frac{\eta}{\gamma}(\delta - \alpha) - \eta - \alpha \frac{\phi}{\gamma}(1 - \frac{\eta}{\gamma}) \\
C = \eta \left(\frac{\phi}{\gamma} + 1 - \frac{\eta}{\gamma}\right) \\
f(s) = As^2 + Bs + c
\]

Note that \( f(0) = c > 0, f'(0) = B < 0, f(1) > 0 \), and \( f'(1) > 0 \). Hence, two solutions \( s^* \) to \( f(s) = 0 \) exist if \( B^2 - 4AC > 0 \). I.e., if

$$R_3 = \frac{\eta}{\gamma}(\alpha - \delta) + \eta + \alpha \frac{\phi}{\gamma}(1 - \frac{\eta}{\gamma}) - 2\sqrt{\alpha \frac{\phi}{\gamma} + 1 - \frac{\eta}{\gamma}}$$

is positive, two EE exist. To determine the stability of these equilibria, we apply the Jury criteria [6]. The Jacobian \( J \) applied to these EE has the following determinant and trace:

\[
tr(J) = (\delta - \alpha)s^* + \phi + (\alpha - \delta)\frac{\eta}{\gamma} + \alpha \frac{\phi}{\gamma}(1 - s^* - \frac{\eta}{\gamma}) + \gamma(1 - s^* - \frac{\eta}{\gamma}) \\
det(J) = (\alpha - \delta)\frac{\eta}{\gamma} + \alpha \frac{\phi}{\gamma}(1 - s^* - \frac{\eta}{\gamma}) - \alpha \xi s^* + \eta
\]

It is easily seen that \( tr(J) \) is always negative. Note that \( det(J) > 0 \) if and only if

$$s^* < \frac{1}{2\alpha \xi} [(\alpha - \delta)\frac{\eta}{\gamma} + \alpha \frac{\phi}{\gamma}(1 - \frac{\eta}{\gamma} + \eta)] = \frac{-B}{2A}.$$
Therefore, the EE with the smaller value of $s^*$, denoted by $(s^*_-, l^*_-)$, is locally asymptotically stable and the EE with the larger value of $s^*$, $(s^*_+, l^*_+)$, is unstable when they exist.

In order to plot the regions of equilibria stability we consider curves $R_1 = 1$, $R_2 = 0$ and $R_3 = 0$, and define two variables $x = \frac{\alpha \xi}{\phi}$ and $y = \frac{\gamma}{\eta}$. By substituting $x$ in $R_1 = 1$ and $R_2 = 1$, we obtain $x = \frac{1}{p}$ and $y = \frac{px}{px - 1}$ for $R_1 = 1$ and $R_2 = 1$, respectively, where $p = \frac{\alpha - \delta}{\alpha \xi}$. Similarly, $y$ expresses $R_3 = 0$ in term of $x$, $p$ and $q$. They are plotted together and all equilibria and stability are classified in Figure 2. The values of parameters used to plot these graphs are estimated in Section 4.

**Figure 2.** Regions and classification of Equilibria Stability with $\alpha = 0.5$, $\delta = 0.15$, $\xi = 3$, $\gamma = 0.1$, $\phi = 0.25$, and $\eta = 0.02$

### 3.3. Sensitivity.

Our model contains two threshold conditions $R_1$ and $R_2$. The sociological terms for these are tipping points because they provide a point at which a stable system turns to an unstable one or vise versa. Hence, in order to determine parameters to which our system is most sensitive we find the sensitivity indices of $R_1$ and $R_2$ for all parameters. To see how a small perturbation made to a parameter $p$ affects a threshold condition $R$, we define the sensitivity index of $R$ for $p$ as

$$S_p = \frac{\partial R}{\partial p} \frac{p}{R}$$

The analytic expressions of the sensitivity indices of all parameters with respect to $R_1$ are

$$S_{\alpha} = \frac{\alpha}{\alpha - \delta}, \quad S_{\delta} = \frac{\delta}{\delta - \alpha} \phi^2, \quad S_{\phi} = -1$$
and ones with respect to $R_2$ are

$$S_\alpha = \frac{1}{R_1(\alpha - \delta - \phi)}, \quad S_\delta = \frac{1}{R_1(\alpha - \delta - \phi)}, \quad S_\phi = -\frac{1}{\alpha - \delta - \phi}, \quad S_\gamma = 1, \quad S_\eta = -\frac{1}{\eta}.$$  

We apply the same set of parameters that generated Figure 2 and varied $\alpha$ to change the conditions of $R_1$ and $R_2$ and conclude that the system is most sensitive to $\alpha$ and $\eta$ with respect to $R_1$ and $R_2$, respectively, regardless of the threshold conditions, less than 1 or greater than 1. It is easily seen that the recruitment rate $\alpha$ dominates the transition from the $S$ class to $D$, but this sensitivity analysis shows how important the role of $\eta$ is. $R_2$ deals with $D$ to $L$ conversions and new individuals in $L$ play a significant role in the acquisition of new protesters. This may explain why some countries tend to apply harsh punishment to protest leaders or negotiate with them as the best way to stop demonstrations.

3.4. Bifurcation. A bifurcation is a point in parameter space where equilibria appear, disappear or change stability. We plot the equilibrium values of $l^*$ versus $R_2$ in Figures 3 and 4. The dotted curves plot $l^*_+$ and are asymptotically stable. Whereas, solid curves plot $l^*_-$ and they are unstable. We vary $\eta$ and fix all other parameters as in Section 4 to create Figure 3 since the system is most sensitive to $\eta$ with respect to $R_2$. Figure 3 shows that the system may have two endemic equilibria if we have sufficient initial number of $L$ individuals even if $R_2 < 1$, $R_2$ is interpreted as the average number of demonstrators that an individual in $L$ would convert. One is stable even when $R_2$ is less than 1 if it exists. Figure 4 has all fixed values of parameters as in Section 4 with varied values of $\alpha$. Both figures show a backward bifurcation and the comparison of two figures also show that the system is more sensitive to $\eta$ than $\alpha$.

Bifurcation Diagrams

![Figure 3. with $\eta = [0.001, 0.5]$](image1)

![Figure 4. with $\alpha = [0.015, 0.8]$](image2)
4. Parameter Estimation

In order to run simulations, all parameters must be estimated. As some of the parameters can be estimated only very roughly, our principle goal is to understand our mathematical analysis better with the simulations and to see how closely the model behavior corresponds to the actual observations.

We estimate the parameters of the model based on a variety of online data through Google-Aleema and international news reports from BBC, CBS, and ABC. Since the numbers of protesters reported vary significantly depending on the sources, we average them. We limit our estimations to the candlelight vigils and demonstrations that occurred in Seoul for 100 days beginning from May 2: $N = 15$ millions, inhabitants in Seoul, Incheon, and satellite cities in Kyeonggi-do who could participate in protests by public transportations [7][8]; $D = 1$ million, total protesters in the area for the durations considered; $L = 15,000$, total number of leaders who organized or staged a protest during the period considered; tens of protesters were arrested every day even if majority of them were released next day. Although there is no accurate form of quantifying data for peer driven recruitment rates and linear terms, by considering the increase or decrease in each class per unit time and the number of arrests, we estimate the following:

$$\alpha = 0.5, \gamma = 0.1, \phi = 0.25, \eta = 0.02, \delta = 0.15, \xi = 3$$

and use these values for all our graphs and simulations.

5. Simulation and Interpretation of Dynamics

We plot the solution curves $D$ and $L$ to see the dynamics of the system. The initial conditions used are $S_0 = 14,902,000$, $D_0 = 100,000$, and $L_0 = 8,000$ and the unit of the vertical axis in the plots is thousands. Figure 5 uses the parameters estimated in Section 4 and we increase the value of $\eta$ to 0.05 in Figure 6.

Both figures describe the rapid decrease of $D$ when the issue is in the process of being resolved, but the decrease of $L$ is very slow even with the increase of $\eta$. This phenomenon explains the sociological circumstances surrounding the protests. When the beef imports issue is resolved, the protesters have no reason to continue any form of demonstrations, but that is not the main issue of the $L$ class members any more. The majority of class $L$ might have used the beef issue as an outlet of their frustrations over the economy and new government along the process of the protests. Since what the individuals of the $L$ class want varies, small protests consisting of individuals in $L$ may occur for a while, but these protests without the support of the general public die out quickly.

6. Conclusion

In this paper we developed an epidemiological model to study the stability, sensitivity, and dynamics of the protests. By examining the property of equilibria and their stability, we derived threshold conditions which can be used to determine the prevalence of protests. If $R_1 < 1$, protests fall to zero. If $R_1 > 1$ and $R_2 < 1$, protests occur, but without leaders who may have
different agenda in addition to the beef issue. These protests are easily controlled when the issue is in the process of being resolved. If $R_3$ is positive and the initial number of leaders is sufficient, we may have a stable endemic equilibrium even when $R_2 < 1$. The system is most sensitive to $\eta$ with respect to $R_2$ and the decrease of the $L$ size is very slow even when there is no public support for the protests that they stage with their additional agenda. One of the main implications of this model is to control the $L$ class by minding $\eta$ if we want to avoid large scale prolonged demonstrations. That is, if we want strong lasting protests, one should recruit a large number of $L$ individuals at the beginning. This model can be easily applied to other protests with a national agenda.

REFERENCES

NEW INTERIOR POINT METHODS FOR SOLVING $P_*(\kappa)$ LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. In this paper we propose new primal-dual interior point algorithms for $P_*(\kappa)$ linear complementarity problems based on a new class of kernel functions which contains the kernel function in [8] as a special case. We show that the iteration bounds are $O((1 + 2\kappa)n^{3/4}\log\frac{n\mu_0}{\epsilon})$ for large-update and $O((1 + 2\kappa)\sqrt{n}\log\frac{n\mu_0}{\epsilon})$ for small-update methods, respectively. This iteration complexity for large-update methods improves the iteration complexity with a factor $n^{5/14}$ when compared with the method based on the classical logarithmic kernel function. For small-update, the iteration complexity is the best known bound for such methods.

1. INTRODUCTION

In this paper, we consider the linear complementarity problem (LCP) as follows:

\[
\begin{cases}
  s = Mx + q, \\
x^s = 0, \\
x \geq 0, \ s \geq 0,
\end{cases}
\]

where $x, s, q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ is a $P_*(\kappa)$ matrix.

Primal-dual interior point method (IPM) is one of the most efficient numerical methods for various optimization problems. ([14]) Even though significant research has been devoted to this topic, the influence on nonlinear programming theory and practice has to be studied. Linear complementarity problems (LCPs) are one of the fundamental problems in mathematical programming and have many applications in science, economics, and engineering. ([7])

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Most of polynomial-time interior point algorithms are based on the logarithmic kernel function. Peng et al.([10] - [12]) proposed a new variant of IPMs based on self-regular kernel functions for linear optimization (LO) problems and extended to semidefinite optimization problems and second order cone optimization problems. They improved the complexity result for large-update methods up to $O(\sqrt{n} \log n \log \frac{n}{\epsilon})$ based on a specific self-regular kernel function. This is the best complexity result for such methods. Bai et al.([3], [4]) proposed new primal-dual interior point methods (IPMs) for LO problems based on eligible kernel functions and the scheme for analyzing the algorithm based on four conditions on the kernel function.([4]) They simplified the analysis and obtained the best known complexity result for a specific eligible kernel function.([4]) Cho([5]) and Cho et al.([6]) extended these algorithms to $P_\kappa$ LCPs and obtained the similar complexity results. Recently, Amini et al.([2]) introduced a generalized algorithm.

In this section we compute the iteration bound for the algorithm based on kernel function. We obtained the iteration complexity with a factor $n$ when compared with the method based the classical logarithmic kernel function.([4]) Cho([5]) and Cho et al.([6]) extended these algorithms to $P_\kappa$ LCPs and obtained the best known complexity result for a specific eligible kernel function.([4]) They simplified the analysis and obtained the best known complexity result for a specific eligible kernel function.([4]) Cho([5]) and Cho et al.([6]) extended these algorithms to $P_\kappa$ LCPs and obtained the similar complexity results. Recently, Amini et al.([2]) introduced a generalized algorithm.

In this section we introduce some basic concepts and a generic primal-dual interior point algorithm. In Section 4 we compute the iteration bound for the algorithm based on kernel function.

We use the following notations throughout the paper. $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$ denote the set of $n$-dimensional nonnegative vectors and positive vectors, respectively. For $x, s \in \mathbb{R}^n$, $x_{\min}$ and $x_{s}$ denote the smallest component of the vector $x$ and the componentwise product of the vectors $x$ and $s$, respectively. $e$ denotes the $n$-dimensional vector of ones. For any $\mu > 0$, we define $v := \sqrt{x s / \mu}, v^{-1} := \sqrt{\mu e / (x s)}$ whose $i$-th components are $\sqrt{x_i s_i / \mu}$ and $\sqrt{\mu / (x_i s_i)}$, respectively. We denote $X$ the diagonal matrix from a vector $x$, i.e., $X = \text{diag}(x)$. $I$ denotes the index set, e.g., $I = \{1, 2, \ldots, n\}$. For $f(x), g(x) : \mathbb{R}_{++}^n \to \mathbb{R}_{++}$, $f(x) = \Theta(g(x))$ if $f(x) \leq c_1 g(x)$ for some positive constant $c_1$ and $f(x) = \Theta(g(x))$ if $c_2 g(x) \leq f(x) \leq c_3 g(x)$ for some positive constants $c_2$ and $c_3$.

2. Preliminaries

In this section we introduce some basic concepts and a generic primal-dual interior point algorithm.
**Definition 2.1.** ([9]) Let \( \kappa \geq 0 \). \( P_\ast(\kappa) \) is the class of matrices \( M \) satisfying
\[
(1 + 4\kappa) \sum_{i \in I_+(\xi)} \xi_i[M\xi]_i + \sum_{i \in I_-(\xi)} \xi_i[M\xi]_i \geq 0, \quad \xi \in \mathbb{R}^n,
\]
where \( [M\xi]_i \) denotes the \( i \)-th component of the vector \( M\xi \) and
\[
I_+(\xi) = \{ i \in I : \xi_i[M\xi]_i \geq 0 \}, \quad I_-(\xi) = \{ i \in I : \xi_i[M\xi]_i < 0 \}.
\]

**Lemma 2.2.** ([9]) Let \( M \in \mathbb{R}^{n \times n} \) be a \( P_\ast(\kappa) \) matrix and \( x, s \in \mathbb{R}^n_+ \). Then for all \( c \in \mathbb{R}^n \) the system
\[
\begin{align*}
- M\Delta x + \Delta s &= 0, \\
S\Delta x + X\Delta s &= c
\end{align*}
\]
has a unique solution \((\Delta x, \Delta s)\).

The basic idea of primal-dual IPMs is to replace the second equation in (1.1) by the parameterized equation \( xs = \mu e \) with \( \mu > 0 \) as follows:
\[
\begin{align*}
s &= Mx + q, \\
x = \mu e, \\
x &> 0, \quad s > 0.
\end{align*}
\]

(2.1)

Without loss of generality, we assume that (1.1) is strictly feasible, i.e., there exists a \((x^0, s^0)\) such that \( s^0 = Mx^0 + q \), \( x^0 > 0 \), \( s^0 > 0 \). For this, the reader refer to [9]. Since \( M \) is a \( P_\ast(\kappa) \) matrix and (1.1) is strictly feasible, the system (2.1) has a unique solution for each \( \mu > 0 \).([9])

We denote the solution \((x(\mu), s(\mu))\), \( \mu > 0 \), which is called the \( \mu \)-center. The set of \( \mu \)-centers \((\mu > 0)\) is the central path of (1.1). IPMs follow the central path approximately and approach the solution of (1.1) as \( \mu \) goes to zero.

For given \((x, s) := (x^0, s^0)\) by applying Newton method to the system (2.1) we have the following Newton system:
\[
\begin{align*}
- M\Delta x + \Delta s &= 0, \\
S\Delta x + X\Delta s &= \mu e - xs.
\end{align*}
\]

(2.2)

By Lemma 2.2, the system (2.2) has a unique search direction \((\Delta x, \Delta s)\). By taking a step along the search direction \((\Delta x, \Delta s)\), one constructs a new iterate \((x_+, s_+)\), where
\[
x_+ = x + \alpha \Delta x, \quad s_+ = s + \alpha \Delta s,
\]

for some \( \alpha \geq 0 \). For the motivation of the new algorithm we define the scaled vectors:
\[
v := \sqrt{\frac{x s}{\mu}}, \quad d := \sqrt{\frac{x}{s}}, \quad d_x := \frac{v \Delta x}{x}, \quad d_s := \frac{v \Delta s}{s}.
\]

(2.3)

Using (2.3), we can rewrite the system (2.2) as follows:
\[
\begin{align*}
- M d_x + d_s &= 0, \\
d_x + d_s &= v^{-1} - v,
\end{align*}
\]

(2.4)
where \( \tilde{M} := DMD \) and \( D := \text{diag}(d) \). Note that the right side of the second equation in (2.4) equals the negative gradient of the logarithmic barrier function \( \Psi_l(v) \), i.e., \( d_x + d_s = -\nabla \Psi_l(v) \),

\[
\Psi_l(v) := \sum_{i=1}^n \psi_l(v_i), \quad \psi_l(t) = \frac{t^2}{2} - \log t, \quad t > 0.
\]  

(2.5)

We call \( \psi_l \) the classical logarithmic kernel function of \( \Psi_l(v) \).

The generic interior point algorithm works as follows. Assume that we are given a strictly feasible point \((x, s)\) which is in a \( \tau \)-neighborhood of the given \( \mu \)-center. Then we decrease \( \mu \) to \( \mu_+ = (1 - \theta)\mu \), for some fixed \( \theta \in (0, 1) \) and solve the Newton system (2.2) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size \( \alpha \) which is defined by some line search rule. This procedure is repeated until we find a new iterate \((x_+, s_+)\) that is in a \( \tau \)-neighborhood of the \( \mu_+ \)-center and then we let \( \mu := \mu_+ \) and \((x, s) := (x_+, s_+)\). Then \( \mu \) is again reduced by the factor \( 1 - \theta \) and we solve the Newton system targeting at the new \( \mu_+ \)-center, and so on. This process is repeated until \( \mu \) is small enough, say until \( n\mu < \varepsilon \).

\begin{center}
\textbf{Generic Primal-Dual Algorithm}
\end{center}

Input:
- A threshold parameter \( \tau \geq 1 \);
- an accuracy parameter \( \varepsilon > 0 \);
- a fixed barrier update parameter \( \theta, \ 0 < \theta < 1 \);
- \((x^0, s^0)\) and \( \mu^0 > 0 \) such that \( \Psi_l(x^0, s^0, \mu^0) \leq \tau \).

begin
\begin{itemize}
    \item \( x := x^0; s := s^0; \mu := \mu^0; \)
    \item while \( n\mu > \varepsilon \) do
        \begin{itemize}
            \item \( \mu := (1 - \theta)\mu; \)
            \item while \( \Psi_l(v) > \tau \) do
                \begin{itemize}
                    \item Solve the system (2.2) for \( \Delta x \) and \( \Delta s \);
                    \item Determine a step size \( \alpha \);
                    \item \( x := x + \alpha \Delta x; \)
                    \item \( s := s + \alpha \Delta s; \)
                    \item \( v := \sqrt{\frac{\Delta s}{\mu}}; \)
                \end{itemize}
            \end{itemize}
        \end{itemize}
    \end{itemize}
end

When \( \tau = O(n) \) and \( \theta = \Theta(1) \), we call the algorithm a large-update method. Taking \( \tau = O(1) \) and \( \theta = \Theta(\frac{1}{\sqrt{n}}) \), we call the algorithm a small-update method.
3. The kernel function

In this section we define a new class of kernel functions and give its properties which are essential to our analysis. We call \( \psi : \mathbb{R}_{++} \to \mathbb{R}_+ \) a kernel function if \( \psi \) is twice differentiable and satisfies the following conditions:

\[
\begin{align*}
\psi'(1) &= \psi(1) = 0, \\
\psi''(t) &= 0, \ t > 0, \\
\lim_{t \to 0^+} \psi(t) &= \lim_{t \to \infty} \psi(t) = \infty. 
\end{align*}
\]  

(3.1)

Now we consider a new class of kernel function \( \psi(t) \) as follows:

\[
\psi(t) := 8t^2 - 11t + 1 + 2t^p - (5 - 2p) \log t, \quad \frac{7}{15} \leq p \leq \frac{5}{2}, \ t > 0.
\]  

(3.2)

For \( \psi(t) \) we have the first three derivatives as follows:

\[
\begin{align*}
\psi'(t) &= 16t - 11 - 2pt^{-p-1} - (5 - 2p)t^{-1}, \\
\psi''(t) &= 16 + 2p(p + 1)t^{-p-2} + (5 - 2p)t^{-2}, \\
\psi'''(t) &= 2p(p + 1)(p + 2)t^{-p-3} - 2(5 - 2p)t^{-3}.
\end{align*}
\]  

(3.3)

From (3.1) and (3.3), \( \psi(t) \) is clearly a kernel function and

\[
\psi''(t) > 16, \quad \frac{7}{15} \leq p \leq \frac{5}{2}, \ t > 0.
\]  

(3.4)

In this paper, we replace the function \( \Psi_l(v) \) in (2.5) with the function \( \Psi(v) \) as follows:

\[
d_x + d_s = -\nabla \Psi(v),
\]  

(3.5)

where \( \Psi(v) = \sum_{i=1}^n \psi(v_i) \), \( \psi(t) \) is defined in (3.2) and assume that \( \tau \geq 1 \). Hence the new search direction \( (\Delta x, \Delta s) \) is obtained by solving the following modified Newton-system:

\[
\begin{cases}
-M \Delta x + \Delta s = 0, \\
S \Delta x + X \Delta s = -\mu v \nabla \Psi(v).
\end{cases}
\]  

(3.6)

Since \( -\mu v \nabla \Psi(v) = -\mu v \left( 16v - 11 - 2pv^{-p-1} - (5 - 2p)v^{-1} \right) \), the second equation in (3.6) can be written as

\[
S \Delta x + X \Delta s = -16xs + 11\sqrt{\mu xs} + 2p\sqrt{\mu^{2+p}(xs)^{-p}} + (5 - 2p)\mu.
\]

Since \( \Psi(v) \) is strictly convex and minimal at \( v = e \), we have

\[
\Psi(v) = 0 \iff v = e \iff x = x(\mu), \ s = s(\mu).
\]

We use \( \Psi(v) \) as the proximity function. Also, we define norm-based proximity measure \( \delta(v) \) as follows:

\[
\delta(v) := \frac{1}{2}||\nabla \Psi(v)|| = \frac{1}{2}||d_x + d_s||.
\]  

(3.7)

In the following we give technical properties of \( \psi(t) \) which are essential to our analysis.
Lemma 3.1. Let $\psi(t)$ be as defined in (3.2). Then we have the following properties:

(i) $\psi(t)$ is exponentially convex, for all $t > 0$,

(ii) $\psi''(t)$ is monotonically decreasing, for all $t > 0$,

(iii) $t \psi''(t) - \psi'(t) > 0$, for all $t > 0$,

(iv) $\psi''(t) \psi'(\beta t) - \beta \psi'(t) \psi''(\beta t) > 0$, for all $t > 1$ and $\beta > 1$.

Proof: For (i): By Lemma 2.1.2 in [12], it suffices to show that $\psi(t)$ satisfies $t \psi''(t) + \psi'(t) \geq 0$ for all $t > 0$. Using (3.3), we have for $\frac{7}{15} \leq p \leq \frac{5}{2}$

$$t \psi''(t) + \psi'(t) = 32t - 11 + 2p^2 t^{p-1}.$$ 

Let $g(p, t) = 32t - 11 + 2p^2 t^{-p-1}$. Then $g_t(p, t) = 32 - 2p^2(p+1)t^{-p-2}$ and $g_{tt}(p, t) = 2p^2(p+1)(p+2)t^{-p-3} > 0$, for $t > 0$. Letting $g_t(p, t) = 0$, we have $t = (\frac{p^2(p+1)}{16})^{\frac{1}{p+2}}$. Since $g(p, t)$ is strictly convex in $t$, $g(p, t)$ has a global minimum at $t^* = (\frac{p^2(p+1)}{16})^{\frac{1}{p+2}}$, i.e., $g(p, t^*) \leq g(p, t)$, for $\frac{7}{15} \leq p \leq \frac{5}{2}$ and $t > 0$. For $\frac{7}{15} \leq p \leq \frac{5}{2}$ and $t := t^*$, we have

$$g_p(p, t^*) = 2p(t^*)^{-p-1}(2 - p \log t^*)$$

$$= 2p \left( \frac{p^2(p+1)}{16} \right)^{-\frac{p+1}{p+2}} \left( 2 - \frac{p}{p+2} \log \left( \frac{p^2(p+1)}{16} \right) \right) > 0,$$

since $\log \left( \frac{p^2(p+1)}{16} \right) < 1$ for $\frac{7}{15} \leq p \leq \frac{5}{2}$. This implies that $g\left( \frac{7}{15}, t^* \right) \leq g(p, t^*)$, for $\frac{7}{15} \leq p \leq \frac{5}{2}$.

Hence $g\left( \frac{7}{15}, t^* \right)$ is the smallest value for $\frac{7}{15} \leq p \leq \frac{5}{2}$ and $t > 0$. Since $g\left( \frac{7}{15}, t^* \right) > 0.0111$, we have the result.

For (ii): From (3.3), it is clear.

For (iii): Using (3.3), we have

$$t \psi''(t) - \psi'(t) = 11 + 2p(p+2)t^{-p-1} + 2(5 - 2p)t^{-1} > 0.$$

For (iv): By Lemma 2.4 in [4], it suffices to show that $\psi(t)$ satisfies Lemma 3.1 (ii) and (iii). This completes the proof. \qed

Lemma 3.2. For $\psi(t)$ we have

(i) $8(t-1)^2 \leq \psi(t) \leq \frac{1}{16} (\psi'(t))^2$, $t > 0$,

(ii) $\psi(t) \leq \frac{21+2p^2}{2}(t-1)^2$, $t \geq 1$.

Proof: For (i): Using the first condition of (3.1) and (3.4), we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi \geq 16 \int_1^t \int_1^\xi d\zeta d\xi = 8(t-1)^2,$$

which proves the first inequality. The second inequality is obtained as follows:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi \leq \frac{1}{16} \int_1^t \int_1^\xi \psi''(\zeta) \psi''(\zeta)d\zeta d\xi$$

$$= \frac{1}{16} \int_1^t \psi''(\xi) \psi'(\xi)d\xi = \frac{1}{16} \int_1^t \psi'(\xi) d\psi'(\xi) = \frac{1}{32} (\psi'(t))^2.$$
For (ii): Using Taylor’s theorem, \( \psi(1) = \psi'(1) = 0, \psi'' < 0, \) and \( \psi'''(1) = 21 + 2p^2 \), we have

\[
\psi(t) = \psi(1) + \psi'(1)(t-1) + \frac{1}{2} \psi''(1)(t-1)^2 + \frac{1}{3!} \psi'''(\xi)(t-1)^3
\]

\[
= \frac{1}{2} \psi''(1)(t-1)^2 + \frac{1}{3!} \psi'''(\xi)(t-1)^3
\]

\[
\leq \frac{21 + 2p^2}{2}(t-1)^2,
\]

for some \( \xi, 1 \leq \xi \leq t \). This completes the proof. \( \square \)

**Remark 3.3.** Define \( \psi_b(t) = -11t + 9 + \frac{2p}{t} - (5 - 2p) \log t \). Then \( \psi(t) := 8t^2 - 8 + \psi_b(t) \). \( \psi'_b(t) = -11 - \frac{2p}{t^2} - \frac{5 - 2p}{t} \) and \( \psi''_b(t) = \frac{2p(p+1)}{tp^2} + \frac{5 - 2p}{t^2} > 0 \). Hence, \( \psi'(t) \) is monotonically increasing with respect to \( t \).

**Lemma 3.4.** Let \( \rho : [0, \infty) \to [1, \infty) \) be the inverse function of \( \psi(t) \), for \( t \geq 1 \), \( \rho \) and \( \rho : [0, \infty) \to (0, 1] \), the inverse functions of \( -\frac{1}{2} \psi'(t) \) and \( -\psi'_b(t) \), for \( 0 < t \leq 1 \), respectively. Then we have

(i) \( \sqrt{\frac{r}{8}} + 1 \leq \rho(r) \leq 1 + \sqrt{\frac{r}{8}}, r \geq 0 \),

(ii) \( \rho(z) \geq \rho(16 + 2z), z \geq 0 \),

(iii) \( \rho(u) = \left( \frac{2p}{u - 16 + 2p} \right)^{1/2}, u \geq 16 \).

**Proof:** For (i): Letting \( r = \psi(t) \) for \( t \geq 1 \), we have \( \rho(r) = t \). By the definition of \( \psi(t) \), \( r = 8t^2 - 11t + 1 + \frac{2p}{t} - (5 - 2p) \log t \). This implies

\[
8t^2 = r + 11t - 1 - \frac{2}{p} + (5 - 2p) \log t \geq r + 8.
\]

Hence we have

\[
t = \rho(r) \geq \sqrt{\frac{r}{8}} + 1.
\]

Using Lemma 3.2 (i), we have \( r = \psi(t) \geq 8(t - 1)^2 \). Then we have

\[
t = \rho(r) \leq 1 + \sqrt{\frac{r}{8}}.
\]

For (ii): Let \( z = -\frac{1}{2} \psi'(t) \), for \( 0 < t \leq 1 \). By Remark 3.3, we have

\[-2z = \psi'(t) = 16t + \psi'_b(t).
\]

This implies that

\[-\psi'_b(t) = 16t + 2z \leq 16 + 2z.
\]

Using Remark 3.3 and \( t = \rho(z) \), we have

\[
t = \rho(z) \geq \rho(16 + 2z).
\]
For (iii): Letting \( u = -\psi_b'(t) \), for \( 0 < t \leq 1 \), we have \( \rho(u) = t \). By the definition of \( -\psi_b'(t) \), we have \( u = 11 + \frac{2p}{t^{p+1}} + \frac{5-2p}{t} \geq 16, \ 0 < t \leq 1 \). This implies
\[
\frac{2p}{t^{p+1}} = u - 11 - \frac{5-2p}{t} \leq u - 16 + 2p.
\]
Hence we have
\[
t = \rho(u) \geq \left( \frac{2p}{u - 16 + 2p} \right)^{\frac{1}{p+1}}, \ u \geq 16.
\]

**Corollary 3.5.** Let \( \rho \) be as defined in Lemma 3.4. Then we have
\[
\rho(z) \geq \left( \frac{p}{z + p} \right)^{\frac{1}{p+1}}, \ z \geq 0.
\]

**Proof:** Using Lemma 3.3 (ii) and (iii), we have
\[
\rho(z) \geq \rho(16 + 2z) = \left( \frac{p}{z + p} \right)^{\frac{1}{p+1}}.
\]
This completes the proof.

**Lemma 3.6.** (Theorem 3.2 in [3]) Let \( \varrho \) be as defined in Lemma 3.4. Then we have
\[
\Psi(\beta v) \leq n\psi \left( \beta \varrho \left( \frac{\Psi(v)}{n} \right) \right), \ v \in \mathbb{R}^n_+, \ \beta \geq 1.
\]

In the following we obtain an estimate for the effect of a \( \mu \)-update on the value of \( \Psi(v) \).

**Theorem 3.7.** Let \( 0 \leq \theta < 1 \) and \( v_+ = \frac{v}{\sqrt{1-\theta}} \). If \( \Psi(v) \leq \tau \), then we have
\[
\Psi(v_+) \leq \frac{21 + 2p^2}{2(1-\theta)} \left( \sqrt{n\theta} + \sqrt{\frac{\tau}{8}} \right)^2.
\]

**Proof:** Since \( \frac{1}{\sqrt{1-\theta}} \geq 1 \) and \( \theta \left( \frac{\Psi(v)}{n} \right) \geq 1 \), we have \( \frac{\theta \left( \frac{\Psi(v)}{n} \right)}{\sqrt{1-\theta}} \geq 1 \). Using Lemma 3.6 with \( \beta = \frac{1}{\sqrt{1-\theta}} \), Lemma 3.2 (ii), Lemma 3.4 (i), and \( \Psi(v) \leq \tau \), we have
\[
\Psi(v_+) \leq n\psi \left( \frac{1}{\sqrt{1-\theta}} \theta \left( \frac{\Psi(v)}{n} \right) \right) \leq \frac{(21 + 2p^2)n}{2} \left( \theta \left( \frac{\Psi(v)}{n} \right) \right)^2 \frac{\sqrt{1-\theta}}{\sqrt{1-\theta}} - 1 \right)^2
\]
\[
= \frac{(21 + 2p^2)n}{2} \left( \theta \left( \frac{\Psi(v)}{n} \right) - \frac{\sqrt{1-\theta}}{\sqrt{1-\theta}} \right)^2
\]
\[
(21 + 2p^2)n \left( \frac{1 + \sqrt{\frac{\tau}{8n}} - \sqrt{1 - \theta}}{\sqrt{1 - \theta}} \right)^2 
\leq \frac{(21 + 2p^2)n}{2} \left( \theta + \sqrt{\frac{\tau}{8n}} \right)^2 = 21 + 2p^2 \left( \sqrt{n\theta} + \sqrt{\frac{\tau}{8}} \right)^2,
\]
where the last inequality holds from \(1 - \sqrt{1 - \theta} = \frac{\theta}{1 + \sqrt{1 - \theta}} \leq \theta, 0 \leq \theta < 1.\) This completes the proof.

Denote
\[
\hat{\Psi}_0 := \frac{21 + 2p^2}{2(1 - \theta)} \left( \sqrt{n\theta} + \sqrt{\frac{\tau}{8}} \right)^2.
\] (3.8)
Then \(\hat{\Psi}_0\) is an upper bound for \(\Psi(v)\) during the process of the algorithm.

**Remark 3.8.** For large-update method with \(\tau = \mathcal{O}(n)\) and \(\theta = \Theta(1)\), \(\hat{\Psi}_0 = \mathcal{O}(n)\) and for small-update method with \(\tau = \mathcal{O}(1)\) and \(\theta = \Theta(\frac{1}{\sqrt{n}})\), \(\hat{\Psi}_0 = \mathcal{O}(1)\).

### 4. Complexity Result

In this section we compute a feasible step size and the decrease of the proximity function during an inner iteration and give the complexity results of the algorithm. For fixed \(\mu\) if we take a step size \(\alpha\), then we have new iterates
\[
x_+ = x + \alpha \Delta x, \quad s_+ = s + \alpha \Delta s.
\]
Using (2.3), we have
\[
x_+ = x \left( e + \alpha \frac{\Delta x}{x} \right) = x \left( e + \alpha \frac{d_x}{v} \right) = \frac{x}{v} (v + \alpha d_x)
\]
and
\[
s_+ = s \left( e + \alpha \frac{\Delta s}{s} \right) = s \left( e + \alpha \frac{d_s}{v} \right) = \frac{s}{v} (v + \alpha d_s).
\]
Thus we have
\[
v_+ = \sqrt{\frac{x_+ + s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.
\]
Define for \(\alpha > 0\)
\[
f(\alpha) = \Psi(v_+) - \Psi(v).
\]
Then \(f(\alpha)\) is the difference of proximities between a new iterate and a current iterate for fixed \(\mu\). Using Lemma 3.1 (i), we have
\[
\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}) \leq \frac{1}{2} (\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).
\]
Hence we have \(f(\alpha) \leq f_1(\alpha)\), where
\[
f_1(\alpha) := \frac{1}{2} (\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v).
\]
Obviously, we have
\[
f(0) = f_1(0) = 0.
\]
By taking the derivative of $f_1(\alpha)$ with respect to $\alpha$, we have

$$f_1'(\alpha) = \frac{1}{2} \sum_{i=1}^{n} (\psi'(v_i + \alpha [d_x]_i) [d_x]_i + \psi'(v_i + \alpha [d_s]_i) [d_s]_i),$$

where $[d_x]_i$ and $[d_s]_i$ denote the $i$-th components of the vectors $d_x$ and $d_s$, respectively. Using (3.5) and (3.7), we have

$$f_1'(0) = \frac{1}{2} \nabla \Psi(v)^T (d_x + d_s) = -\frac{1}{2} \nabla \Psi(v)^T \nabla \Psi(v) = -2\delta(v)^2.$$ 

Differentiating $f_1'(\alpha)$ with respect to $\alpha$, we have

$$f_1''(\alpha) = \frac{1}{2} \sum_{i=1}^{n} (\psi''(v_i + \alpha [d_x]_i) [d_x]_i^2 + \psi''(v_i + \alpha [d_s]_i) [d_s]_i^2).$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in $\alpha$ unless $d_x = d_s = 0$. Since $M$ is a $P_\tau(\kappa)$ matrix and $M \Delta x = \Delta s$ from (3.6), for $\Delta x \in \mathbb{R}^n$,

$$(1 + 4\kappa) \sum_{i \in I_+} [\Delta x]_i [\Delta s]_i + \sum_{i \in I_-} [\Delta x]_i [\Delta s]_i \geq 0,$$

where $I_+ = \{i \in I : [\Delta x]_i [\Delta s]_i \geq 0\}$, $I_- = I - I_+$. Since $d_x d_s = \frac{\Delta x \Delta s}{\mu} = \Delta x \Delta s$ and $\mu > 0$, we have

$$(1 + 4\kappa) \sum_{i \in I_+} [d_x]_i [d_s]_i + \sum_{i \in I_-} [d_x]_i [d_s]_i \geq 0.$$ 

**Lemma 4.1.** Let $\delta(v)$ be as defined in (3.7). Then we have

$$\delta(v) \geq 2\sqrt{2\Psi(v)}.$$ 

**Proof:** Using (3.7) and Lemma 3.2 (i), we have

$$\delta(v)^2 = \frac{1}{4} \|\nabla \Psi(v)\|^2 = \frac{1}{4} \sum_{i=1}^{n} (\psi'(v_i))^2 \geq \frac{1}{4} \sum_{i=1}^{n} 32\psi(v_i) = 8\Psi(v).$$

Hence we have $\delta(v) \geq 2\sqrt{2\Psi(v)}$. \hfill \qed

**Remark 4.2.** Using Lemma 4.1 and the assumption $\Psi(v) \geq \tau \geq 1$, we have

$$\delta(v) \geq 2\sqrt{2\Psi(v)} \geq 2\sqrt{2}. \quad (4.1)$$

For notational convenience we denote $\delta := \delta(v)$ and $\Psi := \Psi(v)$.

**Lemma 4.3.** (Modification of lemma 4.4 in [6]) $f_1'(\alpha) \leq 0$ if $\alpha$ is satisfying

$$-\psi'(v_{min} - 2\alpha \delta \sqrt{1 + 2\kappa}) + \psi'(v_{min}) \leq \frac{2\delta}{\sqrt{1 + 2\kappa}}. \quad (4.2)$$
Lemma 4.4. (Modification of lemma 4.5 in [6]) Let \( \rho \) be as defined in Lemma 3.4. Then the largest step size \( \alpha \) satisfying (4.2) is given by
\[
\hat{\alpha} := \frac{1}{2\delta \sqrt{1 + 2\kappa}} \left( \rho(\delta) - \rho\left(1 + \frac{1}{\sqrt{1 + 2\kappa}}\right) \delta \right).
\]

Lemma 4.5. (Modification of lemma 4.6 in [6]) Let \( \rho \) and \( \hat{\alpha} \) be as defined in Lemma 4.4. Then
\[
\hat{\alpha} \geq \frac{1}{(1 + 2\kappa) \psi''\left(\rho\left(1 + \frac{1}{\sqrt{1 + 2\kappa}}\right)\delta\right)}.
\]
Define
\[
\bar{\alpha} := \frac{1}{(1 + 2\kappa) \psi''\left(\rho\left(1 + \frac{1}{\sqrt{1 + 2\kappa}}\right)\delta\right)}.
\]
Then we have \( \bar{\alpha} \leq \hat{\alpha} \).

Lemma 4.6. (Lemma 1.3.3 in [12]) Suppose that \( h(t) \) is twice differentiable convex function with
\( h(0) = 0, \ h'(0) < 0 \), \( h(t) \) attains its (global) minimum at \( t^* > 0 \), and \( h''(t) \) is increasing with respect to \( t \). Then for any \( t \in [0, t^*] \),
\[
h(t) \leq \frac{th'(0)}{2}.
\]

Lemma 4.7. (Modification of lemma 4.8 in [6]) If the step size \( \alpha \) is such that \( \alpha \leq \bar{\alpha} \), then
\[
f(\alpha) \leq -\alpha \delta^2.
\]

Theorem 4.8. Let \( \bar{\alpha} \) be as defined in (4.3). Then for \( \alpha = 1 + \frac{1}{\sqrt{1 + 2\kappa}} \) and \( \kappa \geq 0 \), we have
\[
f(\bar{\alpha}) \leq -\frac{2}{2(p+1)} \Psi^p \psi^{p+2}
\]
where \( L(p, a) := 4\sqrt{2} + 2p(p+1) \left(\frac{1}{2\sqrt{2}} + \frac{a}{p}\right)^{\frac{p+2}{p+1}} + (5 - 2p) \left(\frac{1}{2\sqrt{2}} + \frac{a}{p}\right)^{\frac{2}{p+1}} \).

Proof: Using Corollary 3.5, we have
\[
\rho(a\delta) \geq \left(\frac{p}{a\delta + p}\right)^{\frac{1}{p+1}}.
\]
Using Lemma 4.7, (4.3), (4.4), and Lemma 3.1 (ii), we obtain
\[
f(\bar{\alpha}) \leq -\bar{\alpha} \delta^2 = -\frac{\delta^2}{(1 + 2\kappa) \psi''\left(\rho(a\delta)\right)} \leq -\frac{\delta^2}{(1 + 2\kappa) \psi''\left(\frac{p}{a\delta + p}\right)^{\frac{1}{p+1}}}. \]
Using (3.3) and (4.1), we have

\[ \psi'' \left( \left( \frac{p}{a \delta + p} \right)^{\frac{p+1}{2}} \right) = 16 + 2p(p+1) \left( 1 + \frac{a \delta}{p} \right)^{\frac{p+2}{p+1}} + (5-2p) \left( 1 + \frac{a \delta}{p} \right)^{\frac{p}{p+1}} \]

\[ \leq 4\sqrt{2}\delta + 2p(p+1) \left( \frac{\delta}{2\sqrt{2}} + \frac{a \delta}{p} \right)^{\frac{p+2}{p+1}} + (5-2p) \left( \frac{1}{2\sqrt{2}} + \frac{a \delta}{p} \right)^{\frac{p}{p+1}} \]

\[ = 4\sqrt{2}\delta + 2p(p+1) \left( \frac{1}{2\sqrt{2}} + \frac{a}{p} \right)^{\frac{p+2}{p+1}} \delta^{\frac{p+2}{p+1}} + (5-2p) \left( \frac{1}{2\sqrt{2}} + \frac{a}{p} \right)^{\frac{p}{p+1}} \delta^{\frac{p}{p+1}} \]

\[ \leq L(p, a) \delta^{\frac{p+2}{p+1}}, \quad (4.6) \]

where

\[ L(p, a) := 4\sqrt{2} + 2p(p+1) \left( \frac{1}{2\sqrt{2}} + \frac{a}{p} \right)^{\frac{p+2}{p+1}} \delta^{\frac{p+2}{p+1}} + (5-2p) \left( \frac{1}{2\sqrt{2}} + \frac{a}{p} \right)^{\frac{p}{p+1}} \delta^{\frac{p}{p+1}}. \]

Using (4.5), (4.6), and Lemma 4.1, we have

\[ f(\bar{\alpha}) \leq -\frac{1}{(1+2\kappa)L(p, a)} \delta^{\frac{p}{p+1}} \leq -\frac{1}{(1+2\kappa)L(p, a)} (2\sqrt{2}\Psi)^{\frac{p}{p+1}} \]

\[ = -\frac{2^{\frac{3p}{p+1}}}{(1+2\kappa)L(p, a)} \Psi^{\frac{p}{p+1}}. \]

This completes the proof. \( \square \)

**Lemma 4.9.** (Lemma 1.3.2 in [12]) Let \( t_0, t_1, \ldots, t_K \) be a sequence of positive numbers such that

\[ t_{k+1} \leq t_k - \gamma t_k^{1-\lambda}, \quad k = 0, 1, \ldots, K-1, \]

where \( \gamma > 0 \) and \( 0 < \lambda \leq 1 \). Then \( K \leq \left\lfloor \frac{t_0}{\gamma^{1/\lambda}} \right\rfloor \).

We define the value of \( \Psi(v) \) after the \( \mu \)-update as \( \Psi_0 \) and the subsequent values in the same outer iteration are denoted as \( \Psi_k, k = 1, 2, \ldots \). Then we have

\[ \Psi_0 \leq \Psi_0, \quad (4.7) \]

where \( \Psi_0 \) is defined in (3.8). Let \( K \) denote the total number of inner iterations per outer iteration. Then we have

\[ \Psi_{K-1} \geq \tau, \quad 0 \leq \Psi_K \leq \tau. \]

**Lemma 4.10.** Let \( \bar{\Psi}_0 \) be as defined in (3.8) and \( K \) the total number of inner iterations in the outer iteration. Then we have for \( \frac{7}{15} \leq p \leq \frac{5}{2} \) and \( a = 1 + \frac{1}{\sqrt{1+2\kappa}} \)

\[ K \leq (1+2\kappa) \bar{L}(p, a) \bar{\Psi}_0^{\frac{p+2}{p+1}}, \]

where \( \bar{L}(p, a) \) is defined in (4.6).
where $\tilde{L}(p, a) := \frac{(p+1)^2}{2} \frac{2(p+1)}{2} - \frac{p^2(p+1)}{p+2} L(p, a)$.

**Proof:** Using Theorem 4.8, Lemma 4.9 with $\gamma := \frac{2\pi(p+1)}{1+2\kappa L(p,a)}$ and $\lambda := \frac{p+2}{2(p+1)}$, and (4.7), we have

$$K \leq \frac{(1 + 2\kappa) L(p, a)}{2\pi(p+1)} \cdot \frac{2(p+1)}{p+2} \Psi_0 \frac{p+2}{p+1} \frac{1+\gamma}{\theta} \leq (1 + 2\kappa) \tilde{L}(p, a) \tilde{\Psi}_0 \frac{p+2}{p+1},$$

where

$$\tilde{L}(p, a) := \frac{(p+1)^2}{2} \frac{2(p+1)}{2} - \frac{p^2(p+1)}{p+2} L(p, a). \quad (4.8)$$

This completes the proof. \qed

**Theorem 4.11.** Let a $P_\kappa$ LCP be given. If $\tau \geq 1$, the total number of iterations to have an approximate solution with $n\mu \leq \epsilon$ is bounded by

$$\left\lceil \frac{(1 + 2\kappa) \tilde{L}(p, a)}{\theta} \frac{p+2}{p+1} \frac{1+\gamma}{\theta} \log \frac{n\mu^0}{\epsilon} \right\rceil,$$

where $\tilde{\Psi}_0$ as defined in (3.8), $\tilde{L}(p, a)$ in (4.8), $\frac{7}{15} \leq p \leq \frac{5}{2}$, and $0 < \theta < 1$.

**Proof:** If the central path parameter $\mu$ has the initial value $\mu^0 > 0$ and is updated by multiplying $1 - \theta$ with $0 < \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\epsilon} \right\rceil$$

iterations we have $n\mu \leq \epsilon$. ([13]) For the total number of iterations, we multiply the number of inner iterations by that of outer iterations. Hence the total number of iterations is bounded by

$$\left\lceil \frac{(1 + 2\kappa) \tilde{L}(p, a)}{\theta} \frac{p+2}{p+1} \frac{1+\gamma}{\theta} \log \frac{n\mu^0}{\epsilon} \right\rceil.$$

This completes the proof. \qed

**Remark 4.12.** By Remark 3.8, for large-update methods by taking $\tau = O(n)$, $\theta = \Theta(1)$, and $p = \frac{5}{2}$, the algorithm has $O((1 + 2\kappa)\sqrt{n} \log \frac{n\mu^0}{\epsilon})$ iteration complexity which improves the iteration complexity with a factor $n^{\frac{\kappa}{2}}$ when compared with the method based on the classical logarithmic kernel function. For small-update methods by taking $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we have $O((1+2\kappa)\sqrt{n} \log \frac{n\mu^0}{\epsilon})$ iteration complexity which is the best known complexity result for such methods.
REFERENCES


EXISTENCE OF SOLUTIONS FOR IMPULSIVE NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In this article, we study the existence and uniqueness of mild and classical solutions for a nonlinear impulsive differential equation with nonlocal conditions

\[ u'(t) = Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 \leq t \leq T_0, \quad t \neq t_i, \]
\[ u(0) + g(u) = u_0, \]
\[ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0, \]

in a Banach space \( X \), where \( A \) is the infinitesimal generator of a \( C_0 \) semigroup, \( g \) constitutes a nonlocal conditions, and \( \Delta u(t_i) = u(t_i^+) - u(t_i^-) \) represents an impulsive conditions.

1. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations whose duration is negligible in comparison with the duration of the processes. Consequently, it is natural to assume that these perturbations act instantaneously, that is in the form of impulses. For more details on this theory and applications, see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [9], and Samoilenko and Perestyuk [14], where numerous properties of their solutions are studied and detailed bibliographies are given.

The starting point of this paper is the works in papers [1, 11, 12]. Especially, the authors in [12] investigated the existence and uniqueness of mild and classical solutions for an impulsive first order system

\[ u'(t) = Au(t) + f(t, u(t)), \quad 0 \leq t \leq T_0, \quad t \neq t_i, \]
\[ u(0)) = u_0, \]
\[ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0. \]
by using semigroup theory and Schauder’s fixed point theorem. And in [11], authors studied existence and uniqueness of mild and classical solutions for the following impulsive system

\[ u'(t) = Au(t) + f(t, u(t)), \quad 0 \leq t \leq K, \quad t \neq t_i, \]
\[ u(0) + g(u) = u_0, \]
\[ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < K. \]

by using the Banach contraction principle and Schauder’s fixed point theorem.

Motivated by the above mentioned works [11, 12], the main purpose of this paper is to prove the existence and uniqueness of mild and classical solutions for the following first order impulsive system

\[ u'(t) = Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 \leq t \leq T_0, \quad t \neq t_i, \]
\[ u(0) + g(u) = u_0, \]
\[ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0. \]

in a Banach space \( X \), where \( A \) is the infinitesimal generator of a strongly continuous semigroup \( \{T(t) \mid t \geq 0\} \), \( f \in C([0, T_0] \times X \times X \times X, X) \), \( g \in \mathcal{PC}([0, T_0], X) \),

\[ Tu(t) = \int_0^t K(t, s)u(s)ds, \quad K \in C[D, R^+], \]
\[ Su(t) = \int_{T_0}^T H(t, s)u(s)ds, \quad H \in C[D_0, R^+], \]

where \( D = \{(t, s) \in R^2 : 0 \leq s \leq t \leq T_0\} \), \( D_0 = \{(t, s) \in R^2 : 0 \leq t, s \leq T_0\} \) and \( \mathcal{PC}([0, T_0], X) \) consist of a functions \( u \) that are a map from \([0, T_0]\) into \( X \), such that \( u(t) \) is continuous at \( t \neq t_i \) and left continuous at \( t = t_i \), and the right limit \( u(t_i^+) \) exists for \( i = 1, 2, \ldots, p \).

Evidently \( \mathcal{PC}([0, T_0], X) \) is a Banach space with the norm

\[ \|u\|_{\mathcal{PC}} = \sup_{t \in [0, T_0]} \|u(t)\|. \]

The nonlocal Cauchy problem was considered by Byszewski [4] and the importance of nonlocal conditions in different fields has been discussed in [4] and [6] and the references therein. For example, in [6] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

\[ g(x) = \sum_{k=0}^{n} c_k \varphi(t_k), \]

where \( c_k, \ k = 0, 1, \ldots, n \) are given constants and \( 0 < t_0 < t_1 < \cdots < t_n < a \). In this case the above equations allows the additional measurements at \( t = t_k, \ k = 0, 1, \ldots, n \). In the past several years theorem about existence and uniqueness of differential, impulsive differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by many authors [3, 5, 7, 8, 10].
In the present paper, we discuss the existence and uniqueness for the impulsive problem (1.1)-(1.3). Our approach here is based on the semigroup theory [13] and fixed point theorem.

2. Existence results

In this section, first we define the concept of mild and classical solutions for the problem (1.1)-(1.3).

**Definition 1.** A function $u(\cdot) \in \mathcal{PC}([0, T_0], X)$ is a mild solution of equations (1.1)-(1.3) if it satisfies

$$
  u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s, u(s), Tu(s), Su(s))ds
  + \sum_{0 < t_i < t} T(t-t_i)I_i(u(t_i)), \ 0 \leq t \leq T_0.
$$

**Definition 2.** A classical solution of equations (1.1)-(1.3) is a function $u(\cdot)$ in $\mathcal{PC}([0, T_0], X) \cap C^1([0, T_0] \setminus \{t_1, t_2, \ldots, t_p\}, X)$, $u(t) \in D(A)$ (the domain of $A$) for $t \in [0, T_0] \setminus \{t_1, t_2, \ldots, t_p\}$, which satisfies equations (1.1)-(1.3) on $[0, T_0]$.

The mild and classical solutions of (1.1)-(1.3) will be established under different conditions of the functions $f, g, I, T$ and the semigroup $T(\cdot)$.

2.1. Lipschitz conditions. Let $B(X)$ be the Banach space of all linear and bounded operators on $X$. Define

$$
  M = \sup_{t \in [0, T_0]} \|T(t)\|_{B(X)}, \tag{2.1}
$$

which is a finite number.

Now we list out the following hypotheses:

(H1) $f : [0, T_0] \times X \times X \times X \to X$, $g : \mathcal{PC}([0, T_0], X) \to X$ and $I_i : X \to X$, $i = 1, 2, \ldots, p$ are continuous and there exists constants $L_1, L_2, L_3 > 0$, $G > 0$, $h_i > 0$, $i = 1, 2, \ldots, p$, such that

$$
  \|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq L_1\|x_1 - y_1\| + L_2\|x_2 - y_2\| + L_3\|x_3 - y_3\|,
$$

$$
  t \in [0, T_0]$, $x_i, y_i \in X$, $i = 1, 2, 3$. 

$$
  \|g(u) - g(v)\| \leq G\|u - v\|$, $u, v \in \mathcal{PC}([0, T_0], X)$,

$$
  \|I_i(x) - I_i(y)\| \leq h_i\|u - v\|$, $x, y \in X$.

(H2) Denote $L = \max\{L_1, L_2, L_3\}$, $K^* = \sup_{t \in [0, T_0]} \int_0^t |K(t, s)|dt < \infty$, and

$$
  H^* = \sup_{t \in [0, T_0]} \int_0^{T_0} |H(t, s)|dt < \infty.
$$
The constants $L, G, K^*, H^*$ satisfy the inequality
\[ M \left[ G + LT_0(1 + K^* + H^*) + \sum_{i=1}^{p} h_i \right] < 1. \]

**Theorem 2.1.** Assume that the hypotheses (H1)-(H3) are satisfied. Then for every $u_0 \in X$, for $t \in [0, T_0]$ the equation
\[
  u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds \\
  + \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)), \quad 0 \leq t \leq T_0.
\]
has a unique mild solution.

**Proof.** Let $u_0 \in X$ be fixed. Define an operator $F$ on $PC([0, T_0], X)$ by
\[
(Fu)(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds \\
+ \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)), \quad 0 \leq t \leq T_0.
\]
Then it is clear that $F : PC([0, T_0], X) \to PC([0, T_0], X)$. Now we show that $F$ is contraction. For any $u, v \in PC([0, T_0], X)$, we have
\[
\| (Fu)(t) - (Fv)(t) \| \leq \| T(t)[g(u) - g(v)] \| + \int_0^t \| T(t - s)\|_{B(X)}\| f(s, u(s), Tu(s), Su(s)) \| ds \\
- f(s, v(s), Tv(s), Sv(s))\|ds \\
+ \sum_{0 < t_i < t} \| T(t - t_i)\|_{B(X)}\| I_i(u(t_i)) - I_i(v(t_i)) \|.
\]
Using the hypothesis (H1) and equation (2.1), we have
\[
\| (Fu)(t) - (Fv)(t) \| \leq MG\| u - v \|_{PC} + M \left[ \int_0^t L_1\| u - v \| + L_2\| Tu - Tv \| + L_3\| Su - Sv \| \right] ds + M\| u - v \|_{PC} \sum_{i=1}^{p} h_i. \tag{2.2}
\]
Now,
\[
\int_0^t L_2\| Tu - Tv \| ds \leq L_2 \int_0^t \int_0^s |K(s, \tau)|\| u(\tau) - v(\tau) \| d\tau ds
\]
\[
\leq L_2 \int_0^t \|u(s) - v(s)\| \int_s^t \|K(s, \tau)\| \, d\tau \, ds \\
\leq L_2 \|u(t) - v(t)\| \int_0^t K^* \, ds \\
\leq L_2 \|u - v\|_{PC} K^* T_0
\]  \hfill (2.3)

Similarly,
\[
\int_0^t L_3 \|Su - Sv\| \, ds \leq L_3 \|u - v\|_{PC} H^* T_0. \hfill (2.4)
\]

Substitute the equations (2.3) and (2.4) into the equation (2.2), we have
\[
\|(Fu)(t) - (Fv)(t)\| \leq MG \|u - v\|_{PC} + M \left[ L_1 T_0 \|u - v\|_{PC} + L_2 \|u - v\|_{PC} K^* T_0 \\
+ L_3 \|u - v\|_{PC} H^* T_0 + \|u - v\|_{PC} \sum_{i=1}^p h_i \right] \\
\leq M \left[ G + L_1 T_0 + L_2 K^* T_0 + L_3 H^* T_0 + \sum_{i=1}^p h_i \right] \|u - v\|_{PC}.
\]

Using the definition of \(L\), we have
\[
\|(Fu)(t) - (Fv)(t)\| \leq M \left[ G + LT_0 (1 + K^* + H^*) + \sum_{i=1}^p h_i \right] \|u - v\|_{PC}
\]

From hypothesis (H3), we have
\[
\|Fu - Fv\|_{PC} \leq \|u - v\|_{PC}, \quad u, v \in PC([0, T_0], X).
\]

Therefore, \(F\) is a contraction operator on \(PC([0, T_0], X)\). Thus \(F\) has a unique fixed point, which gives rise to a unique mild solution. This completes the proof. \(\square\)

**Remark 1.** If \(S = 0\) in equation (1.1) and assume that the hypotheses (H1)-(H2) are satisfied (with simple modifications) and \(M[G + LT_0 (1 + K^* + H^*) + \sum_{i=1}^p h_i] < 1\), then equation (1.1)-(1.3) has a unique mild solution.

2.2. \textbf{g is compact.} It should be pointed out that a compact operator is a continuous operator which maps a bounded set into a precompact set.

Now we assume the following hypotheses:

(H4) \(f\) is continuous and maps a bounded set into a bounded set.

(H5) \(g : PC([0, T_0], X) \rightarrow X\) and \(I_i : X \rightarrow X\), \(i = 1, 2, \ldots p\), are compact operators, and \(T(\cdot)\) is also compact.
(H6) For each $u_0 \in X$, there exists a constant $r > 0$ such that
\[
M(\|u_0\| + \sup_{\varphi \in Y_r} \|g(\varphi)\| + T_0 \sup_{s \in [0, T_0], \varphi \in Y_r} \|f(s, \varphi(s), T\varphi(s), S\varphi(s))\|
+ \sup_{\varphi \in Y_r} \sum_{i=1}^{p} \|I_i(\varphi(t_i))\|) \leq r,
\]
where $Y_r = \{ \varphi \in \mathcal{PC}([0, T_0], X) : \|\varphi(t)\| \leq r \text{ for } t \in [0, T_0] \}$.

Under these hypotheses, we can prove the following result.

**Theorem 2.2.** Let (H4)-(H6) be satisfied. Then for every $u_0 \in X$, for $t \in [0, T_0]$ the equation
\[
u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds
+ \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i))
\]
has at least a mild solution.

**Proof.** Let $u_0 \in X$ be fixed. Define an operator $F$ on $\mathcal{PC}([0, T_0], X)$ by
\[
(Fu)(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds
+ \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)).
\]
The operator $F$ is continuous from $Y_r$ to $Y_r$. In order to use Schauder’s second fixed point theorem to obtain a fixed point and hence a mild solution, we need to prove that $F$ is a compact operator. For this reason, we split $(Fu)(t)$ as $(F_1u)(t) + (F_2u)(t)$. That is $(Fu)(t) = (F_1u)(t) + (F_2u)(t)$, where
\[
(F_1u)(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds, \quad 0 \leq t \leq T_0,
\]
\[
(F_2u)(t) = \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)), \quad 0 \leq t \leq T_0.
\]

Now, we show that $F_1$ and $F_2$ are compact operators. First, we prove that $F_2$ is a compact operator. The operator
\[
(F_2u)(t) = \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)) = \begin{cases} 0, & t \in [0, t_1], \\ T(t - t_1)I_1(u(t_1)), & t \in (t_1, t_2), \\ \vdots \\ \sum_{i=1}^{p} T(t - t_i)I_i(u(t_i)), & t \in (t_p, T_0], \end{cases}
\]
and that the interval $[0, T_0]$ is divided into finite subintervals by $t_i$, $i = 1, 2, \ldots, p$, so that we only need to prove that

$$W = \{T(\cdot - t_1)I_1(u(t_1)) : \cdot \in [t_1, t_2], u \in Y_r\}$$

is precompact in $C([t_1, t_2], X)$, as the cases for other subintervals are the same. From hypothesis (H5), we see that for each $t \in [t_1, t_2]$, the set $\{T(t - t_1)I_1(u(t_1)) : u \in Y_r\}$ is precompact in $X$. Next, for $t_1 \leq s < t \leq t_2$, we have, using the semigroup property,

$$\|T(t - t_1)I_1(u(t_1)) - T(s - t_1)I_1(u(t_1))\| = \|T(s - t_1)[T(t - s) - T(0)]I_1(u(t_1))\|$$

$$\leq M\|T(t - s) - T(0)\|I_1(u(t_1))\|$$  (2.5)

Thus, the functions in $W$ are equicontinuous due to compactness of $I_1$ and the strong continuity of $T(\cdot)$. From the Arzela-Ascoli theorem, we deduce that $F_2$ is a compact operator.

Similarly we can prove compactness of $F_1$. That is, for each $t \in [0, T_0]$, the set $\{T(t)[u_0 - g(u)] : u \in Y_r\}$ is precompact in $X$, since $g$ is compact. Also, for each $t \in (0, T_0]$ and $\epsilon \in (0, t)$,

$$\left\{ \int_0^{t-\epsilon} T(t - s)f(s, u(s), Tu(s), Su(s))ds : u \in Y_r \right\}$$

$$= \left\{ T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon)f(s, u(s), Tu(s), Su(s))ds : u \in Y_r \right\}$$

is precompact in $X$, since $T(\cdot)$ is compact. Then, as

$$\int_0^{t-\epsilon} T(t - s)f(s, u(s), Tu(s), Su(s))ds \to \int_0^{t} T(t - s)f(s, u(s), Tu(s), Su(s))ds$$

as $\epsilon \to 0$.

We can conclude that $\left\{ \int_0^{t} T(t - s)f(s, u(s), Tu(s), Su(s))ds : u \in Y_r \right\}$ is precompact in $X$ using the total boundedness. Therefore, for each $t \in [0, T_0]$, $\{(F_1u)(t) : u \in Y_r\}$ is precompact in $X$.

Next, we show that the equicontinuity of $Q = \{(F_1u)(\cdot) : \cdot \in [0, T_0], u \in Y_r\}$. By using the idea of equation (2.5), we can prove the equicontinuity of $\{T(\cdot)[u_0 - g(u)] : \cdot \in [0, T_0], u \in$
$Y_r$. For the second term in $Q$, we let $0 \leq s_1 < s_2 \leq T_0$ and obtain
\[
\left\| \int_0^{s_2} T(s_2 - s)f(s, u(s), Tu(s), Su(s))ds - \int_0^{s_1} T(s_1 - s)f(s, u(s), Tu(s), Su(s))ds \right\|
\]
\[
= \left\| \int_0^{s_1} [T(s_2 - s) - T(s_1 - s)]f(s, u(s), Tu(s), Su(s))ds \right\|
\]
\[
+ \int_{s_1}^{s_2} T(s_2 - s)f(s, u(s), Tu(s), Su(s))ds
\]
\[
\leq \int_0^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
+ M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
(2.6)

If $s_1 = 0$, then the right-hand side of (2.6) can be made small when $s_2$ is small independently of $u \in Y_r$. If $s_1 > 0$, then we can find a small number $\eta > 0$ so that if $s_1 < \eta$, then the right-hand side of (2.6) can be estimated as
\[
\int_0^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
+ M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
\leq 2\eta M \max\{\|f(s, u(s), Tu(s), Su(s))\| : u \in Y_r, s \in [0, T_0]\}
\]
\[
+ M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\|ds,
\]
which can be made small when $s_2 - s_1$ is small independently of $u \in Y_r$.

If $s_1 > \eta$, then the right-hand side of (2.6) can be estimated as
\[
\int_0^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
+ M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
\leq \int_0^{s_1 - \eta} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
+ \int_{s_1 - \eta}^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
+ M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
\[
\leq \int_0^{s_1-\eta} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds \\
+ 2\eta M \max\{\|f(s, u(s), Tu(s), Su(s))\| : u \in Y_r, s \in [0, T_0]\} \\
+ M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\|ds
\]

Now, as \(T(\cdot)\) is compact, \(T(t)\) is operator norm continuous for \(t > 0\). Thus \(T(t)\) is operator norm continuous uniformly for \(t \in [\eta, T_0]\). Therefore, \(\|T(s_2 - s) - T(s_1 - s)\|_{L(X)}\) and hence
\[
\int_0^{s_1-\eta} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \|f(s, u(s), Tu(s), Su(s))\|ds
\]
can be made small when \(s_2 - s_1\) is small independently of \(u \in Y_r\). Thus the function in \(Q\) are equicontinuous. Therefore, \(\mathcal{F}_1\) is a compact operator by the Arzela-Ascoli theorem, and hence \(\mathcal{F}\) is also a compact operator. Now, Schauder’s second fixed point theorem implies that \(\mathcal{F}\) has a fixed point, which gives rise to a mild solution. This completes the proof.

\[\square\]

2.3. \(g\) is not Lipschitz and not compact. Here, we will prove mild solutions under the following hypotheses:

(H7) The function \(f\) is continuous and there exists a constant \(L > 0\) such that
\[
\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad t \in [0, T_0], \quad x, y \in X.
\]

(H8) The function \(I_i : X \to X, i = 1, 2, \ldots, p\), are compact operators, and \(T(\cdot)\) is also compact.

(H9) For each \(u_0 \in X\), there exists a constant \(r > 0\) such that
\[
M(\|u_0\|) + \sup_{\varphi \in Y_r} \|g(\varphi)\| + T_0 \sup_{s \in [0, T_0], \varphi \in Y_r} \|f(s, \varphi(s), T\varphi(s), Su(s))\| \\
+ \sup_{\varphi \in Y_r} \sum_{i=1}^p \|I_i(\varphi(t_i))\| \leq r.
\]

(H10) The function \(g : \mathcal{PC}([0, T_0], X) \to X\) is continuous, maps \(Y_r\) into a bounded set, and there is a \(\delta = \delta(r) \in (0, t_1)\) such that \(g(\varphi) = g(\varphi)\) for any \(\varphi \in Y_r\) with \(\varphi(s) = (s), s \in [\delta, T_0]\).

**Theorem 2.3.** Let (H7)-(H10) be satisfied. Then for every \(u_0 \in X\), for \(t \in [0, T_0]\) the equation
\[
u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s, u(s), Tu(s), Su(s))ds \\
+ \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i))
\]
has at least a mild solution.

**Proof.** For \(\delta = \delta(r) \in (0, t_1)\), set \(Y(\delta) = \mathcal{PC}([\delta, T_0], X)\) = restrictions of functions in \(\mathcal{PC}([0, T_0], X)\) on \([\delta, T_0]\), \(Y_r(\delta) = \{\varphi \in Y(\delta) ; \|\varphi(t)\| \leq r \text{ for } t \in [\delta, T_0]\}\). For \(u \in Y_r(\delta)\)
fixed, we define a mapping $F_u$ on $Y_r$ by

$$ (F_u\varphi)(t) = T(t)[u_0 - g(\tilde{u})] + \int_0^t T(t-s)f(s,\varphi(s),T\varphi(s),S\varphi(s))ds $$

$$ + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [0,T_0], $$

where

$$ \tilde{u}(t) = \begin{cases} u(t), & \text{if } t \in [\delta,T_0], \\ u(\delta), & \text{if } t \in [0,\delta]. \end{cases} $$

By hypothesis (H9), the mapping $F_u$ maps $Y_r$ into itself. Moreover, by hypothesis (H7) we deduce inductively that for $m \in \mathbb{N}$,

$$ \|\left( F_u^m \varphi \right)(t) - \left( F_u^m \varphi \right)(t) \| \leq \frac{[ML(1 + K^* + H^*)]^m}{m!} \sup_{s \in [0,t]} \|\varphi(s) - \varphi(s)\|, $$

$$ t \in [0,T_0], \varphi, \in Y_r, m = 1,2,\ldots. $$

Hence, we infer that for $m$ large enough, the mapping $F_u^m$ is a contractive mapping. Thus, by a well-known extension of the Banach contraction mapping principle, $F_u$ has a unique fixed point $\varphi_u \in Y_r$, i.e.,

$$ \varphi_u(t) = T(t)[u_0 - g(\tilde{u})] + \int_0^t T(t-s)f(s,\varphi_u(s),T\varphi_u(s),S\varphi_u(s))ds $$

$$ + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [0,T_0]. $$

Based on this fact, we define a mapping $\mathcal{F}$ from $Y_r(\delta)$ into itself by

$$ (\mathcal{F}u)(t) = \varphi_u(t), \quad t \in [\delta,T_0] $$

$$ = T(t)[u_0 - g(\tilde{u})] + \int_0^t T(t-s)f(s,\varphi_u(s),T\varphi_u(s),S\varphi_u(s))ds $$

$$ + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [\delta,T_0]. $$

Similarly to the above and [10], we can use compactness and equicontinuity and then apply the Arzela-Ascoli theorem to prove that $\mathcal{F}$ is a compact operator. Therefore, we can use Schauder’s second fixed point theorem to conclude that $\mathcal{F}$ has a fixed point $u_* \in Y_r(\delta)$. We set $u = \varphi_{u_*}$. Then

$$ u(t) = T(t)[u_0 - g(\tilde{u}_*)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds $$

$$ + \sum_{0<t_i<t} T(t-t_i)I_i(u_*(t_i)), \quad t \in [0,T_0]. $$

(2.7)
But \( g(\tilde{u}_*) = g(u) \) and \( u_*(t_i) = u(t_i) \), since \( u_*(t) = (\mathcal{F}u_*)(t) = \varphi_{u_*}(t) = u(t), t \in [\delta, T_0] \), by the definition of \( \mathcal{F} \). This concludes, together with (2.7), that \( u(t) \) is a solution of (1.1)-(1.3). This completes the proof. □

2.4. Classical solutions. Now, we recall the following result.

**Lemma 1.** [1] Assume that \( u_0 \in D(A), q_i \in D(A), i = 1, 2, \ldots, p \) and that \( f \in C^1([0, T_0] \times X \times X \times X, X) \). Then the impulsive equation

\[
\begin{align*}
  u'(t) = Au(t) + f(t, u(t), Tu(t), Su(t)), & \quad 0 < t < T_0, t \neq t_i, \\
  u(0) = u_0, & \\
  \Delta u(t_i) = q_i, & \quad i = 1, 2, 3, \ldots, p.
\end{align*}
\]

has a unique classical solution \( u(\cdot) \) which, for \( t \in [0, T_0] \), satisfies

\[
u(t) = T(t)u_0 + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds + \sum_{0 < t_i < t} T(t - t_i)q_i.
\]

Now, we make the following hypothesis:

(H11) There exists a constant \( L > 0 \) such that

\[
\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad t \in [0, T_0], x, y \in X.
\]

**Theorem 2.4.** Let (H11) be satisfied and \( u(\cdot) \) be a mild solution of (1.1)-(1.3). Assume that \( u_0 \in D(A), I_i(u(t_i)) \in D(A), i = 1, 2, \ldots, p \), and that \( f \in C^1([0, T_0] \times X \times X \times X, X) \). Then \( u(\cdot) \) gives rise to a unique classical solution of (1.1)-(1.3).

**Proof.** Let \( u(\cdot) \) be the mild solution. Let \( q_i = I_i(u(t_i)), i = 1, 2, \ldots, p \). Then from Lemma 2.1,

\[
\begin{align*}
  v'(t) = Av(t) + f(t, v(t), Tv(t), Sv(t)), & \quad 0 < t < T_0, t \neq t_i, \\
  v(0) = u(0) = u_0 - g(u) & \\
  \Delta v(t_i) = q_i, & \quad i = 1, 2, 3, \ldots, p.
\end{align*}
\]

has a unique classical solution \( v(\cdot) \) which satisfies for \( t \in [0, T_0] \),

\[
v(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, v(s), Tv(s), Sv(s))ds + \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)).
\]

Since \( u(\cdot) \) is the mild solution of (1.1)-(1.3), for \( t \in [0, T_0] \),

\[
u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds + \sum_{0 < t_i < t} T(t - t_i)I_i(u(t_i)).
\]

Thus, we get

\[
v(t) - u(t) = \int_0^t T(t - s)[f(s, v(s), Tv(s), Sv(s)) - f(s, u(s), Tu(s), Su(s))]ds,
\]
which gives, by hypothesis (H11) and an application of Gronwall’s inequality,
\[ \|v - u\|_{PC} = 0. \]
This implies that \(u(\cdot)\) gives rise to a classical solution. This completes the proof. \(\square\)

**Remark 2.** If \(S = 0\) in equation (1.1) and assume that the hypothesis (H11) be satisfied and \(u(\cdot)\) be a mild solution of (1.1)-(1.3). Assume that \(u_0 \in D(A), I_i(u(t_i)) \in D(A), i = 1, 2, \ldots, p, \) and that \(f \in C^1([0, T_0] \times X \times X, X)\). Then \(u(\cdot)\) gives rise to a unique classical solution of (1.1)-(1.3).

Now, we consider the simple case on \(g\). If \(g = 0\) in (1.2), then the problem (1.1)-(1.3) is reduced to initial value problem
\[
\begin{align*}
    u'(t) &= Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 \leq t \leq T_0, \quad t \neq t_i, \\
    u(0) &= u_0, \\
    \Delta u(t_i) &= I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0, 
\end{align*}
\]
where \(A, f, I_i\) are defined as in problem (1.1)-(1.3).

Now, we prove the existence and uniqueness of mild and classical solution for the problem (1.4)-(1.6).

For this reason, we list the following hypotheses:

(H1') \(f : [0, T_0] \times X \times X \times X \to X\), and \(I_i : X \to X, i = 1, 2, \ldots, p\) are continuous and there exists constants \(L_1, L_2, L_3 > 0, h_i > 0, i = 1, 2, \ldots, p\), such that
\[
\begin{align*}
    \|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| &\leq L_1\|x_1 - y_1\| + L_2\|x_2 - y_2\| + L_3\|x_3 - y_3\|, \\
    &\quad t \in [0, T_0], \quad x_i, y_i \in X, \quad i = 1, 2, 3, \\
    \|I_i(x) - I_i(y)\| &\leq h_i\|x - y\|, \quad x, y \in X. 
\end{align*}
\]

(H3') The constants \(L, K^*, H^*\) satisfy the inequality
\[
M \left[ LT_0(1 + K^* + H^*) + \sum_{i=1}^{p} h_i \right] < 1.
\]

Now, we state the following theorems without proof. The proof is similar to Theorem 2.1 and Theorem 2.4.

**Theorem 2.5.** Assume that the hypotheses (H1'), (H2) and (H3') are satisfied. Then for every \(u_0 \in X, \) for \(t \in [0, T_0]\) the equation
\[
\begin{align*}
    u(t) &= T(t)u_0 + \int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds \\
    &\quad + \sum_{0 \leq t_i < t} T(t - t_i)I_i(u(t_i))
\end{align*}
\]
has a unique mild solution.
Theorem 2.6. Let (H11) be satisfied and $u(\cdot)$ be a mild solution of (1.4)-(1.6). Assume that $u_0 \in D(A)$, $I_i(u(t_i)) \in D(A)$, $i = 1, 2, \ldots, p$, and that $f \in C^1([0, T_0] \times X \times X \times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of (1.4)-(1.6).

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REFERENCES

Approximation of Quadric Surfaces Using Splines

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Abstract. In this paper we present an approximation method of quadric surface using quartic spline. Our method is based on the approximation of quadratic rational Bézier patch using quartic Bézier patch. We show that our approximation method yields $G_1$ (tangent plane) continuous quartic spline surface. We illustrate our results by the approximation of helicoid-like surface.

1. Introduction

Approximations of conic section and quadric surface by Bézier curve and surface are important tasks in CAGD (Computer Aided Geometric Design) or CAD/CAM. In the recent twenty years, many works on the approximation of conic section including circular arc or quadric surface by Bézier curve or surface with high order approximation have been developed [1, 6, 7, 8, 17, 19, 20, 22].

In particular, Fang presented the quintic Bézier curve approximation of circular arc [10] and of conic section [11], and presented rational quartic representation for conic sections [12]. Floater found the approximation of conic section by quadratic Bézier curve [14] and Bézier curve of odd degree $n$ with error bound analysis [15], and presented the approximation of rational curve by Bézier curve having the optimal approximation order $2n$ [16]. Recently, Ahn has presented the quartic Bézier curve approximation of conic section with error bound analysis [3].

In this paper we present an approximation method of quadric surface using quartic spline using the approximation method in [3]. We find the necessary and sufficient condition that the quartic approximate spline is $G_1$ (tangent plane) continuous. We also obtain the error bound of our approximation method using the error analysis in [15].

In §2, we present the method of the quartic spline approximation of the quadric surface, and find the some properties of our approximation method. In §3, we illustrate our assertions by some examples.
2. APPROXIMATION METHOD OF QUADRIC SURFACE USING QUARTIC BÉZIER SURFACE

In this section we present an approximation method of quadric spline by the \( G^1 \) quartic spline. Quadric spline consists of quadratic rational Bézier patches, and the quadratic rational Bézier patch is the extension of quadratic rational Bézier curves to two variables, which is also called by conic section. Conic section is represented in the standard rational quadratic Bézier form

\[
S(t, s) = \sum_{i=0}^{4} \sum_{j=0}^{4} b_{ij} B_i^4(t) B_j^4(s), \quad t, s \in [0, 1]
\]

where \( b_{ij}, 0 \leq i, j \leq 4 \), are the control points, and \( b_{ij} \) also called control net of \( S(t, s) \). Actually, the quartic rational Bézier patch \( S(t, s) \) can be expressed by

\[
S(t, s) = \sum_{i=0}^{4} \sum_{j=0}^{4} b_{ij} B_i^4(t) B_j^4(s), \quad t, s \in [0, 1]
\]

where \( b_{ij} \), \( 0 \leq i, j \leq 4 \), are the control points, and \( b_{ij} \) also called control net of \( S(t, s) \). Actually, the quartic Bézier approximation method for given quadratic rational Bézier patch means looking for the control net \( b_{ij} \) for the given control points \( p_{ij} \) and weights \( w_{ij} \).

Recently, we found an approximation method of conic section by quartic Bézier curve in [3]. In the method, for given conic section \( r(t) \) having the control points \( p_i, i = 0, 1, 2 \), and weight \( w \), the quartic Bézier approximation \( b(t) = \sum_{i=0}^{4} B_i^4(t)b_i \) has the control points

\[
\begin{align*}
b_0 & = p_0 \\
b_1 & = (1 - \alpha)p_0 + \alpha p_1 \\
b_2 & = \frac{1 - \beta}{2}p_0 + \beta p_1 + \frac{1 - \beta}{2}p_2 \\
b_3 & = \alpha p_1 + (1 - \alpha)p_2 \\
b_4 & = p_2
\end{align*}
\]
where
\[ \alpha = \frac{2w}{w+1} - \frac{3}{4}\beta \]
\[ \beta = \frac{2(w^2 - w + 2 + 2(w - 1)\sqrt{w + 1})}{3w(w + 1)}. \]

Also, we had the error bound analysis[3],
\[ d_H(b, r) \leq E_4(w)|p_0 - 2p_1 + p_2| \quad (2.4) \]
where
\[ E_4(w) = \frac{1}{2^8} \max \left( \frac{1}{w^2}, 1 \right) \frac{|w - 1|^3(w + 2 - 2\sqrt{w + 1})^2}{w^2(w + 1)} \]
for \( 0 < w < \frac{7 + \sqrt{17}}{2} \approx 5.562 \).

Now, we construct the quartic Bézier surface approximation of quadratic rational Bézier surface. Let \( R(t, s) \) be given quadratic rational Bézier patch as in Equation (2.1). We use the quartic Bézier curve approximation method[3] of conic section as two variables case. Thus we have the matrix form of the quartic Bézier curve approximation in Equation (2.3)
\[ (b_0, \cdots, b_4)^T = A_w(p_0, p_1, p_2)^T \quad (2.5) \]
or equivalently, \((b_0, \cdots, b_4) = (p_0, p_1, p_2)A_w^T\), where \( A_w \) is \( 5 \times 3 \) matrix
\[ A_w = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \alpha & 0 \\ \frac{1-\beta}{\beta} & \frac{1-\beta}{\alpha} & 1-\alpha \\ 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \end{pmatrix}. \]

Finally, we present the quartic Bézier surface approximation
\[ S(t, s) = \sum_{i=0}^{4} \sum_{j=0}^{4} b_{ij} B_i(t) B_j(s) \]
where the control net \( b_{ij} \) is
\[ (b_{ij})_{i=0, \cdots, 4}^{j=0, \cdots, 4} = A_{w1}(p_{ij})_{i=0,1,2}^{j=0,1,2} A_{w2}. \]

As an example, we plot the quadratic rational Bézier patch and its quartic Bézier approximation in Figure 1. For more detailed description, we define the intermediate surface \( H(t, s) \) as the quartic Bézier curve approximation of \( R(t, s) \) in view point of the variable \( s \). Then
\[ H(t, s) = \sum_{j=0}^{4} \frac{\sum_{j=0}^{4} h_{ij} w_{0j} B_j^2(t)}{\sum_{j=0}^{4} w_{0j} B_j^2(t)} B_j(s) \quad (2.6) \]
where the control points \((h_{i0}, \cdots, h_{i4})^T = A_{w2}(p_{i0}, p_{i1}, p_{i2})^T\) for \( i = 0, 1, 2 \), or equivalently,
\[ (h_{i0}, \cdots, h_{i4}) = (p_{i0}, p_{i1}, p_{i2}) A_{w2}^T. \quad (2.7) \]
\( \mathbf{H}(t, s) \) is the conic section in direction of \( t \) and quartic Bézier curve in direction of \( s \). Also, \( \mathbf{S}(t, s) \) is obtained as the quartic Bézier curve approximation of \( \mathbf{H}(t, s) \) in view point of the variable \( t \). Thus \((\mathbf{b}_{0j}, \ldots, \mathbf{b}_{4j})^T = A_w (\mathbf{h}_{0j}, \mathbf{h}_{1j}, \mathbf{h}_{2j})^T \) for \( j = 0, \ldots, 4 \).

We will find a sufficient condition that our approximation method yields \( G^1 \) continuous quartic Bézier surface. Let \( \mathbf{R}_l(t, s) \) and \( \mathbf{R}_r(t, s) \) be two consecutive quadratic rational Bézier patches with common boundary curve \( \mathbf{R}_l(1, s) = \mathbf{R}_r(0, s), s \in [0, 1] \). Let \( \mathbf{p}_{lj} \) and \( \mathbf{p}_{rj} \) be the control points of \( \mathbf{R}_l(t, s) \) and \( \mathbf{R}_r(t, s) \), respectively. Clearly, \( \mathbf{p}_{lj} = \mathbf{p}_{r0j}, j = 0, 1, 2 \). If the two patches satisfy

\[
\mathbf{p}_{lj}^1 \mathbf{p}_{lj}^2 = \lambda \mathbf{p}_{r0j} \mathbf{p}_{r1j}, \quad (j = 0, 1, 2)
\]

(2.8) for some constant \( \lambda > 0 \), then our approximation method yields \( G^1 \) continuous quartic spline.

**Proposition 2.1.** If any consecutive quadratic rational Bézier patches satisfy Equation (2.8) on their common boundary, then our approximation method yields \( G^1 \) continuous quartic spline.

**Proof.** Let \( \mathbf{S}_l(t, s) \) and \( \mathbf{S}_r(t, s) \) be the quartic Bézier surface approximations of \( \mathbf{R}_l(t, s) \) and \( \mathbf{R}_r(t, s) \), respectively, with common boundary curve \( \mathbf{S}_l(1, s) = \mathbf{S}_r(0, s), s \in [0, 1] \). To show that they are \( G^1 \) continuous at the common boundary, it is sufficient[13, 18, 21] to show that \( \mathbf{b}_{lj}^1 \mathbf{b}_{lj}^2 = \lambda \mathbf{b}_{r0j} \mathbf{b}_{r1j}, (j = 0, \ldots, 4) \), where \( \mathbf{b}_{lj} \) and \( \mathbf{b}_{rj} \) are the control nets of \( \mathbf{S}_l \) and \( \mathbf{S}_r \), respectively.
Let \( h^l_{ij} \) and \( h^r_{ij} \), \((i = 0, 1, 2 \text{ and } j = 0, \cdots, 4)\), be the control net of intermediate surfaces \( H^l \) and \( H^r \), respectively. Since \( w^l_2 = w^r_2 \), we obtain \( A^l_{w^l_2} = A^r_{w^r_2} \) and \( h^l_{2j} = h^r_{0j} \) for \( j = 0, \cdots, 4 \).

Let \( A_{w^l_2} = (a_{ij})_{i=0,1,2} \).

By Equation (2.7), for \( i = 0, 1, 2 \) and \( j = 0, \cdots, 4 \),
\[
    h^l_{1j}h^l_{2j} = \sum_{k=0}^{2} a_{jk}p^l_{2k} - \sum_{k=0}^{2} a_{jk}p^l_{1k} = \sum_{k=0}^{2} a_{jk}p^l_{1k}p^l_{2k}
\]
\[
    = \lambda \sum_{k=0}^{2} a_{jk}p^r_{0k}p^r_{1k} = \lambda h^r_{0j}h^r_{1j}.
\]

For \( i = 0, \cdots, 4 \) and \( j = 0, \cdots, 4 \), \( b^l_{ij} = \sum_{k=0}^{2} a_{ik}h^l_{kj} \). And for each \( j = 0, \cdots, 4 \),
\[
    b^l_{3j}b^l_{4j} = \sum_{k=0}^{2} a_{4k}h^l_{kj} - \sum_{k=0}^{2} a_{3k}h^l_{kj} = \sum_{k=0}^{2} (a_{4k} - a_{3k})h^l_{kj}
\]
\[
    b^r_{0j}b^r_{1j} = \sum_{k=0}^{2} a_{1k}h^r_{kj} - \sum_{k=0}^{2} a_{0k}h^r_{kj} = \sum_{k=0}^{2} (a_{1k} - a_{0k})h^r_{kj}
\]

Since \( a_{40} - a_{30} = a_{12} - a_{02} = 0, a_{41} - a_{31} = a_{10} - a_{00} = -\alpha \) and \( a_{42} - a_{32} = a_{11} - a_{01} = \alpha \),
we have
\[
    b^l_{3j}b^l_{4j} = \alpha (h^l_{2j} - h^l_{1j}) = \alpha h^l_{1j}h^l_{2j}
\]
\[
    b^r_{0j}b^r_{1j} = \alpha (h^r_{1j} - h^r_{0j}) = \alpha h^r_{0j}h^r_{1j}
\]

By Equation (2.9) we finally have
\[
    b^l_{3j}b^l_{4j} = \lambda b^r_{0j}b^r_{1j}.
\]

**Corollary 2.2.** If we use this approximation method for a quadric surface, then the \( G^1 \) continuity at the boundary is automatically achieved.

We also present the error analysis of our approximation method as the following proposition. Under the assumption Equation (2.2), the proof of the following proposition can be obtained by the similar way in that of Theorem 4.2 in Floater[15] for odd degree. Thus we omit the proof.

**Proposition 2.3.** For given quadric surface \( R(t, s) \), the quartic Bézier surface approximation \( S(t, s) \) has the error bound
\[
    d_H(R, S) \leq E_4(w_2) \max_{i=0,1,2} |p_{i0} - 2p_{i1} + p_{i2}| + E_4(w_1) \max_{j=0,1,2} |p_{0j} - 2p_{1j} + p_{2j}|,
\]
where \( d_H(R, S) \) is the Hausdorff distance between two surfaces \( R \) and \( S \). (For more knowledge of the Hausdorff distance refer to [4, 9, 15].)
3. EXAMPLES AND COMMENTS

In this section we apply our approximation method to two examples. One is to approximate the quadratic rational Bézier patch by quartic Bézier surface. Let \( R(t, s) \) be the quadratic rational Bézier patch with the control points

\[
\begin{pmatrix}
0, 5, \frac{\pi}{2} - 1 \\
-5, 5, \frac{3\pi}{4} - 1 \\
-5, 0, \pi - 1
\end{pmatrix}
\]

and \( w_1 = w_2 = \frac{1}{\sqrt{2}} \), as shown in Figure 1(a). Using our approximation method, the quartic Bézier surface \( S(t, s) \) is obtained as shown in Figure 1(b), and Proposition 2.2 yields the upper bound of error

\[
d_H(R, S) \leq E_4(\frac{1}{\sqrt{2}}) \times (6\sqrt{2} + \sqrt{3}) \approx 2.08 \times 10^{-5}.
\]

The other is to approximate the helix-like surface by quartic spline. The helix-like surface was constructed by the quadratic rational spline such as in [2]. The quadratic rational spline satisfies the condition in (2.8) with \( \lambda = 1 \) and \( w_0 = w_1 = \frac{1}{\sqrt{2}} \). As shown in Figure 2(a), the quadratic rational spline consists of \( 4 \times 2 \) quadratic rational Bézier patches. Using our approximation method, we have the quartic spline consisting of \( 4 \times 2 \) quartic Bézier patches as shown in Figure 2(b). On each common boundaries between consecutive quartic Bézier patches, they are \( G^1 \) continuous. Thus the quartic Bézier spline is an \( G^1 \) approximation of the quadric spline.

Our approximation method can be also applied to approximate spheres and torii. It is easy to see that the approximate quartic splines of these surface are also \( G^1 \) continuous and have very small upper bounds of error.

ACKNOWLEDGEMENTS

The authors are very grateful to the anonymous referees for their valuable suggestions.

REFERENCES

Figure 2. (a) The helix-like surface constructed by $4 \times 2$ quadratic rational Bézier patches, whose boundaries are plotted by thick lines. (b) The quartic approximate spline constructed by $4 \times 2$ quartic Bézier patches, whose boundaries are also plotted by thick lines. The quartic spline is $G^1$ continuous on every common boundaries of the consecutive quartic Bézier patches.

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FAST AND AUTOMATIC INPAINTING OF BINARY IMAGES USING A
PHASE-FIELD MODEL

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ABSTRACT. Image inpainting is the process of reconstructing lost or deteriorated parts of images using information from surrounding areas. We propose a computationally efficient and fast phase-field method which uses automatic switching parameter, adaptive time step, and automatic stopping of calculation. The algorithm is based on an energy functional. We demonstrate the performance of our new method and compare it with a previous method.

1. INTRODUCTION

Image inpainting[4, 1, 6] is the process of reconstructing lost or deteriorated parts of images using information from surrounding areas. Let $f(x)$, where $x = (x, y)$, be a given image in a domain $\Omega$. Let $c(x, t)$ be a phase-field which is governed by the following modified Cahn-Hilliard (CH) equation[5]:

\begin{align}
ct &= \Delta \mu + \lambda(x)(f(x) - c), \\
\mu &= F'(c) - \epsilon^2 \Delta c,
\end{align}

where $F(c) = 0.25c^2(1 - c)^2$. In the examples considered here, we use binary images in which most of the pixels are either exactly black or white. Eqs. (1.1) and (1.2) are the modified CH equation, due to the added fidelity term $\lambda(x)(f(x) - c)$ [2]. Image inpainting using phase-field methods is recently investigated by authors in [2, 3]. It is a good starting point for using partial differential equations in inpainting images, however, we found there are a couple of defects. First of all, switching parameter $\epsilon$ and stopping the calculation are done by trial and error. Furthermore, large time step $\Delta t$ is more or less time step rescaling and it turns out that it is equivalent to using smaller time step than usual usage. In this paper, we propose a phase-field method which uses automatic varying $\epsilon$, adaptive time step, and a stopping criterion based on energy functional.

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The outline of this paper is the following. In Sec. 2, the discrete equations for the governing equations are presented. In Sec. 3, we present computational examples. We propose a new automatic controlled algorithm in Sec. 4. Finally, in Sec. 5, conclusions are drawn.

2. Discrete equations and a numerical solution

In this section, we present fully discrete schemes for the CH equation in two dimensional space, i.e., \( \Omega = (a, b) \times (c, d) \). Let \( N_x \) and \( N_y \) be positive even integers, \( h = (b - a)/N_x \) be the uniform mesh size, and \( \Omega_h = \{(x_i, y_j) : x_i = (i - 0.5)h, y_j = (j - 0.5)h, 1 \leq i \leq N_x, 1 \leq j \leq N_y \} \) be the set of cell-centers. Let \( c_{ij} \) and \( \nu_{ij} \) be approximations of \( c(x_i, y_j) \) and \( \nu(x_i, y_j) \). Then, a semi-implicit time and centered difference space discretization of Eqs. (1.1) and (1.2) is

\[
\frac{c_{ij}^{n+1} - c_{ij}^n}{\Delta t} = \Delta_d \mu_{ij}^{n+\frac{1}{2}} + \lambda_{ij}(f_{ij} - c_{ij}^n),
\]

\[
\mu_{ij}^{n+\frac{1}{2}} = \varphi(c_{ij}^{n+1}) - \frac{c^n}{4} - \epsilon^2 \Delta_d c_{ij}^{n+1},
\]

where \( \varphi(c) = F'(c) + \frac{c}{4} \).

We can rewrite Eqs. (2.1) and (2.2) as follows:

\[
\frac{c_{ij}^{n+1} - c_{ij}^n}{\Delta t} = \Delta_d \nu_{ij}^{n+1} - \frac{1}{4} \Delta_d c_{ij}^n + \lambda_{ij}(f_{ij} - c_{ij}^n),
\]

\[
\nu_{ij}^{n+1} = \varphi(c_{ij}^{n+1}) - \epsilon^2 \Delta_d c_{ij}^{n+1}.
\]

For completeness, the numerical solution using a nonlinear multigrid method is described. We use nonlinear Full Approximation Storage (FAS) multigrid method to solve the nonlinear discrete system (2.3) and (2.4) at the implicit time level. The nonlinearity is treated using one step of Newton’s iteration and a pointwise Gauss-Seidel relaxation scheme is used as the smoother in the multigrid method. See the reference text [10] for additional details and backgrounds. The algorithm of the nonlinear multigrid method for solving the discrete CH system is:

First, let us rewrite Eqs. (2.3) and (2.4) as follows.

\[\text{NSO}(c^{n+1}, \nu^{n+1}) = (\phi^n, \ n),\]

where \( \text{NSO}(c^{n+1}, \nu^{n+1}) = \left( \frac{c^{n+1}}{\Delta t} - \Delta_d \nu^{n+1} - \varphi(c^{n+1}) + \epsilon^2 \Delta_d c^{n+1}, \ 0 \right) \)

and the source term is

\[\left( \phi^n, \ n \right) = \left( \frac{c^n}{\Delta t} + \lambda(f - c^n) - \frac{1}{4} \Delta_d c^n, \ 0 \right).\]

In the following description of one FAS cycle, we assume a sequence of grids \( \Omega_k \) (\( \Omega_{k-1} \) is coarser than \( \Omega_k \) by factor 2). Given the number \( \beta \) of pre- and post-smoothing relaxation sweeps, an iteration step for the nonlinear multigrid method using the V-cycle is formally

\[\text{NSO}(c^{n+1}, \nu^{n+1}) = (\phi^n, \ n),\]

where \( \text{NSO}(c^{n+1}, \nu^{n+1}) = \left( \frac{c^{n+1}}{\Delta t} - \Delta_d \nu^{n+1} - \varphi(c^{n+1}) + \epsilon^2 \Delta_d c^{n+1}, \ 0 \right) \)

and the source term is

\[\left( \phi^n, \ n \right) = \left( \frac{c^n}{\Delta t} + \lambda(f - c^n) - \frac{1}{4} \Delta_d c^n, \ 0 \right).\]
written as follows [10]:

**FAS multigrid cycle**

\[
\{c^{m+1}_k, \nu^{m+1}_k\} = FAScycle(k, c^m_k, \nu^m_k, NSO_k, \phi^n_k, n_k, \beta).
\]

That is, \(\{c^m_k, \nu^m_k\}\) and \(\{c^{m+1}_k, \nu^{m+1}_k\}\) are the approximations of \(c^{n+1}(x_i, y_j)\) and \(\nu^{n+1}(x_i, y_j)\) before and after a FAS cycle. Now, define the FAS cycle.

1) **Presmoothing**

\[
\{c^m_k, \nu^m_k\} = SMOOTH^\beta(c^m_k, \nu^m_k, NSO_k, \phi^n_k, n_k),
\]

which means performing \(\beta\) smoothing steps with the initial approximations \(c_0^m, \nu_0^m\), source terms \(\phi^n_k, n_k\), and **SMOOTH** relaxation operator to get the approximations \(c^m_k, \nu^m_k\). One **SMOOTH** relaxation operator step consists of solving the system (2.4) and (2.5) given below by \(2 \times 2\) matrix inversion for each \(i\) and \(j\). Here, we derive the smoothing operator in two dimensions. Rewriting Eq. (2.3), we get

\[
\frac{c_{ij}^{n+1}}{\Delta t} + 4\Delta t_{ij} + \frac{\nu_{ij}^{n+1}}{h^2} = \phi_{ij}^n + \frac{\nu_{i+1,j}^{n+1} + \nu_{i-1,j}^{n+1} + \nu_{i,j+1}^{n+1} + \nu_{i,j-1}^{n+1} - 4c_{ij}^{n+1}}{h^2}.
\]

Since \(\varphi(c_{ij}^{n+1})\) is nonlinear with respect to \(c_{ij}^{n+1}\), we linearize \(\varphi(c_{ij}^{n+1})\) at \(c_{ij}^m\), i.e.,

\[
\varphi(c_{ij}^{n+1}) \approx \varphi(c_{ij}^m) + \frac{d\varphi(c_{ij}^m)}{dc}(c_{ij}^{n+1} - c_{ij}^m).
\]

After substitution of this into (2.4), we get

\[
- \left(\frac{d\varphi(c_{ij}^m)}{dc} + \frac{4c_{ij}^m}{h^2}\right)c_{ij}^{n+1} + \nu_{ij}^{n+1} = \frac{n_{ij} + \varphi(c_{ij}^m) - \frac{d\varphi(c_{ij}^m)}{dc}c_{ij}^m}{h^2}(c_{ij}^{n+1} - c_{ij}^m) - \frac{\nu_{i+1,j}^{n+1} + \nu_{i-1,j}^{n+1} + \nu_{i,j+1}^{n+1} + \nu_{i,j-1}^{n+1} - 4c_{ij}^{n+1}}{h^2}.
\]

Next, we replace \(c_{kl}^{n+1}\) and \(\nu_{kl}^{n+1}\) in the Eqs. (2.5) and (2.6) with \(c_{kl}^m\) and \(\nu_{kl}^m\) if \(k \leq i\) and \(l \leq j\), otherwise with \(c_{kl}^m\) and \(\nu_{kl}^m\), i.e.,

\[
\frac{c_{ij}^m}{\Delta t} + 4\Delta t_{ij} + \frac{\nu_{ij}^m}{h^2} = \phi_{ij}^n + \frac{\nu_{i+1,j}^m + \nu_{i-1,j}^m + \nu_{i,j+1}^m + \nu_{i,j-1}^m}{h^2},
\]

\[
- \left(\frac{d\varphi(c_{ij}^m)}{dc} + \frac{4c_{ij}^m}{h^2}\right)c_{ij}^m + \nu_{ij}^m = \frac{n_{ij} + \varphi(c_{ij}^m) - \frac{d\varphi(c_{ij}^m)}{dc}c_{ij}^m}{h^2}(c_{ij}^{n+1} - c_{ij}^m) - \frac{\nu_{i+1,j}^m + \nu_{i-1,j}^m + \nu_{i,j+1}^m + \nu_{i,j-1}^m}{h^2}.
\]

2) **Compute the defect**

\[
(d_{1,k}, d_{2,k}) = (\phi^n_k, n_k) - NSO_k(c_{ij}^m, \nu_{ij}^m).
\]

3) **Restrict the defect and**

\[
(d_{1,k-1}, d_{2,k-1}) = I_{k-1}(d_{1,k}, d_{2,k}), (c_{ij}^m, \nu_{ij}^m) = I_{k-1}(c_{ij}^m, \nu_{ij}^m).
\]
The restriction operator $I_{k}^{k-1}$ maps $k$-level functions to $(k-1)$-level functions.

\[ d_{k-1}(x_i, y_j) = I_{k}^{k-1}d_k(x_i, y_j) = \frac{1}{4}[d_k(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}) + d_k(x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}}) + d_k(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}) + d_k(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})]. \]

4) Compute the right-hand side

\[(\phi_{k-1}^n, \frac{n}{k-1}) = (a_{1,m}^{n,k-1}, a_{2,m}^{n,k-1}) + NSO_{k-1}(c_{m}^{n,k-1}, \hat{v}_{m}^{n,k-1}). \]

5) Compute an approximate solution \{\hat{c}_{m}^{n,k-1}, \hat{v}_{m}^{n,k-1}\} of the coarse grid equation on $\Omega_{k-1}$, i.e.

\[ NSO_{k-1}(c_{m}^{n,k-1}, \nu_{m}^{n,k-1}) = (\phi_{k-1}^n, \frac{n}{k-1}). \tag{2.7} \]

If $k = 1$, we apply the smoothing procedure in (1) to obtain the approximate solution. If $k > 1$, we solve (2.7) by performing a FAS $k$-grid cycle using \{\hat{c}_{m}^{n,k-1}, \hat{v}_{m}^{n,k-1}\} as an initial approximation:

\[ \{\hat{c}_{m}^{n,k-1}, \hat{v}_{m}^{n,k-1}\} = \text{FAScycle}(k-1, \hat{c}_{m}^{n,k-1}, \hat{v}_{m}^{n,k-1}, NSO_{k-1}, \phi_{k-1}^n, \frac{n}{k-1}, \beta). \]

6) Compute the coarse grid correction (CGC):

\[ \hat{v}_{m}^{k,k-1} = c_{m}^{k,k-1} - c_{m}^{k,k-1}, \hat{v}_{m}^{2,k-1} = v_{m}^{k,k-1} - \hat{v}_{m}^{k,k-1}. \]

7) Interpolate the correction: \[ c_{m}^{k} = I_{k}^{k-1}c_{m}^{k,k-1}, v_{m}^{k} = I_{k}^{k-1}v_{m}^{k,k-1}. \]

Here, the coarse values are simply transferred to the four nearby fine grid points, i.e. $v_{k}(x_i, y_j) = I_{k}^{k-1}v_{k-1}(x_{i, y_j}) = v_{k-1}(x_{i+\frac{1}{2}, y_{j+\frac{1}{2}}})$ for $i$ and $j$ odd-numbered integers.

8) Compute the corrected approximation on $\Omega_k$

\[ c_{m}^{k,k-1} = c_{m}^{k,k-1} + \hat{v}_{m}^{k,k-1}, \nu_{m}^{k,k-1} = \nu_{m}^{k,k-1} + \hat{v}_{m}^{k,k-1}. \]

9) Postsmoothing

\[ \{c_{m}^{k+1,k}, \nu_{m}^{k+1,k}\} = SMOOTH^{\beta}(c_{m}^{k,k}, \nu_{m}^{k,k}, \text{after CGC}, \nu_{m}^{k,k}, \text{after CGC}, NSO_{k}, \phi_{k}^n, \frac{n}{k}). \]

This completes the description of a nonlinear FAS cycle for the discrete modified CH equation. Let us define a maximum norm

\[ \|c_{m}\|_{\infty} = \max_{1 \leq i \leq N_x, 1 \leq j \leq N_y} |c_{i,j}|. \]

3. Computational Examples

In this section, we will compare the numerical scheme of the previous Bertozzi’s paper [2] with our scheme. First we refer to the discrete Eq. (9) in the paper [2]

\[ \frac{u^{n+1} - u^n}{\Delta t} + \varepsilon \Delta^2_du^{n+1} - C_1 \Delta_du^{n+1} + C_2 u^{n+1} = \Delta_d \left( \frac{1}{\varepsilon} W'(u^n) \right) + \lambda(x)(f(x) - u^n) - C_1 \Delta_d u^n + C_2 u^n, \tag{3.1} \]
where \( W(u) = u^2(1 - u)^2 \) and the constants \( C_1 \) and \( C_2 \) are large enough so that the equation is convex for the range of \( u \) in the simulation. Next, we rewrite Eq. (3.1) as follows:

\[
\frac{u^{n+1} - u^n}{4\Delta t} = \Delta d \left( \frac{1}{4} W'(u^n) - \frac{\varepsilon^2}{4} \Delta d u^{n+1} + \frac{\varepsilon}{4} C_1 (u^{n+1} - u^n) \right) + \frac{\varepsilon}{4} \lambda(x)(f(x) - u^n).
\]

Table 1 shows that two schemes are equivalent.

**Table 1.** Equivalent forms of two schemes.

<table>
<thead>
<tr>
<th></th>
<th>Bertozzi’s numerical scheme</th>
<th>Our numerical scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{u^{n+1} - u^n}{4\Delta t} )</td>
<td>( \Delta d \left( \frac{1}{4} W'(u^n) - \frac{\varepsilon^2}{4} \Delta d u^{n+1} + \frac{\varepsilon}{4} C_1 (u^{n+1} - u^n) \right) + \frac{\varepsilon}{4} \lambda(x)(f(x) - u^n) )</td>
<td>( \frac{c^{n+1} - c^n}{\Delta t} ) = ( \Delta d \left( F'(c^n) - \varepsilon^2 \Delta d c^{n+1} + \frac{1}{4} (c^{n+1} - c^n) \right) + \lambda(x)(f(x) - c^n) )</td>
</tr>
</tbody>
</table>

We perform two test problems such as inpaintings of a double stripe and of a cross to show that two schemes are equivalent.

![Figure 1](image.png)

**Figure 1.** (a) Bertozzi’s result. (b) Our result. Left column is initial data, middle column is results at iteration 50, and right column is results at iteration 700.

### 3.1. Inpainting of a double stripe.

In this test problem, the computational domain \( \Omega = (0, 1.28) \times (0, 1.28) \) and 128 x 128 mesh size are taken. The initial configurations are shown in the first column in Fig. 1. In the first and second rows, figures are results using the previous
scheme and our proposed scheme, respectively. The gray region in the initial configuration denotes the inpainting region. In the previous scheme, we start calculations with a large $\epsilon = 0.8$ value, then switch its value to $\epsilon = 0.01$ after 50 iterations, and stop the calculation at 700 iterations. We repeat the same calculations with equivalent values which are summarized in Table 2. The prime notations of the parameters are from previous method and right arrow implies changing values when switching happens. By comparing two results, we see that our scheme is equivalent to the previous Bertozzi’s scheme.

<table>
<thead>
<tr>
<th>Previous Value</th>
<th>Current Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta t'$</td>
<td>$\Delta t = \frac{4\Delta t'}{e'(C'_2\Delta t' + 1)}$</td>
</tr>
<tr>
<td>$\lambda'$</td>
<td>$\lambda = \frac{1}{4}\lambda'$</td>
</tr>
<tr>
<td>$\epsilon'$</td>
<td>$\epsilon = \frac{\epsilon}{2}$</td>
</tr>
</tbody>
</table>

3.2. **Inpainting of a cross.** For the second test problem, the initial configuration is a cross with an inpainting region as shown in the first column in Fig. 2. The computational domain $\Omega = (0, 1.28) \times (0, 1.28)$ and $128 \times 128$ mesh size are taken. In the first and second rows, figures are results using the previous scheme and our proposed scheme, respectively. In the previous scheme, we start calculations with a large $\epsilon = 0.8$ value, switch its value to $\epsilon = 0.01$ after 300 iterations, and stop the calculation at 1000 iterations. We repeat the same calculations with equivalent values which are summarized in Table 3. By comparing two results, we see that our scheme is equivalent to the previous Bertozzi’s scheme.

<table>
<thead>
<tr>
<th>Previous Value</th>
<th>Current Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta t'$</td>
<td>$\Delta t = \frac{4\Delta t'}{e'(C'_2\Delta t' + 1)}$</td>
</tr>
<tr>
<td>$\lambda'$</td>
<td>$\lambda = \frac{1}{4}\lambda'$</td>
</tr>
<tr>
<td>$\epsilon'$</td>
<td>$\epsilon = \frac{\epsilon}{2}$</td>
</tr>
</tbody>
</table>

From these two test problems, we can conclude that two schemes are equivalent. However, in the previous algorithm, when to switch the parameters and when to stop the calculation are from trial and error. Therefore, it is our main purpose to propose an automatic switching and stopping algorithm based on an energy functional.
In this section, we propose an automatic switching and stopping algorithm based on an energy functional. Let us reconsider the first test problem which is the inpainting of a double stripe. Fig. 3 shows the temporal evolution of contour plots of the phase-field. Around the switching time, we can observe the phase separation which means that the inpainting region separates into white and dark regions. Then we switched the $\epsilon$ parameter. Next, let us take a look at the time evolution of the energy functional. In Fig. 4, the energy is increased at the initial stage and it is decreased. Similar phenomena in the second test problem which is the inpainting of a cross are observed. See the Figs. 5 and 6. Therefore, it is natural to monitor the energy functional for switching the parameter and stopping the calculation.

4.1. Inpainting of damaged images. Fig. 7(a) and (c) show the initial images of damaged double stripes and cross and Fig. 7(b) and (d) show the results with our proposed automatic algorithm to a double stripe and a cross inpainting problems. In the case of double stipes (see Fig. 7(a) and (b)), it only requires 16 iterations to recover the damaged images. Also in the other case (cross image, see Fig. 7(c) and (d)) we obtain the recovered image after 15 iterations. We use $\Delta t = 1/128$, $\epsilon = 0.038424$, $\lambda = 3/\Delta t$ and when $\text{diff}$, the difference of energies of $e^{n+1}$ and $e^n$, is smaller than $tol1 (= 0.08)$ is equal to 3 times, that is, at the number of iteration is 3 in both cases, we switch the parameter as $\Delta t' = 2.0\Delta t$, $\epsilon' = 0.0875\epsilon$, $\lambda' = 1.8/\Delta t'$. When $\text{diff}$ is smaller than prescribed tolerance, $tol2 (= 1.0E − 6)$, we stop this algorithm.
4.2. Inpainting of obscured text. In order to recover obscured text (see Fig. 8(a)), we use our inpainting algorithm. Fig. 8(a) shows the initial image which is obscured text by lines...
and Fig. 8(b) shows the recover result image. We use the parameter as $\Delta t = 1/128$, $\epsilon = 0.038424$, $\lambda = 3/\Delta t$ and $diff$ is smaller than $tol1 (= 0.08)$ is equal to 3 times, that is, at the number of iteration is 4 in this case, we switch the parameter as $\Delta t' = 1.8\Delta t$, $\epsilon' = 8\epsilon$. 

**Figure 5.** Temporal evolution of contour plots of the phase-field for the cross inpainting.

**Figure 6.** Temporal evolution of contour plots of the energy functional for the cross inpainting.
The proposed automatic switching and stopping algorithm is as follows. Algorithm:
Given a maximum iteration number $N$, tolerances $tol_1$ and $tol_2$
• Set $k = 1$, $flag = 0$.
• While ($k \leq N$) do Steps 1-4
  Step 1 Compute $c^{n+1}$ from $c^n$ by solving Eqs. (2.3) and (2.4).
  Step 2 Check the difference of energies of $c^n$ and $c^{n+1}$
    \[\text{diff} = |\mathcal{E}(c^n) - \mathcal{E}(c^{n+1})|\]
    If ($flag < 3$ and diff < $tol_1$)
      $flag = flag + 1$
    End
  Step 3 Switch parameters and reset data
    If ($flag = 3$)
      If ($c^{n+1} > 0.5$)
        $c^{n+1} = 1$
      Else
        $c^{n+1} = 0$
      End
      $\Delta t = 2\Delta t$
      do Step 1 twice
      $\epsilon = 0.0875\epsilon$
      $\lambda = 1.8/\Delta t'$
    End
  Step 4 Stop loop
    If (diff < $tol_2$ and $flag = 3$)
      Stop loop
    End
End

0.0875$\epsilon$, $\lambda' = 1.8/\Delta t'$. Our automatic switching method of the modified CH equation is faster than the previous model.

5. CONCLUSION

We have shown that our automatic switching algorithm achieves faster inpainting of binary images than the previous trial and error algorithm. Therefore, inpainting region is reconstructed more efficiently and faster than previous method. The developed automatic algorithm can be applied to calculating option pricing such as the Black-Scholes equations accurately and efficiently.
Figure 7. Recovery of damaged images. The computational domain is $\Omega = (0, 1.28) \times (0, 1.28)$ and mesh size is $128 \times 128$.

Figure 8. Recovery of damaged text. The computational domain is $\Omega = (0, 2.56) \times (0, 1.28)$ and mesh size is $256 \times 128$.

Acknowledgments

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References
