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ENERGY DECAY RATES FOR THE KELVIN-VOIGT TYPE WAVE EQUATION WITH ACOUSTIC BOUNDARY

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ABSTRACT. In this paper, we study uniform exponential stabilization of the vibrations of the Kelvin-Voigt type wave equation with acoustic boundary in a bounded domain in R^n. To stabilize the systems, we incorporate separately, the internal material damping in the model as like Gannesh C. Gorain [1]. Energy decay rates are obtained by the exponential stability of solutions by using multiplier technique.

1. INTRODUCTION

In this paper, we consider the uniform stability of a mathematical problems governed by the following a nonlinear wave equations of the Kelvin-Voigt type with acoustic boundary conditions:

$$u'' = (a^2 + b\int_{\Omega} |\nabla u|^2 dx)\triangle u + 2\lambda\triangle u' \text{ in } \Omega \times R^+, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_0 \times R^+, \tag{1.2}$$

$$(a^2 + b\int_{\Omega} |\nabla u|^2 dx)\frac{\partial u}{\partial \nu} + \underline{2\lambda\frac{\partial u'}{\partial \nu}} = y' \text{ on } \Gamma_1 \times R^+, \tag{1.3}$$

$$u' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+, \tag{1.4}$$

$$u(0) = u_0, u'(0) = u_1 \text{ in } \Omega, \tag{1.5}$$

where Ω is a bounded, connected set in $R^n (n \geq 1)$ having a smooth boundary $\Gamma = \partial\Omega$, consisting of two parts Γ_0 and Γ_1 such that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \Gamma$. Primes denote the time derivative, Δ the Laplacian in R^n taken in space variables, ν the unit normal of Γ pointing towards exterior of Ω and $R^+ := (0, \infty)$. The parameters $\lambda > 0$ is a small internal material damping coefficient, and $a > 0, b > 0$ are constant real

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numbers. p and q are functions satisfying some conditions to be specified later. Physically, the integro-differential equations (1.1)-(1.5) occurs in the study of vibrations of damped flexible space structures in bounded domain in R^n. The term $2\lambda\Delta u'$ is the internal material damping of Kelvin-Voigt type of the structure. The boundary conditions considered here are of mixed Dirichlet and Neumann type and acoustic boundary. The analytical studies in the area of stabilization of distributed parameter system is currently of interest in view of application to vibration control of various structural elements. The phenomenon was first observed by Hunton as reported by Harrison [7]. The nonlinear model like (1.1) for transverse vibrations was originally derived by Kirchhoff [3]. Beale and Rosencrans[5] introduced acoustic boundary conditions of the general form

$$\frac{\partial u}{\partial \nu} = y' \text{ on } \Gamma_1 \times R^+$$
$$\gamma u' + m(x)y'' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+.$$

Recently, wave equations with acoustic boundary conditions have been treated by many authors [4, 5, 6, 8, 9, 11, 12, 13]. In [4], the authors studied the nonlinear wave equations

$$u'' - M(\int_\Omega |u|^2 dx)\Delta u + |u'|^\alpha u' = 0 \text{ in } \Omega \times R^+,$$
$$u = 0 \text{ on } \Gamma_0 \times R^+,$$
$$\frac{\partial u}{\partial \nu} = y' \text{ on } \Gamma_1 \times R^+,$$
$$\gamma u' + m(x)y'' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+.$$

They proved the existence of solutions, but did not give decay rate for solutions. As regards uniform decay rates for solutions to problems with acoustic boundary conditions, there are not much literature [2],[4],[9],[11],[12]. Frota and Larkin[9] established global solvability and decay estimates for a linear wave equation with boundary conditions

$$\frac{\partial u}{\partial \nu} = h(x)y' \text{ on } \Gamma_1 \times R^+$$
$$\gamma u' + p(x)y' + q(x)y = 0 \text{ on } \Gamma_1 \times R^+.$$

In this paper we are motivated by boundary conditions of Park[4] and results of Gorain[1] and Kang[10]. The aim of this paper is to study uniform stabilization of the generalized nonlinear Kirchhoff type wave equations governed by (1.1)-(1.5) with the mixed boundary conditions. To our knowledge, this problem has not been considered by predecessors and is studied first, as a Kirchhoff of kelvin-Voigt type model, in this paper. The plan of this paper as follows. In section 2, we give some notation, some conditions and material needed for our work. In section 3, we drive the uniform stability on account of internal material damping of Kelvin-Voigt type with acoustic boundary. The notation used in this paper is standard and can be found in Gorain[1].

2. PRELIMINARIES AND SOME NOTATIONS

In this section, we present some notations and some material in the proof of our result. Throughout this paper, we use the notation $V = \{u \in H^1(\Omega) : u = 0 \; on \; \Gamma_0\}$ the subspace of the classical Sobolev space $H^1(\Omega)$ of real valued functions of order one. Let k be the smallest positive constant independent of t (depends only on Ω) satisfying the Poincare inequality

$$\int_\Omega u^2 dx \leq k \int_\Omega |\nabla u|^2 dx \text{ for every } u \in V. \tag{2.1}$$

And also let $\overline{k}$ be the smallest positive constant independent of t (depends only on Γ_1) satisfying the embedding inequality

$$\int_{\Gamma_1} u^2 d\Gamma \leq \overline{k} \int_\Omega |\nabla u|^2 dx \text{ for every } u \in V. \tag{2.2}$$

For the functions p and q, we assume that $p, q \in C(\Gamma_1)$ and $p(x) > 0$ and $q(x) > 0$ for all $x \in \Gamma_1$. This assumption implies that there exist positive constants $p_i, q_i (i = 0, 1)$ such that

$$p_0 \leq p(x) \leq p_1, \; q_0 \leq q(x) \leq q_1 \; \text{ for all } x \in \Gamma_1. \tag{2.3}$$

By using Gälerkin's approximation and the methods of Gorain[1] and Park[12], we can obtain the following existence result for the solution subject to (1.1)-(1.5) under the conditions on p and q as above. For the initial data $(u_0, u_1) \in (V \cap H^2(\Omega)) \times V$, there exists a unique pair of functions (u, y), which is a solution to the problem (1.1)-(1.5) in the class

$$\begin{aligned} u \in L^\infty(0, T; V \times H^2(\Omega)), \quad & u' \in L^\infty(0, T; V), \\ u'' \in L^\infty(0, T; L^2(\Omega)), \quad & y, y' \in L^2(0, \infty; L^2(\Gamma_1)). \end{aligned}$$

In the order to state our main results, we define the energy of problem (1.1)-(1.5) by

$$E(t) = \frac{1}{2} \int_\Omega (u')^2 dx + \frac{a^2}{2} \int_\Omega |\nabla u|^2 dx + \frac{b}{4} (\int_\Omega |\nabla u|^2 dx)^2 + \frac{1}{2} \int_{\Gamma_1} q(x)(y)^2 d\Gamma. \tag{2.4}$$

3. UNIFORM STABILITY ON ACCOUNT OF INTERNAL DAMPING OF KELVIN-VOIGT TYPE

If we differentiate (2.4) with respect to t and use the governing Eq.(1.1) we get

$$E'(t) = \int_{\Omega}[u'(a^2 + b\int_{\Omega}|\nabla u|^2)\triangle u + 2\lambda u'\Delta u' + a^2\nabla u \cdot \nabla u']dx$$
$$+b\int_{\Omega}|\nabla u|^2dx\int_{\Omega}\nabla u \cdot \nabla u'dx + \int_{\Gamma_1} q(x)yy'd\Gamma.$$

Application of Green's formula and using the boundary conditions (1.2)-(1.4) and then a simplification, we get

$$E'(t) = -2\lambda\int_{\Omega}|\nabla u'|^2dx - \int_{\Gamma_1} p(x)(y')^2d\Gamma \leq 0 \quad \forall t \in R^+. \tag{3.1}$$

We see from (3.1) that the energy E is a decreasing function of time and hence

$$E(t) \leq E(0) \quad \forall t \geq 0, \tag{3.2}$$

where

$$E(0) = \frac{1}{2}\int_{\Omega}[(u_1)^2 + a^2|\nabla u_0|^2]dx + \frac{b}{4}(\int_{\Omega}|\nabla u_0|^2dx)^2$$
$$+\frac{1}{2}\int_{\Gamma_1} q(x)(y(x,0))^2d\Gamma.$$

Under what conditions does this energy E decay with time uniformly? An affirmative answer is contained in the following theorem.

Theorem 3.1. *If $u = u(x,t)$ is a regular solution of the system (1.1)-(1.5) with initial values $(u_0, u_1) \in V \times L^2(\Omega)$, then the energy $E(t)$ of the system defined by (2.4) satisfies*

$$E(t) < Me^{-\mu t}E(0), t \in R^+$$

for some real constants $M > 1$(3.24) and $\mu > 0$(3.21), dependent on the damping parameter λ(3.19).

Firstly, we need to prove the following lemma.

Lemma 3.1. *For every solution $u = u(x,t)$ of the system (1.1)-(1.5), the time derivative of the functional G defined by*

$$\Psi(t) = \int_{\Omega}(uu' + \lambda|\nabla u|^2)dx + \int_{\Gamma_1} uyd\Gamma + \frac{1}{2}\int_{\Gamma_1} p(x)y^2d\Gamma \tag{3.3}$$

satisfies

$$\Psi'(t) \leq 2k\int_{\Omega}|\nabla u'|^2dx + 2\int_{\Gamma_1} uy'd\Gamma - \frac{b}{2}(\int_{\Omega}|\nabla u|^2dx)^2 - 2E(t), \forall t \in R^+. \tag{3.4}$$

Proof. If we differentiate (3.3) with respect to t and replace u'' by the relation (1.1), then we get

$$\begin{aligned}\Psi'(t) &= \int_\Omega (uu'' + (u')^2 + 2\lambda \nabla u \cdot \nabla u')dx + \int_{\Gamma_1} (u'y + uy')d\Gamma + \int_{\Gamma_1} p(x)yy'd\Gamma \quad (3.5)\\ &= \int_\Omega [u(a^2 + b\int_\Omega |\nabla u|^2 dx)\triangle u dx + 2\lambda(u\triangle u' + \nabla u \cdot \nabla u')]dx \\ &\quad + \int_\Omega (u')^2 dx + \int_{\Gamma_1} uy'd\Gamma + \int_{\Gamma_1} y(u' + p(x)y')d\Gamma.\end{aligned}$$

Applying Green's formula, we have

$$\begin{aligned}\Psi'(t) &= \int_\Omega (u')^2 dx - (a^2 + b\int_\Omega |\nabla u|^2 dx)\int_\Omega |\nabla u|^2 dx \quad (3.6)\\ &+ \int_\Gamma u[(a^2 + b\int_\Omega |\nabla u|^2 dx)\frac{\partial u}{\partial \nu} + 2\lambda \frac{\partial u'}{\partial \nu}]d\Gamma + \int_{\Gamma_1} uy'd\Gamma + \int_{\Gamma_1} y(u' + p(x)y')d\Gamma.\end{aligned}$$

Using the boundary conditions (1.2)-(1.4) and the energy (2.4), relation (3.6) can be written as

$$\Psi'(t) = 2\int_\Omega (u')^2 dx + 2\int_{\Gamma_1} uy'd\Gamma - \frac{b}{2}(\int_\Omega |\nabla u|^2 dx)^2 - 2E(t), \forall t \in R^+. \quad (3.7)$$

Hence the proof of lemma complete. □

Proof of Theorem 1. We introduce a modified energy like Lyapunov functional V by

$$V(t) = E(t) + \frac{\lambda}{k}\Psi(t) \text{ for } t \geq 0. \quad (3.8)$$

Now, using the Cauchy-Schwarz's inequality, the Poincare inequality(2.1)-(2.2) and the defined of energy (2.4), we obtain estimate as follow

$$\begin{aligned}|\int_\Omega uu'dx| &\leq \frac{\sqrt{k}}{2a}\int_\Omega ((u')^2 + \frac{a^2}{k}u^2)dx \quad (3.9)\\ &\leq \frac{\sqrt{k}}{2a}(\int_\Omega (u')^2 dx + a^2\int_\Omega |\nabla u|^2 dx) \leq \frac{\sqrt{k}}{a}E(t),\end{aligned}$$

$$0 \leq \lambda \int_\Omega |\nabla u|^2 dx \leq \frac{2\lambda}{a^2}E(t), \quad (3.10)$$

$$\begin{aligned}|\int_{\Gamma_1} uyd\Gamma| &\leq \int_{\Gamma_1} \frac{1}{2q(x)}u^2 d\Gamma + \frac{1}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma \quad (3.11)\\ &\leq \frac{\bar{k}}{2q_0}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_{\Gamma_1} q(x)y^2 d\Gamma \leq (\frac{\bar{k}}{a^2 q_0} + 1)E(t),\end{aligned}$$

and

$$\frac{1}{2}\int_{\Gamma_1} p(x)y^2 d\Gamma \leq \frac{p_1}{2q_0}\int_{\Gamma_1} q(x)y^2 d\Gamma \leq \frac{p_1}{q_0}E(t). \quad (3.12)$$

Thus the inequality (3.9)-(3.12) yield for Ψ that estimates

$$-(\frac{\sqrt{k}}{a}+\frac{\bar{k}}{a^2q_0}+1)E(t)\le\Psi(t)\le(\frac{\sqrt{k}}{a}+\frac{2\lambda}{a^2}+\frac{\bar{k}}{a^2q_0}+\frac{p_1}{q_0}+1)E(t). \tag{3.13}$$

Then it follows from (3.13) that

$$\left(1-\frac{\lambda}{k}(\frac{\sqrt{k}}{a}+\frac{\bar{k}}{a^2q_0}+1)\right)E(t)\le V(t)\le \tag{3.14}$$
$$\left(1+\frac{\lambda}{k}(\frac{\sqrt{k}}{a}+\frac{2\lambda}{a^2}+\frac{\bar{k}}{a^2q_0}+\frac{p_1}{q_0}+1)\right)E(t)\quad\forall t\ge 0,$$

where we assume that

$$0<\lambda<\frac{k}{\sqrt{k}/a+\bar{k}/(a^2q_0)+1}, \tag{3.15}$$

so that left hand side of (3.14) is positive.

Next, differentiating $V(t)$(defined by (3.8)) with respect to t using expression $E'(t)$(defined by (3.1)) and Lemma 3.1, we have

$$V'(t)=-\int_{\Gamma_1}p(x)(y')^2d\Gamma-\frac{b\lambda}{2k}(\int_{\Omega}|\nabla u|^2dx)^2+\frac{2\lambda}{k}\int_{\Gamma_1}uy'd\Gamma-\frac{2\lambda}{k}E(t). \tag{3.16}$$

Now, using the Cauchy-Schwarz's inequality, the Poincare inequality, the conditions (2.2)-(2.3) and the definition of energy (2.4), we obtain estimate

$$|\frac{2\lambda}{k}\int_{\Gamma_1}uy'd\Gamma|\le\int_{\Gamma_1}p(x)(y')^2d\Gamma+\frac{\lambda^2}{k^2}\int_{\Gamma_1}\frac{1}{p(x)}u^2d\Gamma \tag{3.17}$$
$$\le\int_{\Gamma_1}p(x)(y')^2d\Gamma+\frac{2\bar{k}\lambda^2}{a^2k^2p_0}E(t)$$

From (3.16)-(3.17) and condition (3.15), we have

$$V'(t)\le-\frac{b\lambda}{2k}(\int_{\Omega}|\nabla u|^2dx)^2-\frac{2\lambda}{k}(1-\frac{\bar{k}\lambda}{a^2kp_0})E(t) \tag{3.18}$$
$$<-\frac{2\lambda}{k}(1-\frac{\bar{k}\lambda}{a^2kp_0})E(t)\quad\forall t\in R^+,$$

where we assume that

$$0<\lambda<\min\{\frac{a^2kp_0}{\bar{k}},\frac{k}{\frac{\sqrt{k}}{a}+\frac{\bar{k}}{a^2q_0}+1}\}. \tag{3.19}$$

With the help of (3.14), the above yields the differential inequality

$$V'(t)+\mu V(t)<0\quad\forall t\in R^+, \tag{3.20}$$

where

$$0 < \mu = \frac{2\lambda}{k}(1 - \frac{\bar{k}\lambda}{a^2 k p_0}) \frac{1}{1 + \frac{\lambda}{k}(\frac{\sqrt{k}}{a} + \frac{2\lambda}{a^2} + \frac{p_1 \bar{k}}{a^2 q_0} + \frac{p_1}{q_0} + 1)}. \tag{3.21}$$

Multiplying (3.20) by $e^{\mu t}$ and integrating over the time interval $[0, t]$, we get the estimate

$$V(t) < e^{-\mu t} V(0) \quad \forall t \in R^+. \tag{3.22}$$

Invoking the inequality (3.14) again in (3.22), we have

$$E(t) < M e^{-\mu t} E(0) \quad \forall t \in R^+, \tag{3.23}$$

where

$$M = \frac{1 + \frac{\lambda}{k}(\frac{\sqrt{k}}{a} + \frac{2\lambda}{a^2} + \frac{\bar{k}}{a^2 q_0} + \frac{p_1}{q_0} + 1)}{1 - \frac{\lambda}{k}(\frac{\sqrt{k}}{a} + \frac{\bar{k}}{a^2 q_0} + 1)}. \tag{3.24}$$

The finishes the proof of the theorem. □

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CONTROLLABILITY RESULTS FOR IMPULSIVE NEUTRAL EVOLUTION DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper, we consider the controllability of a certain class of impulsive neutral evolution differential equations in Banach spaces. Sufficient conditions for controllability are obtained by using the Hausdorff measure of noncompactness and Monch fixed point theorem under the assumption of noncompactness of the evolution system.

1. INTRODUCTION

Impulsive differential equations form an appropriate model for describing phenomena where systems instantaneously change their state. Because of this reason they have numerous applications in several fields of applied sciences, such as Biology, Economics and Physics. There has been a significant development in impulsive theory in recent years, especially in the area of impulsive differential equations with fixed moments, see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [17] and Samoilenko and Perestyuk [25].

The study of the existence and stability of the differential equations with delay was initiated by Travis and Webb [26] and Webb [28]. In many areas of science there has been an increasing interest in the investigation of functional differential equations, incorporating memory or aftereffect, that is, there is an effect of infinite delay on state equations. Related to this, we refer the reader to Kolmanovskii and Myshkis [15, 16] and Wu [29]. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. For the literature relative to impulsive neutral differential systems with infinite delay, we refer the reader to [4, 5, 8, 12, 30, 31].

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On the other hand, the concept of controllability is of great importance in mathematical control theory. Controllability for differential systems in Banach spaces under the assumption of compactness and noncompactness of the operator semigroups has been studied by many authors [1, 6, 7, 9, 10, 13, 18, 20, 21, 23, 24, 27] by using various fixed point theorems. In particular, by using Monch fixed point theorem, Guo et al. [10] established the sufficient conditions for the controllability of the following class of impulsive evolution inclusions with nonlocal conditions:

$$\begin{aligned}
&x'(t) - A(t)x(t) \in F(t, x(t)) + Bu(t), \text{ a.e. on } T = [0, b],\\
&\Delta x(t_i) = I_i(x(t_i)),\ i = 1, 2, \cdots, s,\\
&x(0) + M(x) = x_0,
\end{aligned}$$

under the assumption of noncompactness of the semigroup generated by the evolution system. Very recently, by using the same fixed point theorem, Ji et al. [13] extended the controllability results of Guo et al. [10] into the following impulsive differential systems:

$$\begin{aligned}
&x'(t) = A(t)x(t) + f(t, x(t)) + (Bu)(t), \text{ a.e. on } [0, b],\\
&\Delta x|_{t=t_i} = I_i(x(t_i)),\ i = 1, 2, \ldots, s,\\
&x(0) + M(x) = x_0,
\end{aligned}$$

under the assumption that the evolution system generated by $A(t)$ is equicontinuous.

Motivated by the above mentioned works [10, 13, 31], in this paper, we establish the sufficient conditions for controllability of the impulsive neutral evolution differential equations with infinite delay of the form:

$$\begin{aligned}
&\frac{d}{dt}\Big[x(t) - g(t, x_t)\Big] + A(t)x(t) = f(t, x_t) + (Bu)(t),\\
&t \in J = [0, b],\ t \neq t_k,\ k = 1, 2, \cdots, m, && (1.1)\\
&\Delta x|_{t=t_k} = I_k(x_{t_k}),\ k = 1, 2, \cdots, m, && (1.2)\\
&x_0 = \varphi \in \mathcal{B}, && (1.3)
\end{aligned}$$

where $\{A(t)\}_{t\in J}$ is a family of linear operators in a Banach space X generating an evolution operator $U : \Delta = \{(t, s) \in [0, b] \times [0, b] : 0 \leq s \leq t \leq b\} \to \mathcal{L}(X)$, here X is a Banach space and $\mathcal{L}(X)$ is the Banach space of all bounded linear operators in X; the history $x_t : (-\infty, 0] \to X,\ x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $f, g : J \times \mathcal{B} \to X$ are appropriate functions; the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are given and $I_k : \mathcal{B} \to X,\ k = 1, 2, \cdots, m$, are given impulsive functions; the control function $u(\cdot)$ is considered in the space $L^2(J, V)$, where V is a Banach space of controls and $B : V \to X$ is a bounded linear operator.

2. PRELIMINARIES

Let $(X, \|\cdot\|)$ be a real Banach space. We denote $\mathcal{C}([0,b],X)$ the space of all X-valued functions on $[0,b]$ with norm $\|x\| = \sup\{\|x(t)\| : t \in [0,b]\}$ and by $L^1([0,b],X)$ the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\|dt$.

To describe appropriately our problems, we say that a function $u : [\sigma,\tau] \to X$ is a normalized piecewise continuous function on $[\sigma,\tau]$ if u is piecewise continuous and left continuous on $(\sigma,\tau]$. By the symbol $\mathcal{PC}([\sigma,\tau];X)$, we denote the space of normalized piecewise continuous functions from $[\sigma,\tau]$ into X. In particular, we denote the space $\mathcal{PC}$ formed by all functions $u : [0,b] \to X$ such that u is continuous at $t \neq t_k$, $u(t_k^-) = u(t_k)$ and $u(t_k^+)$ exists, for all $k = 1,2,\cdots,m$. It is easy to see that $\mathcal{PC}$ is a Banach space with the norm $\|x\|_{\mathcal{PC}} = \sup_{s\in[0,b]} \|x(s)\|$.

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced by Hale and Kato [11] and it is appropriate to treat retarded impulsive differential equations.

Let $\mathcal{B}$ will be a linear space of functions mapping from $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, and satisfies the following axioms:

(A) If $x : (-\infty,\sigma+b] \to X,\ b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma,\sigma+b];X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma,\sigma+b]$ the following conditions hold:
  - (i) $x_t \in \mathcal{B}$,
  - (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  - (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty),\ K$ is continuous, M is locally bounded, and $H,\ K,\ M$ are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

For the family of linear operators $\{A(t) : t \in J\}$, we assume the following hypotheses.

(A1) The domain $D(A(t))$ of $A(t)$ is dense in X and independent of t.

(A2) For each $t \in J$, the resolvent $R(\lambda : A(t))$ of $A(t)$ exists for all λ with $Re\lambda \leq 0$ and there exists a constant $M > 0$ such that $\|R(\lambda : A(t))\| \leq M(|\lambda|+1)^{-1}$

(A3) There exist constants $L > 0$ and $0 < \mu \leq 1$ such that $\|(A(t)-A(s))A^{-1}(\tau)\| \leq L|t-s|^\mu$ for $t,s,\tau \in J$.

Under the assumptions $(A1)-(A3)$, the family $\{A(t) : t \in J\}$ generates an unique evolution system $\{U(t,s) : 0 \leq s \leq t \leq b\}$ satisfying:

(a) There exists a positive constant M_0 such that $\|U(t,s)\| \leq M_0$ for $0 \leq s \leq t \leq b$.

(b) For every $v \in D(A(t))$ and $t \in J,\ \ U(t,s)v$ is differential with respect to s on $0 \leq s \leq t \leq b$ and $\frac{\partial}{\partial s}U(t,s)v = U(t,s)A(s)v$.

Definition 2.1. *A two parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq b$ on X is called an evolution system if the following two conditions are satisfied:*

(i) *$U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq b$;*

(ii) $(t,s) \to U(t,s)$ *is strongly continuous on* Δ, *i.e., for each* $x \in X$, *the function* $(t,s) \in \Delta \to U(t,s)x$ *is continuous.*

More details about evolution system can be found in Pazy [22].

Definition 2.2. *(*[3]*) Let* E^+ *be the positive cone of an order Banach space* $(E, \leq)$. *A function* Φ *defined on the set of all bounded subsets of the Banach space* X *with values in* E^+ *is called a measure of noncompactness(MNC) on* X *if* $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ *for all bounded subsets* $\Omega \subseteq X$, *where* $\overline{co}\Omega$ *stands for the closed convex hull of* Ω.

The MNC Φ *is said:*

(1) *Monotone if for all bounded subsets* Ω_1, Ω_2 *of* X *we have:* $(\Omega_1 \subseteq \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))$;
(2) *Nonsingular if* $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ *for every* $a \in X$, $\Omega \subset X$;
(3) *Regular if* $\Phi(\Omega) = 0$ *if and only if* Ω *is relatively compact in* X.

One of the most examples of MNC is the noncompactness measure of Hausdorff β *defined on each bounded subset* Ω *of* X *by*

$\beta(\Omega) = \inf\{\epsilon > 0;\ \Omega$ *can be covered by a finite number of balls of radii smaller than* $\epsilon\}$.

It is well known that MNC β *enjoys the above properties and other properties see* [3, 14]*: For all bounded subsets* $\Omega, \Omega_1, \Omega_2$ *of* X,

(4) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, *where* $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$;
(5) $\beta(\Omega_1 \cup \Omega_2) \leq max\{\beta(\Omega_1), \beta(\Omega_2)\}$;
(6) $\beta(\lambda\Omega) \leq |\lambda|\beta(\Omega)$ *for any* $\lambda \in \mathbb{R}$;
(7) *If the map* $Q : D(Q) \subseteq X \to Z$ *is Lipschitz continuous with constant* k, *then* $\beta_Z(Q\Omega) \leq k\beta(\Omega)$ *for any bounded subset* $\Omega \subseteq D(Q)$, *where Z is a Banach space.*

Lemma 2.1. *(*[3]*) If* $W \subset C([a,b], X)$ *is bounded and equicontinuous, then* $\beta(W(t))$ *is continuous for* $t \in [a,b]$ *and*

$$\beta(W) = \sup\{\beta(W(t)),\ t \in [a,b]\}, \quad \textit{where} \quad W(t) = \{x(t) : x \in W\} \subseteq X.$$

Lemma 2.2. *(*[14]*) Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions in* $L^1([0,b], \mathbb{R}^+)$. *Assume that there exist* $\mu, \eta \in L^1([0,b], \mathbb{R}^+)$ *satisfying* $\sup\limits_{n\geq 1}\|f_n(t)\| \leq \mu(t)$ *and* $\beta(\{f_n(t)\}_{n=1}^{\infty}) \leq \eta(t)$ *a.e.* $t \in [0,b]$, *then for all* $t \in [0,b]$, *we have*

$$\beta\Big(\Big\{\int_0^t U(t,s)f_n(s)ds : n \geq 1\Big\}\Big) \leq 2M_0 \int_0^t \eta(s)ds.$$

The following fixed-point theorem, a nonlinear alternative of Monch type, plays a key role in our proof of controllability of the system $(1.1) - (1.3)$.

Lemma 2.3. *(*[19, Theorem 2.2]*) Let* D *be a closed convex subset of a Banach space* X *and* $0 \in D$. *Assume that* $F : D \to X$ *is a continuous map which satisfies Monch's condition, that is* ($M \subseteq D$ *is countable,* $M \subseteq \overline{co}(\{0\} \cup F(M)) \Rightarrow \overline{M}$ *is compact*). *Then* F *has a fixed point in* D.

3. CONTROLLABILITY RESULTS

In this section, we present and prove the controllability results for the system $(1.1)-(1.3)$. First, we give the mild solution of the problem $(1.1)-(1.3)$.

Definition 3.3. *A function $x:(-\infty,b]\to X$ is a mild solution of the initial value problem $(1.1)-(1.3)$, if $x_0=\varphi\in\mathcal{B}$, $x(\cdot)|_J\in\mathcal{PC}$ and*

$$x(t)=U(t,0)[\varphi(0)-g(0,\varphi)]+g(t,x_t)+\int_0^t U(t,s)A(s)g(s,x_s)ds$$
$$+\int_0^t U(t,s)\Big[f(s,x_s)+(Bu)(s)\Big]ds+\sum_{0<t_k<t}U(t,t_k)I_k(x_{t_k}),\ t\in J.$$

Definition 3.4. *The system $(1.1)-(1.3)$ is said to be controllable on the interval J if for every initial function $\varphi\in\mathcal{B}$ and $x_1\in X$, there exists a control $u\in L^2(J,V)$ such that the mild solution $x(\cdot)$ of $(1.1)-(1.3)$ satisfies $x(b)=x_1$.*

We will study the problem $(1.1)-(1.3)$ under the following hypotheses:

(H1) The evolution system $\{U(t,s)\}_{(t,s)\in\Delta}$ generated by the family of linear operators $\{A(t)\}_{t\in J}$ is equicontinuous. i.e., $(t,s)\to\{U(t,s)x : x\in E\}$ is equicontinuous for $t>0$ and for all bounded subsets E.

(H2) The function $f: J\times\mathcal{B}\to X$ satisfies:
  - (i) For a.e. $t\in J$, the function $f(t,\cdot):\mathcal{B}\to X$ is continuous and for all $\varphi\in\mathcal{B}$, the function $f(\cdot,\varphi): J\to X$ is strongly measurable.
  - (ii) For each positive integer r, there exists an integrable function $\alpha_r: J\to[0,+\infty)$ such that

$$\sup_{\|\varphi\|_{\mathcal{B}}\le r}\|f(t,\varphi)\|\le\alpha_r(t),\ \text{ for a.e. } t\in J,$$
$$\text{and}\quad \liminf_{r\to\infty}\int_0^b\frac{\alpha_r(t)}{r}dt=\delta<+\infty.$$

  - (iii) There exists integrable function $\eta: J\to[0,+\infty)$ such that

$$\beta(f(t,D))\le\eta(t)\sup_{-\infty<\theta\le 0}\beta(D(\theta))\quad\text{for a.e. } t\in J \text{ and } D\subset\mathcal{B},$$

    where $D(\theta)=\{v(\theta): v\in D\}$ and β is the Hausdorff MNC.

(H3) The linear operator $W: L^2(J,V)\to X$ is defined by

$$Wu=\int_0^b U(b,s)Bu(s)ds\quad\text{such that}$$

  - (i) W has an invertible operator W^{-1} which take values in $L^2(J,V)\setminus$ ker$W$, and there exist positive constants M_1, M_2 such that $\|B\|\le M_1$ and $\|W^{-1}\|\le M_2$.
  - (ii) There is $K_W\in L^1(J,\mathbb{R}^+)$ such that, for every bounded set $Q\subset X$,

$$\beta(W^{-1}Q)(t)\le K_W(t)\beta(Q).$$

(H4) There exists a positive constant $M_3 > 0$ such that $\|A(t)A^{-1}(0)\| \leq M_3$ for $t \in J$.

(H5) The function $g : J \times \mathcal{B} \to X$ is continuous and there exist positive constants L_0, C_1, C_2 such that,

(i)

$$\|A(0)g(t,\varphi_1) - A(0)g(t,\varphi_2)\| \leq L_0(\|\varphi_1 - \varphi_2\|_{\mathcal{B}}),\ \forall\, t \in J,\ \varphi_1, \varphi_2 \in \mathcal{B},$$

(ii)

$$\|A(0)g(t,\varphi)\| \leq C_1\|\varphi\|_{\mathcal{B}} + C_2,\ \forall \varphi \in \mathcal{B},\ t \in J.$$

(H6) (i) There exist positive constants γ_k such that

$$\|I_k(\varphi_1) - I_k(\varphi_2)\| \leq \gamma_k(\|\varphi_1 - \varphi_2\|_{\mathcal{B}}),\ \ \forall\, \varphi_1, \varphi_2 \in \mathcal{B}$$

.

(ii) There exist continuous nondecreasing functions $L_k : [0, +\infty) \to (0, +\infty)$ such that

$$\|I_k(\varphi)\| \leq L_k(\|\varphi\|_{\mathcal{B}}),\ \forall \varphi \in \mathcal{B},$$

$$\text{and} \quad \liminf_{\rho \to \infty} \frac{L_k(\rho)}{\rho} = \lambda_k < +\infty,\ \text{where } \sum_{k=1}^{m} \lambda_k = \lambda.$$

(H7) The following estimation holds true:
$N + \tilde{N} < 1$ where,

$$N = K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big[\|A^{-1}(0)\|L_0 + M_0 M_3 b L_0 + M_0 \sum_{k=1}^{m} \gamma_k\Big],$$

$$\tilde{N} = (2M_0 + 4M_0^2 M_1 \|K_W\|_{L^1})\|\eta\|_{L^1}.$$

Remark 3.1. *From* $(A3)$, *we obtain* $\|A(t)A^{-1}(0)\| \leq L|b|^{\mu} + 1$. *Thus we can choose a positive constant* $M_3 = L|b|^{\mu} + 1$ *satisfying* $(H4)$.

Theorem 3.1. *Assume that the hypotheses* $(H1) - (H7)$ *are satisfied. Then the system* $(1.1) - (1.3)$ *is controllable on J provided that,*

$$K_b(1 + M_0 M_1 M_2 b^{\frac{1}{2}})\Big[C_1(\|A^{-1}(0)\| + M_0 M_3 b) + M_0(\delta + \lambda)\Big] < 1. \tag{3.1}$$

Proof. Using the hypothesis $(H3)$ for an arbitrary function $x : (-\infty, b] \to X$, define the control

$$u_x(t) = W^{-1}\Big[x_1 - U(b,0)[\varphi(0) - g(0,\varphi)] - g(b, x_b) - \int_0^b U(t,s)A(s)g(s,x_s)ds$$
$$- \int_0^b U(b,s)f(s,x_s)ds - \sum_{k=1}^{m} U(b,t_k)I_k(x_{t_k})\Big](t).$$

We shall now show that using this control the operator defined by

$$\Phi x(t) = \begin{cases} \varphi(t),\ t \in (-\infty, 0], \\ U(t,0)[\varphi(0) - g(0,\varphi)] + g(t, x_t) + \int_0^t U(t,s)A(s)g(s, x_s)ds \\ \quad + \int_0^t U(t,s)\Big[f(s, x_s) + Bu_x(s)\Big]ds,\ t \in J \end{cases}$$

has a fixed point. This fixed point is then a solution of $(1.1) - (1.3)$. Clearly, $(\Phi x)(b) = x_1$, which implies that the system $(1.1) - (1.3)$ is controllable.
Suppose that $x(t) = z(t) + y(t),\ t \in (-\infty, b]$, where $y : (-\infty, 0] \to X$ be a function defined by $y_0 = \varphi$ and $y(t) = U(t,0)\varphi(0)$ on J. Then by the axioms of phase space, it is easy to see that $\|z_t + y_t\|_{\mathcal{B}} \le (K_b M_0 H + M_b)\|\varphi\|_{\mathcal{B}} + K_b\|z\|_t$, where $\|z\|_t = \sup_{0\le s\le t}\|z(s)\|$, $K_b = \sup_{0\le t\le b} K(t)$ and $M_b = \sup_{0\le t\le b} M(t)$.

Define $S(b) = \Big\{z : (-\infty, b] \to X$ such that $z_0 = 0,\ z|_J \in \mathcal{PC}\Big\}$ be the space endowed with the supremum norm $\|\cdot\|_b$. Then$(S(b), \|\cdot\|_b)$ is a Banach space. Let $\Gamma : S(b) \to S(b)$ be the operator defined by

$$(\Gamma z)(t) = \begin{cases} 0,\ t \in (-\infty, 0], \\ -U(t,0)g(0,\varphi) + g(t, z_t + y_t) + \int_0^t U(t,s)A(s)g(s, z_s + y_s)ds \\ \quad + \int_0^t U(t,s)\Big[f(s, z_s + y_s) + Bu_z(s)\Big]ds + \sum\limits_{0<t_k<t} U(t,t_k)I_k(z_{t_k} + y_{t_k}), \\ \quad t \in J, \end{cases}$$

where $u_z(\cdot) \in L^2(J, V)$,

$$\begin{aligned} u_z(t) = W^{-1}\Big[&x_1 - U(b,0)[\varphi(0) - g(0,\varphi)] - g(b, z_b + y_b) \\ &- \int_0^b U(t,s)A(s)g(s, z_s + y_s)ds - \int_0^b U(b,s)f(s, z_s + y_s)ds \\ &- \sum_{k=1}^m U(b,t_k)I_k(z_{t_k} + y_{t_k})\Big](t). \end{aligned}$$

Clearly, Γ is well defined and with values in $S(b)$. It is easy to see that if z is a fixed point of Γ, then $z + y$ is a fixed point of Φ. So our aim is to find a fixed point of Γ.
Set $B_q = \{z \in S(b) : \|z\|_b \le q\}$ for some $q > 0$. Clearly, B_q is a nonempty, closed, convex and bounded set in $S(b)$. Then for any $z \in B_q$,

$$\|z_t + y_t\|_{\mathcal{B}} \le (K_b M_0 H + M_b)\|\varphi\|_{\mathcal{B}} + K_b q = q'. \tag{3.2}$$

For better readability, we break the proof into sequence of steps.
Step 1: There exists $q \ge 1$ such that $\Gamma(B_q) \subseteq B_q$.

Suppose the contrary. Then for each positive integer q, there exists $z \in B_q$ such that $\|(\Gamma z)(t)\| > q$ for some $t \in J$. It follows from the hypotheses $(H1) - (H6)$ and (3.2) we have

$$
\begin{aligned}
q &< \|(\Gamma z)(t)\| \\
&\leq M_0\|A^{-1}(0)\|(C_1\|\varphi\|_{\mathcal{B}} + C_2) + \|A^{-1}(0)\|(C_1q' + C_2) + M_0M_3b(C_1q' + C_2) \\
&\quad + M_0\int_0^t \alpha_{q'}(s)ds + M_0\sum_{k=1}^{m} L_k(q') + M_0M_1b^{\frac{1}{2}}\|u_z\|_{L^2} \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
\text{where } \|u_z\|_{L^2} \leq M_2\Big[&\|x_1\| + M_0[\|\varphi(0)\| + \|A^{-1}(0)\|(C_1\|\varphi\|_{\mathcal{B}} + C_2)] \\
&+ \|A^{-1}(0)\|(C_1q' + C_2) + M_0M_3b(C_1q' + C_2) \\
&+ M_0\int_0^b \alpha_{q'}(s)ds + M_0\sum_{k=1}^{m} L_k(q')\Big] \qquad (3.4)
\end{aligned}
$$

Hence by using (3.4) in (3.3), we have

$$
\begin{aligned}
q \leq \tilde{L} + (1 + M_0M_1M_2b^{\frac{1}{2}})\Big[&\|A^{-1}(0)\|C_1q' + M_0M_3bC_1q' \\
&+ M_0\int_0^b \alpha_{q'}(s)ds + M_0\sum_{k=1}^{m} L_k(q')\Big], \qquad (3.5)
\end{aligned}
$$

where $\tilde{L}$ is independent of q.
Noting that $q' = (K_bM_0H + M_b)\|\varphi\|_{\mathcal{B}} + K_bq \to +\infty$ as $q \to +\infty$, we obtain by hypotheses $(H2)(ii)$ and $(H6)(ii)$,

$$
\lim_{q\to+\infty}\inf\Big(\frac{\int_0^b \alpha_{q'}(s)ds}{q}\Big) = \lim_{q\to+\infty}\inf\Big(\frac{\int_0^b \alpha_{q'}(s)ds}{q'}.\frac{q'}{q}\Big) = \delta K_b,
$$

$$
\lim_{q\to+\infty}\inf\Big(\frac{\sum_{k=1}^{m} L_k(q')}{q}\Big) = \lim_{q\to+\infty}\inf\Big(\frac{\sum_{k=1}^{m} L_k(q')}{q'}.\frac{q'}{q}\Big) = \lambda K_b.
$$

Dividing both sides of (3.5) by q and employing the above two equalities, we have that

$$
1 \leq K_b(1 + M_0M_1M_2b^{\frac{1}{2}})\Big[C_1(\|A^{-1}(0)\| + M_0M_3b) + M_0(\delta + \lambda)\Big].
$$

This contradicts (3.1). Thus, there exists $q \geq 1$ such that $\Gamma(B_q) \subseteq B_q$.
Step 2: $\Gamma : S(b) \to S(b)$ is continuous.

Let $(z^n)_{n\in\mathbb{N}}$ be a sequence in $S(b)$ such that $z^n \to z$ in $S(b)$. Then by hypotheses $(H2)(i)$, $(H5)(i)$ and $(H6)(i)$, we can prove that $f(s, z_s^n + y_s) \to f(s, z_s + y_s)$, $g(s, z_s^n + y_s) \to g(s, z_s + y_s)$ and $I_k(z_{t_k}^n + y_{t_k}) \to I_k(z_{t_k} + y_{t_k})$ uniformly on J.
Then by hypotheses $(H2)(i, ii)$ and $(H5)(i)$ with Dominated convergence theorem,
we conclude that

$$
\int_0^t U(t,s)f(s, z_s^n + y_s)ds \to \int_0^t U(t,s)f(s, z_s + y_s)ds,
$$

and

$$\int_0^t A(s)U(t,s)g(s,z_s^n+y_s)ds \to \int_0^t A(s)U(t,s)g(s,z_s+y_s)ds, \text{ as } n\to\infty.$$

Which implies together with the continuity of the operators B, W^{-1} that,
we have $\|\Gamma z^n - \Gamma z\| \to 0$, as $n \to \infty$. Hence Γ is continuous on $S(b)$.

Step 3: The Monch condition holds:

To prove this, we decompose Γ in the form $\Gamma = \Gamma_1 + \Gamma_2$, for $t \in J$, where

$$\begin{aligned}
&(\Gamma_1 z)(t)\\
&= -U(t,0)g(0,\varphi) + g(t,z_t+y_t) + \int_0^t U(t,s)A(s)g(s,z_s+y_s)ds\\
&\quad + \sum_{0<t_k<t} U(t,t_k)I_k(z_{t_k}+y_{t_k}) + \int_0^t U(t,\zeta)BW^{-1}\Big[x_1 - U(b,0)[\varphi(0)-g(0,\varphi)]\\
&\quad - g(b,z_b+y_b) - \int_0^b U(b,s)A(s)g(s,z_s+y_s)ds \sum_{k=1}^m U(b,t_k)I_k(z_{t_k}+y_{t_k})\Big](\zeta)d\zeta,
\end{aligned}$$

and

$$\begin{aligned}
&(\Gamma_2)z(t)\\
&= \int_0^t U(t,s)f(s,z_s+y_s)ds - \int_0^t U(t,\zeta)BW^{-1}\Big[\int_0^b U(b,s)f(s,z_s+y_s)ds\Big](\zeta)d\zeta.
\end{aligned}$$

Firstly, we prove that Γ_1 is Lipschitz continuous.

Take $z_1, z_2 \in S(b)$. Then by the axioms of phase space and hypotheses $(H5)\&(H6)$, we get that

$$\begin{aligned}
&\|\Gamma_1 z_1(t) - \Gamma_1 z_2(t)\|\\
&\le \|A^{-1}(0)\|L_0\|z_{1t}-z_{2t}\|_{\mathcal{B}} + M_0M_3bL_0\|z_{1s}-z_{2s}\|_{\mathcal{B}}\\
&\quad + M_0\sum_{k=1}^m \gamma_k\|z_{1t_k}-z_{2t_k}\|_{\mathcal{B}} + M_0M_1M_2b^{\frac{1}{2}}\Big[\|A^{-1}(0)\|L_0\|z_{1b}-z_{2b}\|_{\mathcal{B}}\\
&\quad + M_0M_3bL_0\|z_{1s}-z_{2s}\|_{\mathcal{B}} + M_0\sum_{k=1}^m \gamma_k\|z_{1t_k}-z_{2t_k}\|_{\mathcal{B}}\Big]\\
&\le K_b(1+M_0M_1M_2b^{\frac{1}{2}})\Big(\|A^{-1}(0)\|L_0 + M_0M_3bL_0 + M_0\sum_{k=1}^m\gamma_k\Big)\|z_1-z_2\|_b.
\end{aligned}$$

$$\text{That is, } \|\Gamma_1 z_1(t) - \Gamma_1 z_2(t)\|_b \le N\|z_1-z_2\|_b, \tag{3.6}$$

where $N = K_b(1+M_0M_1M_2b^{\frac{1}{2}})\Big(\|A^{-1}(0)\|L_0 + M_0M_3bL_0 + M_0\sum_{k=1}^m\gamma_k\Big)$

Hence, Γ_1 is Lipschitz continuous with Lipschitz constant N.
Next we prove that, Γ_2 maps B_q into an equicontinuous family on J.
Indeed let $t_1, t_2 \in J,\ 0 < t_1 < t_2$. Then for arbitrary $z \in B_q$, we have

$$\begin{aligned}
&\|\Gamma_2 z(t_2) - \Gamma_2 z(t_1)\| \\
&\leq \int_0^{t_1} \|[U(t_2,s) - U(t_1,s)]f(s, z_s + y_s)\|ds + \int_{t_1}^{t_2} \|U(t_2,s)f(s,z_s+y_s)\|ds \\
&\quad + \int_0^{t_1} \|[U(t_2,\zeta) - U(t_1,\zeta)]BW^{-1}\Big[\int_0^b U(b,s)f(s,z_s+y_s)\Big](\zeta)\|d\zeta \\
&\quad + \int_{t_1}^{t_2} \|U(t_2,\zeta)BW^{-1}\Big[\int_0^b U(b,s)f(s,z_s+y_s)\Big](\zeta)\|d\zeta.
\end{aligned}$$

Let $Y(\zeta) = BW^{-1}\Big[\int_0^b U(b,s)f(s,z_s+y_s)\Big](\zeta)$, then

$$\begin{aligned}
&\|\Gamma_2 z(t_2) - \Gamma_2 z(t_1)\| \\
&\leq \int_0^{t_1} \|U(t_2,s) - U(t_1,s)\|\alpha_{q'}(s)ds + \int_{t_1}^{t_2} \|U(t_2,s)\|\alpha_{q'}(s)ds \\
&\quad + \int_0^{t_1} \|U(t_2,\zeta) - U(t_1,\zeta)\|\|Y(\zeta)\|d\zeta + \int_{t_1}^{t_2} \|U(t_2,\zeta)\|\|Y(\zeta)\|d\zeta. \qquad (3.7)
\end{aligned}$$

By the equicontinuity property of $\{U(t,s) : (t,s) \in \Delta\}$ and the absolute continuity of the Lebesgue integral, we can see that the right hand side of (3.7) tends to zero and independent of z as $t_2 \to t_1$. Hence, $\Gamma_2(B_q)$ is equicontinuous on J.
To prove the Monch condition, let $W \subseteq B_q$ is countable and $W \subseteq \overline{co}(\{0\} \cup \Gamma(W))$. We shall show that $\beta(W) = 0$. Without loss of generality, we may suppose that $W = \{z^n\}_{n\in\mathbb{N}}$.
Then by the hypothesis $(H2)(iii),\ (H3)(ii)$ and Lemma 2.2, we have

$$\begin{aligned}
\beta\Big(\Gamma_2 W(t)\Big) &= \beta\Big(\Big\{\Gamma_2 z^n(t)\Big\}_{n=1}^{\infty}\Big) \\
&\leq \beta\Big(\Big\{\int_0^t U(t,s)f(s,z_s^n + y_s)ds\Big\}_{n=1}^{\infty}\Big) \\
&\quad + \beta\Big(\Big\{\int_0^t U(t,\zeta)BW^{-1}\Big[\int_0^b U(b,s)f(s,z_s^n+y_s)ds\Big](\zeta)d\zeta\Big\}_{n=1}^{\infty}\Big) \\
&\leq 2M_0 \int_0^b \eta(s) \sup_{-\infty<\theta\leq 0} \beta\Big(\Big\{z^n(s+\theta) + y(s+\theta)\Big\}_{n=1}^{\infty}\Big)ds \\
&\quad + 2M_0M_1 \int_0^b \beta\Big(W^{-1}\Big[\Big\{\int_0^b U(b,s)f(s,z_s^n+y_s)ds\Big\}_{n=1}^{\infty}\Big](\zeta)\Big)d\zeta
\end{aligned}$$

$$\leq 2M_0 \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} \beta(z^n(\tau)) + 2M_0M_1\Big(\int_0^b K_W(\zeta)d\zeta\Big)$$
$$\times \beta\Big(\Big\{\int_0^b U(b,s)f(s,z_s^n+y_s)ds\Big\}_{n=1}^{\infty}\Big),$$
$$\leq 2M_0 \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} \beta(W(\tau)) + 4M_0^2M_1\Big(\int_0^b K_W(\zeta)d\zeta\Big)$$
$$\times \int_0^b \eta(s)ds. \sup_{0\leq\tau\leq s} \beta(W(\tau)),$$
$$= (2M_0 + 4M_0^2M_1\|K_W\|_{L^1})\|\eta\|_{L^1} \sup_{0\leq\tau\leq s} \beta(W(\tau)).$$

That is, $$\beta(\Gamma_2 W(t)) \leq \tilde{N} \sup_{0\leq\tau\leq s} \beta(W(\tau)), \tag{3.8}$$
where $\tilde{N} = (2M_0 + 4M_0^2M_1\|K_W\|_{L^1})\|\eta\|_{L^1}$.

Since Γ_2 maps B_q into an equicontinuous family on J, $\Gamma_2(W)$ is equicontinuous on J and so W is equicontinuous on J. Then by Lemma 2.1, taking supremum on both sides of (3.8) over J, we have

$$\beta(\Gamma_2(W)) \leq \tilde{N}\beta(W). \tag{3.9}$$

By the property (7) of Definition 2.2,

$$\beta(\Gamma_1(W)) \leq N\beta(W). \tag{3.10}$$

Hence $\beta(\Gamma(W)) \leq \beta(\Gamma_1(W)) + \beta(\Gamma_2(W)) \leq (N+\tilde{N})\beta(W)$.

From the Monch condition, we get that

$$\beta(W) \leq \beta(\overline{co}(\{0\}\cup\Gamma(W))) = \beta(\Gamma(W)) \leq (N+\tilde{N})\beta(W).$$

By $(H7)$, $N+\tilde{N} < 1$, which implies that $\beta(W) = 0$. In the view of Lemma 2.3, i.e., Monch fixed point theorem, we conclude that Γ has a fixed point z in W. Then $x = z + y$ is a fixed point of Φ and thus the system $(1.1)-(1.3)$ is controllable on $[0,b]$. □

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OPTIMAL CONSUMPTION AND SLUTSKY EQUATION WITH EPSTEIN-ZIN TYPE PREFERENCE

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ABSTRACT. In this paper we conduct comparative statics for optimal consumption and portfolio selection of an agent who has a utility function of Epstein and Zin type. We derive the Slutsky equations and decompose the total effects of changes into the substitution effects and the income effects. We identify the role of the elasticity of intertemporal substitution and the coefficient of relative risk aversion.

1. INTRODUCTION

It has been long noted that portfolio selection is closely related to consumer choice. An economic agent finances future consumption by investing in assets. Two different allocations of investments will result in different consumption profiles for the agent. Thus, one can apply consumer choice theory to the study of portfolio selection. Pye [1] noted the importance of this relationship and hinted at the applicability of the Slutsky equation of consumer choice theory to comparative statics analysis for portfolio selection problems. Bierwag and Grove [2], Sandmo [3, 4], Fisher [5], Dalal [6], and Eichner [7, 8] have derived Slutsky equations for asset demands.

Despite the extensive literature on the subject, most research has been concerned with a static portfolio selection problem, ignoring intertemporal allocation of consumption. Fisher [5] was an exception and considered the intertemporal contingent consumption choice problem. In this paper we extend his analysis to the case where the economic agent has a recursive utility function of the Epstein-Zin type [9] and derive Slutsky equations for consumption and asset demand. This utility function allows separation of the elasticity of intertemporal substitution and the coefficient of relative risk aversion. This feature allows us to identify the roles of

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elasticity of intertemporal substitution and the coefficient of relative risk aversion on the overall effect of the income and substitution effects.

This paper is organized as follows. In section 2 we describe our model and solve our model in section 3. Section 4 contains analysis of the substitution and income effects and section 5 concludes.

2. The Model

We consider a consumption and portfolio selection problem of an economic agent in a 2-period discrete-time model. There exist two possible states of the world, u and d, at time 1.

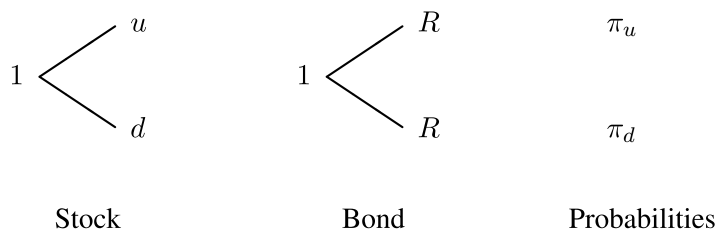

FIGURE 1. Asset Dynamics

The agent has the following utility function of the Epstein-Zin type:

$$U(c_0, c_1) = \left[u(c_0) + \beta u\left(\mathbb{E}\left[\mu(\tilde{c}_1)\right]^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\rho}} \tag{2.1}$$

where

$$u(c) = c^\rho,$$
$$\mu(c) = c^\alpha.$$

$\rho < 1, \alpha < 1, \rho \neq 0, \alpha \neq 0$, c_0 is the agent's consumption at time 0, c_u and c_d are her consumptions at state u and d at time 1, respectively, and $\mathbb{E}$ is her expectation at time 0.[1] It is well-known that the Epstein-Zin utility function has a constant elasticity of intertemporal substitution (EIS) equal to $\sigma \triangleq \frac{1}{1-\rho}$ and a coefficient of relative risk aversion (CRRA) equal to $\gamma \triangleq 1 - \alpha$.

[1] If $\rho = 0$, the utility function takes the form:

$$U(c_0, c_1) = \log c_0 + \beta \log\left(\mathbb{E}\left[\mu(\tilde{c}_1)\right]^{\frac{1}{\alpha}}\right).$$

If $\alpha = 0$, the utility function takes the form:

$$U(c_0, c_1) = \left[u(c_0) + \beta u\left(\exp\left(\mathbb{E}\log(\tilde{c}_1)\right)\right)\right]^{\frac{1}{\rho}},$$

where $u(c) = c^\rho$. The case $\rho = 0$ or $\alpha = 0$ can be regarded as the limiting case of the utility function in equation (2.1). We will derive our results based on this utility function. All our results, however, can be extended to the case $\rho = 0$ or $\alpha = 0$.

The investment opportunity consists of a stock and a bond. The stock price at time 0 is normalized to be 1 and becomes u in state u and d in state d with $u > d$.[2] The bond is risk-free with an interest rate equal to r. Thus, the risk-free return, R, is equal to $1 + r$. We assume a frictionless market. That is, there are no trading costs, taxes, and short-selling restrictions.

The agent wishes to maximize her utility given in (2.1) with her initial wealth W, choosing her consumption, c_0, c_u, c_d, and her investments, θ_s in the stock and θ_b in the bond. Her value function (indirect utility function) is defined as follows:

$$V(W) \triangleq \max_{\{c\},\theta_s,\theta_b} \left[(c_0)^\rho + \beta\left\{\pi_u(c_u)^\alpha + \pi_d(c_d)^\alpha\right\}^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}$$

subject to the budget constraints

$$c_0 \leq W - \theta_s - \theta_b, \tag{2.2}$$

$$c_u \leq \theta_s u + \theta_b R, \tag{2.3}$$

$$c_d \leq \theta_s d + \theta_b R. \tag{2.4}$$

The financial market is complete, since there exist two possible future states and two assets with linearly independent payoffs. Thus, we can rewrite (2.2) - (2.4) as the following one constraint,

$$c_0 + p_u c_u + p_d c_d \leq W, \tag{2.5}$$

where

$$p_u = \frac{R-d}{(u-d)R}, \tag{2.6}$$

$$p_d = \frac{u-R}{(u-d)R}. \tag{2.7}$$

Here p_u and p_d are equal to the prices of the Arrow-Debreu securities for states u and d, respectively. Notice that because of the market completeness the problem is essentially that of consumption choices and portfolio selection is a consequence of the consumption choices.

3. Individual Choice

The agent's problem can be stated as follows:

Problem: *The Agent's Consumption and Portfolio Choice Problem*

The agent chooses her consumption bundle, $\{c_0^1, c_u^1, c_d^1\}$ *and portfolio holdings,* $\{\theta_s, \theta_b\}$, *to maximize her utility given by* (2.1) *subject to budget constraint in* (2.5).

The Lagrangian of the problem is

$$\mathcal{L} = \left[(c_0)^\rho + \beta\left\{\pi_u(c_u)^\alpha + \pi_d(c_d)^\alpha\right\}^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} + \lambda\left\{W - c_0 - p_u c_u - p_d c_d\right\}.$$

[2] Here we use the same notation for the stock return and the state with a slight abuse of notation. The same notation for two different objects is for convenience and will not generate confusion.

The first-order conditions are given as follows

$$\frac{\partial \mathcal{L}}{\partial c_0} = \frac{\partial U}{\partial c_0} - \lambda = 0, \tag{3.1}$$

$$\frac{\partial \mathcal{L}}{\partial c_i} = \frac{\partial U}{\partial c_i} - p_i\lambda = 0,\ i = u, d, \tag{3.2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = W - c_0 - p_u c_u - p_d c_d = 0. \tag{3.3}$$

From (3.2) we get

$$\frac{\frac{\partial U}{\partial c_u}}{\frac{\partial U}{\partial c_d}} = \frac{\pi_u}{\pi_d}\left(\frac{c_u}{c_d}\right)^{\alpha-1} = \frac{p_u}{p_d}$$

and thus

$$\frac{c_u}{c_d} = \left(\frac{\pi_d}{\pi_u}\frac{p_u}{p_d}\right)^{\frac{1}{\alpha-1}} \triangleq \left(\frac{\zeta_u}{\zeta_d}\right)^{\frac{1}{\alpha-1}} \tag{3.4}$$

where ζ_i is the state price density of state i,

$$\zeta_i = \frac{p_i}{\pi_i}.$$

And from (3.1) and (3.2),

$$p_d = \frac{1}{(c_0)^{\rho-1}} \cdot \beta\pi_d(c_d)^{\alpha-1}\left\{\pi_u\left(\frac{\zeta_u}{\zeta_d}\right)^{\frac{\alpha}{\alpha-1}}(c_d)^{\alpha} + \pi_d(c_d)^{\alpha}\right\}^{\frac{\rho}{\alpha}-1}$$

$$\Rightarrow \quad \frac{c_d}{c_0} = \beta^{\frac{1}{1-\rho}}\zeta_d^{\frac{1}{\alpha-1}}\left(\pi_u\zeta_u^{\frac{\alpha}{\alpha-1}} + \pi_d\zeta_d^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\rho-\alpha}{\alpha(1-\rho)}}. \tag{3.5}$$

Symmetrically,

$$\frac{c_u}{c_0} = \beta^{\frac{1}{1-\rho}}\zeta_u^{\frac{1}{\alpha-1}}\left(\pi_u\zeta_u^{\frac{\alpha}{\alpha-1}} + \pi_d\zeta_d^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\rho-\alpha}{\alpha(1-\rho)}}. \tag{3.6}$$

Substituting (3.5) and (3.6) into budget constraint (3.3), we get the following optimal consumption choice of the agent:

$$c_0^{*m} = \frac{W}{1+\mathcal{A}}, \tag{3.7}$$

$$c_u^{*m} = \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{W}{1+\mathcal{A}}, \tag{3.8}$$

$$c_d^{*m} = \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_d^{\frac{1}{\alpha-1}}\frac{W}{1+\mathcal{A}}, \tag{3.9}$$

where

$$\mathcal{A} \triangleq \beta^{\frac{1}{1-\rho}}\left(\pi_u\zeta_u^{\frac{\alpha}{\alpha-1}} + \pi_d\zeta_d^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\rho(1-\alpha)}{\alpha(1-\rho)}} > 0. \tag{3.10}$$

The agent's demand for consumptions in times and states is *Marshallian* and thus we have used superscript m to stand for Marshallian demand.

The agent's optimal portfolio can be found by substituting (3.7) - (3.9) into the original budget constraints, (2.2) - (2.4).

$$\theta_s^{*m} = \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(u-d)(1+\mathcal{A})}, \tag{3.11}$$

$$\theta_b^{*m} = \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})}. \tag{3.12}$$

If the risk premium on the stock is positive, i.e., $\zeta_u^{\frac{1}{\alpha-1}} > \zeta_d^{\frac{1}{\alpha-1}}$, then $\theta_s^{*m} > 0$. From now on, we will assume that the risk premium is positive.

Value function $V(W)$ takes the following simple form

$$V(W) = (1+\mathcal{A})^{\frac{1}{\rho}-1}W. \tag{3.13}$$

We now consider the dual problem of the agent's optimization problem, the *expenditure minimization problem*. From (3.13) we derive the following expenditure function for utility level $\bar{u}$

$$E(\bar{u}) = (1+\mathcal{A})^{1-\frac{1}{\rho}}\,\bar{u}. \tag{3.14}$$

And the agent's Hicksian demand for consumptions and assets can be found from the optimal choices of utility maximization problem by substituting W by $E(\bar{u})$ in (3.14). The Hicksian demand for consumptions is given by the following

$$c_0^{*h} = \frac{\bar{u}}{(1+\mathcal{A})^{\frac{1}{\rho}}}, \tag{3.15}$$

$$c_u^{*h} = \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{\bar{u}}{(1+\mathcal{A})^{\frac{1}{\rho}}}, \tag{3.16}$$

$$c_d^{*h} = \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_d^{\frac{1}{\alpha-1}}\frac{\bar{u}}{(1+\mathcal{A})^{\frac{1}{\rho}}}, \tag{3.17}$$

and the Hicksian demand for assets is given as

$$\theta_s^{*h} = \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}, \tag{3.18}$$

$$\theta_b^{*h} = \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}. \tag{3.19}$$

Superscript h stands for *Hicksian*.

We will use (3.15) - (3.19) to easily derive Slutsky equations in the following section.

4. THE SUBSTITUTION EFFECT AND THE INCOME EFFECT

We now proceed to do comparative statics of the agent's consumption and portfolio choice. A well-known mathematical representation of the substitution effect and the income effect of a parameter change is the *Slutsky equation*. Cook [10] gave one-line proof of the Slutsky equation as follows. If we define the Marshallian demand function as

$$M(p_u, p_d, W) \triangleq \arg\max_c U(c_0, c_u, c_d),$$

and the Hicksian demand function as

$$H(p_u, p_d, \bar{u}) \triangleq \arg\min_c \{c_0 + p_u c_u + p_d c_d\},$$

by definition, $H(p_u, p_d, \bar{u}) = M(p_u, p_d, E(\bar{u}))$. Thus Slutsky equation is

$$\frac{\partial M}{\partial p_i} = \frac{\partial H}{\partial p_i} - \frac{\partial M}{\partial W} \cdot \frac{\partial E}{\partial p_i}. \tag{4.1}$$

The first term on the right hand side in (4.1) represents the *substitution effect* and the second term represents the *income effect*.

4.1. **The Effect of a Change in a State Price.** We analyze the substitution effect and the income effect of state prices, p_u and p_d on consumption at time 0.

We decompose the effect of a change in p_u on consumption at time 0 by deriving the Slutsky equation. Using (3.15) and (3.7), we derive the following Slutsky equation

$$\begin{aligned}\frac{\partial c_0^{*m}}{\partial p_u} &= \frac{\partial c_0^{*h}}{\partial p_u} - \frac{\partial c_0^{*m}}{\partial W}\frac{\partial E}{\partial p_u} \\ &= -\frac{\bar{u}}{\rho\,(1+\mathcal{A})^{\frac{1}{\rho}+1}}\frac{\partial \mathcal{A}}{\partial p_u} - \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{W}{(1+\mathcal{A})^2}\end{aligned} \tag{4.2}$$

where

$$\frac{\partial \mathcal{A}}{\partial p_u} = \frac{\rho}{\rho-1}\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}. \tag{4.3}$$

We see that ρ determines the sign of (4.3).

$$\begin{cases}\dfrac{\partial \mathcal{A}}{\partial p_u} > 0, & \text{if } \rho < 0, \\[2ex] \dfrac{\partial \mathcal{A}}{\partial p_u} < 0, & \text{if } 0 < \rho < 1.\end{cases}$$

From (4.2), it is obvious that the income effect is negative. Since her purchasing power decreases due to an increased price, the effect is to reduce the agent's current consumption.

The sign of the substitution effect depends on the magnitude of EIS, σ. Since $\frac{1}{1-\rho} = \sigma$, EIS < 1 if and only if $\rho < 0$ and EIS > 1 if and only if $0 < \rho < 1$. A large EIS (EIS > 1)

implies the agent is willing to substitute future consumption for current consumption, and thus the substitution effect of the change in the price of future consumption is positive. However, if EIS < 1, the agent is not very willing to substitute future consumption for current consumption, the substitution effect is negative.

We can derive the total effect from (3.7).

$$\frac{\partial c_0^{*m}}{\partial p_u} = -\frac{W}{(1+\mathcal{A})^2}\frac{\partial \mathcal{A}}{\partial p_u}. \tag{4.4}$$

Therefore, if the agent's EIS is less than 1, the total effect is negative, since both the substitution effect and the income effect are negative. If the EIS is greater than 1, the total effect is positive and the substitution effect dominates the income effect.

Case	Sub. effect	Income effect	Total effect
EIS < 1	–	–	–
EIS > 1	+	–	+

TABLE 1. The signs of the substitution and income effects for $\partial c_0^{*m}/\partial p_u$

The effect of a change in p_d on current consumption is similar to that of p_u.

Now we consider the effect of a change in one state price on optimal consumption in the other state, e.g., $\frac{\partial c_u^{*m}}{\partial p_d}$. The Slutsky equation is derived from (3.16) and (3.8).

$$\begin{aligned}\frac{\partial c_u^{*m}}{\partial p_d} &= \frac{\partial c_u^{*h}}{\partial p_d} - \frac{\partial c_u^{*m}}{\partial W}\frac{\partial E}{\partial p_d}\\ &= \frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{\bar{u}}{(1+\mathcal{A})^{\frac{1}{\rho}}}\frac{\partial \mathcal{A}}{\partial p_d}\\ &\quad -\beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\zeta_d^{\frac{1}{\alpha-1}}\frac{W}{(1+\mathcal{A})^2}.\end{aligned} \tag{4.5}$$

And it is simple to show that

$$\begin{cases}\dfrac{\partial \mathcal{A}}{\partial p_d} > 0, & \text{if } \rho < 0,\\[2ex] \dfrac{\partial \mathcal{A}}{\partial p_d} < 0, & \text{if } 0 < \rho < 1.\end{cases}$$

The term, $(\rho - \alpha + (\rho-1)\mathcal{A})/\rho$, plays a key role determining the sign of the substitution effect. However, we are not able to determine the sign of the substitution effect unambiguously. Instead, we can claim that with a large relative risk aversion coefficient γ, the substitution effect is negative. That is, if the agent is highly risk-averse, regardless of the agent's EIS, the substitution effect is negative. On the other hand, if the agent is not very risk-averse and her

EIS is large, as in (4.2), then she is willing to substitute consumption in state d for consumption at time 0, and the substitution effect on consumption in state u can be negative.

The income effect is unambiguously negative; since the price has increased, the agent's purchasing power has declined.

We can derive the total effect from (3.8).

$$\frac{\partial c_u^{*m}}{\partial p_d} = \frac{\rho - \alpha + \alpha(\rho - 1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})} \beta^{\frac{\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}} \zeta_u^{\frac{1}{\alpha-1}} \frac{W}{1+\mathcal{A}} \frac{\partial \mathcal{A}}{\partial p_d}. \tag{4.6}$$

If $\rho < 0$ (EIS < 1), since the sign of $(\rho - \alpha + \alpha(\rho-1)\mathcal{A})/\rho$ is not determined, we are not able to determine the sign of the total effect. However, for a large risk aversion, it is negative. And when $0 < \rho < 1$ (EIS > 1), $(\rho - \alpha + \alpha(\rho-1)\mathcal{A})/\rho$ is positive and $\frac{\partial \mathcal{A}}{\partial p_d}$ is negative and thus the sign of the total effect is negative. When p_d increases, the agent with an EIS > 1 or with large risk aversion will reduce her consumption in state u.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	–	+/–
EIS > 1	+/–	–	–

TABLE 2. The signs of the substitution and income effects for $\partial c_u^{*m}/\partial p_d$

A symmetric argument can be applied for $\frac{\partial c_d^{*m}}{\partial p_u}$.

We now analyze the effect of a change in a state price on the agent's portfolio choice. For investment in risky asset, the Slutsky equation is derived from (3.18) and (3.11) as follows

$$\begin{aligned} \frac{\partial \theta_s^{*m}}{\partial p_u} &= \frac{\partial \theta_s^{*h}}{\partial p_u} - \frac{\partial \theta_s^{*m}}{\partial W} \frac{\partial E}{\partial p_u} \\ &= \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right) \beta^{\frac{\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}} \bar{u}}{(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}} \left(\frac{\rho - \alpha + (\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right) \frac{\partial \mathcal{A}}{\partial p_u} \\ &\quad - \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}} \zeta_u^{\frac{1}{\alpha-1}} \bar{u}}{p_u(1-\alpha)(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}} - \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right) \beta^{\frac{2\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}} \zeta_u^{\frac{1}{\alpha-1}} W}{(u-d)(1+\mathcal{A})^2} \end{aligned} \tag{4.7}$$

The first two terms exhibit the substitution effect. The second term is unambiguously negative, but the sign of the first term is not determined due to term, $(\rho - \alpha + (\rho-1)\mathcal{A})/\rho$. However, if $0 < \rho < 1$ and γ is large, the substitution effect is negative. This is because when the price of a future consumption increases, an agent with a large EIS will increase current consumption and reduce savings. And the income effect is negative since the agent's purchasing power has declined.

We can derive the total effect of $\frac{\partial \theta_s^{*m}}{\partial p_u}$ from (3.11).

$$\frac{\partial \theta_s^{*m}}{\partial p_u} = \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(u-d)(1+\mathcal{A})}\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial p_u}$$
$$-\frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}W}{p_u(1-\alpha)(u-d)(1+\mathcal{A})}.$$

If $\rho < 0$ (EIS < 1), the sign of $(\rho-\alpha+\alpha(\rho-1)\mathcal{A})/\rho$ is not determined and thus the sign of the total effect is ambiguous. Again, with a large γ, it is negative. And if $0 < \rho < 1$ (EIS > 1), the total effect is negative. Therefore, when the price of future consumption increases, an agent with an EIS > 1 or with large risk aversion will reduce her investment in the risky asset.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	–	+/–
EIS > 1	+/–	–	–

TABLE 3. The signs of the substitution and income effects for $\partial \theta_s^{*m}/\partial p_u$

The Slutsky equation for investment in the bond is derived from (3.19) and (3.12) as follows

$$\frac{\partial \theta_b^{*m}}{\partial p_u} = \frac{\partial \theta_b^{*h}}{\partial p_u} - \frac{\partial \theta_b^{*m}}{\partial W}\frac{\partial E}{\partial p_u}$$
$$= \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial p_u}$$
$$+\frac{d\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\bar{u}}{p_u(1-\alpha)R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}} - \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}W}{R(u-d)(1+\mathcal{A})^2}.$$

The first two terms show the substitution effect and the signs of the substitution effect and the income effect are ambiguous, since the sign of term $\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)$ is not determined.

And the total effect can be derived from (3.12) as follows

$$\frac{\partial \theta_b^{*m}}{\partial p_u} = \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})}\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial p_u}$$
$$+\frac{d\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}W}{p_u(1-\alpha)R(u-d)(1+\mathcal{A})}.$$

The sign of the total effect is also is ambiguous.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	+/–	+/–
EIS > 1	+/–	+/–	+/–

TABLE 4. The signs of effect of $\partial\theta_b^{*m}/\partial p_u$

The effect of a change in p_d on portfolio choice is similar to that of p_u.

4.2. **The Effect of a Change in the Interest Rate.** Now let us consider the effect of a change in the interest rate. We first make the following observation which can be derived from (2.6) and (2.7):

$$\frac{\partial p_u}{\partial R} = -\frac{d}{(u-d)R^2} < 0,$$
$$\frac{\partial p_d}{\partial R} = -\frac{u}{(u-d)R^2} < 0.$$

When the interest rate increases, both state prices decline so that the equality, $p_u + p_d = 1/R$, still holds.

We can derive the following Slutsky equation for the effect of a change in the interest rate on consumption at time 0:

$$\begin{aligned}\frac{\partial c_0^{*m}}{\partial R} &= \frac{\partial c_0^{*h}}{\partial R} - \frac{\partial c_0^{*m}}{\partial W}\frac{\partial E}{\partial R}\\ &= -\frac{\bar{u}}{\rho\,(1+\mathcal{A})^{\frac{1}{\rho}+1}}\frac{\partial \mathcal{A}}{\partial R} - \frac{1}{1+\mathcal{A}}\left(\frac{\partial p_u}{\partial R}c_u^{*m} + \frac{\partial p_d}{\partial R}c_d^{*m}\right)\\ &= -\frac{\bar{u}}{\rho\,(1+\mathcal{A})^{\frac{1}{\rho}+1}}\frac{\partial \mathcal{A}}{\partial R} - \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\frac{W}{(1+\mathcal{A})^2}\left(\zeta_u^{\frac{1}{\alpha-1}}\frac{\partial p_u}{\partial R} + \zeta_d^{\frac{1}{\alpha-1}}\frac{\partial p_d}{\partial R}\right)\end{aligned} \tag{4.8}$$

where

$$\frac{\partial \mathcal{A}}{\partial R} = \frac{\rho}{\rho-1}\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\left(\zeta_u^{\frac{1}{\alpha-1}}\frac{\partial p_u}{\partial R} + \zeta_d^{\frac{1}{\alpha-1}}\frac{\partial p_d}{\partial R}\right). \tag{4.9}$$

We have applied the envelop theorem to obtain the second equality. And we know that

$$\begin{cases}\dfrac{\partial \mathcal{A}}{\partial R} < 0, & \text{if } \rho < 0,\\[2ex] \dfrac{\partial \mathcal{A}}{\partial R} > 0, & \text{if } 0 < \rho < 1.\end{cases}$$

In (4.8) the substitution effect is always negative regardless of whether the EIS is smaller than or larger than 1. If the interest rate increases, the price of future consumption decreases and the agent will increase future consumption and reduce current consumption. And the income effect is positive, since the agent's purchasing power increases with an increase in the interest rate.

From (3.7) we derive the following total effect:

$$\frac{\partial c_0^{*m}}{\partial R} = -\frac{W}{(1+\mathcal{A})^2}\frac{\partial \mathcal{A}}{\partial R}.$$

The EIS determines the sign of the total effect. When the EIS is less than 1, the total effect is positive. Since the EIS is small, the income effect dominates the substitution effect. And if when EIS is larger than 1, the total effect is negative and thus the substitution effect dominates the income effect. Therefore, when the interest rate increases, an agent with an EIS < 1 will increase her current consumption and an agent with an EIS > 1 will reduce her current consumption.

Case	Sub. effect	Income effect	Total effect
EIS < 1	−	+	+
EIS > 1	−	+	−

TABLE 5. The signs of the substitution and income effects for $\partial c_0^{*m}/\partial R$

We can now derive the following Slutsky equation for c_u^{*m} from (3.16) and (3.8):

$$\begin{aligned}\frac{\partial c_u^{*m}}{\partial R} &= \frac{\partial c_u^{*h}}{\partial R} - \frac{\partial c_u^{*m}}{\partial W}\frac{\partial E}{\partial R} \\ &= \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{\bar{u}}{(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\frac{\partial \mathcal{A}}{\partial R} - \frac{1}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R}\right) \\ &\quad -\beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}\frac{W}{(1+\mathcal{A})^2}\left(\frac{\partial p_u}{\partial R}\zeta_u^{\frac{1}{\alpha-1}} + \frac{\partial p_d}{\partial R}\zeta_d^{\frac{1}{\alpha-1}}\right). \end{aligned} \tag{4.10}$$

It is clear that the income effect is positive with an increased purchasing power. For the substitution effect, the second term, $-\frac{1}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R}$, in the parenthesis of the first term is positive. Due to term $(\rho-\alpha+(\rho-1)\mathcal{A})/\rho$, however, the sign of the substitution effect is ambiguous. But when the agent's risk aversion is large, the substitution effect is positive.

We can derive the total effect from (3.8).

$$\frac{\partial c_u^{*m}}{\partial R} = \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{W}{1+\mathcal{A}}\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\frac{\partial \mathcal{A}}{\partial R} - \frac{1}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R}\right).$$

If $\rho < 0$ (EIS < 1) the sign of the total effect is ambiguous. With large risk aversion, it is positive. If $0 < \rho < 1$ (EIS > 1), the total effect is positive. Therefore, when the interest rate increases, the agent with a large EIS or with large risk aversion will increase her consumption in state u.

Case	Sub. effect	Income effect	Total effect
EIS < 1	–	+	+/–
EIS > 1	–	+	+

TABLE 6. The signs of the substitution and income effects for $\partial c_u^{*m}/\partial R$

We now analyze the effect of a change in the interest rate on the agent's portfolio choice. We can derive the following Slutsky equation

$$\begin{aligned}\frac{\partial \theta_s^{*m}}{\partial R} &= \frac{\partial \theta_s^{*h}}{\partial R} - \frac{\partial \theta_s^{*m}}{\partial W}\frac{\partial E}{\partial R} \\ &= \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial R} \\ &\quad + \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{\zeta_d^{\frac{1}{\alpha-1}}}{p_d(1-\alpha)}\frac{\partial p_d}{\partial R} - \frac{\zeta_u^{\frac{1}{\alpha-1}}}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R}\right) \\ &\quad - \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}W}{(u-d)(1+\mathcal{A})^2}\left(\zeta_u^{\frac{1}{\alpha-1}}\frac{\partial p_u}{\partial R} + \zeta_d^{\frac{1}{\alpha-1}}\frac{\partial p_d}{\partial R}\right).\end{aligned}$$

The first two terms exhibit the substitution effect and the signs of both terms are ambiguous. But the income effect is unambiguously positive, since the agent's purchasing power increases with an increase in the interest rate.

We derive the total effect from (3.11).

$$\begin{aligned}\frac{\partial \theta_s^{*m}}{\partial R} &= \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(u-d)(1+\mathcal{A})}\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial R} \\ &\quad + \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(u-d)(1+\mathcal{A})}\left(\frac{\zeta_d^{\frac{1}{\alpha-1}}}{p_d(1-\alpha)}\frac{\partial p_d}{\partial R} - \frac{\zeta_u^{\frac{1}{\alpha-1}}}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R}\right).\end{aligned}$$

The sign of the total effect is ambiguous in both cases, $\rho < 0$ and $0 < \rho < 1$.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	+	+/–
EIS > 1	+/–	+	+/–

TABLE 7. The signs of the substitution and income effects for $\partial\theta_s^{*m}/\partial R$

For investment in the risk-free bond, we can derive the following Slutsky equation from (3.19) and (3.12):

$$\begin{aligned}\frac{\partial\theta_b^{*m}}{\partial R} &= \frac{\partial\theta_b^{*h}}{\partial R} - \frac{\partial\theta_b^{*m}}{\partial W}\frac{\partial E}{\partial R}\\ &= \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left[\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial\mathcal{A}}{\partial R} + \frac{1}{R}\right]\\ &\quad + \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{d\zeta_u^{\frac{1}{\alpha-1}}}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R} - \frac{u\zeta_d^{\frac{1}{\alpha-1}}}{p_d(1-\alpha)}\frac{\partial p_d}{\partial R}\right)\\ &\quad - \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})^2}\left(\zeta_u^{\frac{1}{\alpha-1}}\frac{\partial p_u}{\partial R} + \zeta_d^{\frac{1}{\alpha-1}}\frac{\partial p_d}{\partial R}\right).\end{aligned} \tag{4.11}$$

The first two terms show the substitution effect, and we are not able to determine the sign of each term. And also the sign of the income effect is not determined.

The indeterminacy of the sign of the substitution effect in (4.11) can be explained by Dalal [6]'s intuitive argument. His model considers only portfolio choice but the argument is still applicable here. The substitution effect is measured in a situation where the agent minimizes expenditure to keep utility level constant. Suppose that the interest rate increases but the risk premium stays still positive. Suppose also that the agent is not very risk averse and thus reduces her bond investment by a small amount without changing her stock investment in an attempt to keep her expected utility unchanged and take advantage of the risk premium. Since the overall risk of return has decreased, however, her action would probably raise her expected utility. To keep the utility level constant she might need to reduce her bond holding further. In the other case where the agent is severely risk-averse, the agent will increase her bond investment and reduce her stock investment and thus lowers the expected return and risk of her portfolio so that her expected utility is kept constant.

Though Dalal's argument is for portfolio choice, we can apply it to consumption choice as well. For example, (4.10) shows a similar behavior. If the agent is heavily risk-averse, i.e., γ is large, $\rho - \alpha + (\rho - 1)\mathcal{A}$ is positive and thus the substitution effect becomes positive in both cases EIS < 1 and when EIS > 1.

The total effect of the interest rate change on the agent's investment in the risk-free bond is given as

$$\frac{\partial \theta_b^{*m}}{\partial R} = \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})}\left[\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial\mathcal{A}}{\partial R} + \frac{1}{R}\right]$$
$$+\frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})}\left(\frac{d\zeta_u^{\frac{1}{\alpha-1}}}{p_u(1-\alpha)}\frac{\partial p_u}{\partial R} - \frac{u\zeta_d^{\frac{1}{\alpha-1}}}{p_d(1-\alpha)}\frac{\partial p_d}{\partial R}\right). \tag{4.12}$$

The sign of the total effect is ambiguous.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	+/–	+/–
EIS > 1	+/–	+/–	+/–

TABLE 8. The signs of the substitution and income effects for $\partial\theta_b^{*m}/\partial R$

4.3. **The Effect of a Change in the Return on the Stock.** We now analyze the effect of a change in the return on the stock. When the return changes, it changes the state prices. From (2.6) and (2.7) we observe

$$\frac{\partial p_d}{\partial u} = -\frac{\partial p_u}{\partial u} = \frac{R-d}{(u-d)^2 R} \triangleq \frac{\partial p}{\partial u} > 0. \tag{4.13}$$

We derive the following Slutsky equation for $\frac{\partial c_0^{*m}}{\partial u}$ from (3.15) and (3.7):

$$\frac{\partial c_0^{*m}}{\partial u} = \frac{\partial c_0^{*h}}{\partial u} - \frac{\partial c_0^{*m}}{\partial W}\frac{\partial E}{\partial u}$$
$$= -\frac{\bar{u}}{\rho\,(1+\mathcal{A})^{\frac{1}{\rho}+1}}\frac{\partial\mathcal{A}}{\partial u} + \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\frac{W}{(1+\mathcal{A})^2}\frac{\partial p}{\partial u},$$

where

$$\frac{\partial\mathcal{A}}{\partial u} = \frac{\rho}{\rho-1}\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\left(\zeta_d^{\frac{1}{\alpha-1}} - \zeta_u^{\frac{1}{\alpha-1}}\right)\frac{\partial p}{\partial u}.$$

It is simple to show that

$$\begin{cases} \dfrac{\partial\mathcal{A}}{\partial u} < 0, & \text{if } \rho < 0, \\ \dfrac{\partial\mathcal{A}}{\partial u} > 0, & \text{if } 0 < \rho < 1. \end{cases}$$

Therefore, the substitution effect is negative and the income effect is positive. When u increases, the expected return on the stock increases and the agent will increase her investment in it and has incentive to reduce current consumption. Since the consumption in state u is now cheaper, her purchasing power increases.

Let us now compute the total effect from (3.7). Since

$$\frac{\partial c_0^{*m}}{\partial u} = -\frac{W}{(1+\mathcal{A})^2}\frac{\partial \mathcal{A}}{\partial u},$$

If $\rho < 0$ (EIS < 1), the income effect dominates the substitution effect and thus the total effect is positive. If $0 < \rho < 1$ (EIS > 1), the substitution effect dominates the income effect and the total effect is negative. Thus, when u increases, an agent with an EIS < 1 will increase her consumption at time 0 due to increased purchasing power, she is not much willing to substitute current consumption for future consumption. An agent with an EIS > 1 will reduce her current consumption due to the lower price of consumption in state u.

Case	Sub. effect	Income effect	Total effect
EIS < 1	−	+	+
EIS > 1	−	+	−

TABLE 9. The signs of the substitution and income effects for $\partial c_0^{*m}/\partial u$

A similar argument can be applied to a change of d.

We now consider the effect of a change in the stock's return on future consumption. Here we will study the effect of a change in d. The Slutsky equation for $\frac{\partial c_u^{*m}}{\partial d}$ can derived as follows

$$\begin{aligned}\frac{\partial c_u^{*m}}{\partial d} &= \frac{\partial c_u^{*h}}{\partial d} - \frac{\partial c_u^{*m}}{\partial W}\frac{\partial E}{\partial d}\\ &= \beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\frac{\bar{u}}{(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\frac{\partial \mathcal{A}}{\partial d} + \frac{1}{p_u(1-\alpha)}\frac{\partial p}{\partial d}\right)\\ &\quad + \beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}\zeta_u^{\frac{1}{\alpha-1}}\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\frac{W}{(1+\mathcal{A})^2}\frac{\partial p}{\partial d}\end{aligned}$$

where

$$\frac{\partial p_d}{\partial d} = -\frac{\partial p_u}{\partial d} = \frac{u-R}{(u-d)^2 R} \triangleq \frac{\partial p}{\partial d} > 0. \tag{4.14}$$

We observe that the term, $(\rho - \alpha + (\rho-1)\mathcal{A})/\rho$, determines the sign of the substitution effect, and thus we are not able to easily determine the sign of the substitution effect. However, with a large γ it is positive. It is because an increase in return d results in a decrease in p_u and thus the agent will increase consumption in state u. And the income effect is positive, since the return on the risky asset has increased and the agent has a larger purchasing power.

We can derive the total effect from (3.8).

$$\frac{\partial c_u^{*m}}{\partial d} = \beta^{\frac{\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}} \zeta_u^{\frac{1}{\alpha-1}} \left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})} \frac{\partial \mathcal{A}}{\partial u} + \frac{1}{p_u(1-\alpha)} \frac{\partial p}{\partial d} \right) \frac{W}{1+\mathcal{A}}.$$

When $\rho < 0$ (EIS < 1), the sign of the term, $\rho - \alpha + \alpha(\rho - 1)\mathcal{A}$, is ambiguous. With large risk aversion, however, the total effect is positive. And when $0 < \rho < 1$ (EIS > 1), the total effect is clearly positive. It is interesting that when d increases, the agent increases consumption in state u. Therefore, when d increases, an agent with a large EIS or with large risk aversion will increase consumption in state u.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	+	+/–
EIS > 1	+/–	+	+

TABLE 10. The signs of the substitution and income effects for $\partial c_u^{*m}/\partial d$

Now we analyze the effect of a change in the stock's return on the agent's portfolio choice. For investment in risky asset, the following Slutsky equation can be derived from (3.18) and (3.11):

$$\begin{aligned}
\frac{\partial \theta_s^{*m}}{\partial u} &= \frac{\partial \theta_s^{*h}}{\partial u} - \frac{\partial \theta_s^{*m}}{\partial W}\frac{\partial E}{\partial u} \\
&= \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right) \beta^{\frac{\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}} \bar{u}}{(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}} \left[\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})} \right) \frac{\partial \mathcal{A}}{\partial u} + \frac{1}{u-d} \right] \\
&\quad + \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}} \bar{u}}{(1-\alpha)(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}} \left(\frac{\zeta_d^{\frac{1}{\alpha-1}}}{p_d} + \frac{\zeta_u^{\frac{1}{\alpha-1}}}{p_u} \right) \frac{\partial p}{\partial u} \\
&\quad + \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)^2 \beta^{\frac{2\alpha}{\rho(1-\alpha)}} \mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}} W}{(u-d)(1+\mathcal{A})^2} \frac{\partial p}{\partial u}.
\end{aligned}$$

The first two terms show the substitution effect. The second term is positive, but the sign of first term is not determined, and thus, the sign of substitution effect is ambiguous. However, with large risk aversion, it is positive. The income effect is always positive.

The total effect on the investment in the stock can be derived as follows

$$\frac{\partial \theta_s^{*m}}{\partial u} = \frac{\left(\zeta_u^{\frac{1}{\alpha-1}} - \zeta_d^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(u-d)(1+\mathcal{A})}\left[\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial u} + \frac{1}{u-d}\right]$$
$$+\frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(1-\alpha)(u-d)(1+\mathcal{A})}\left(\frac{\zeta_d^{\frac{1}{\alpha-1}}}{p_d} + \frac{\zeta_u^{\frac{1}{\alpha-1}}}{p_u}\right)\frac{\partial p}{\partial u}.$$

If $\rho < 0$ (EIS < 1), the sign of $(\rho-\alpha+\alpha(\rho-1)\mathcal{A})/\rho$ is not determined and thus the sign of total effect is ambiguous. But with large risk aversion, the total effect is positive. If $0 < \rho < 1$ (EIS > 1), the total effect is positive.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	+	+/–
EIS > 1	+/–	+	+

TABLE 11. The signs of the substitution and income effects for $\partial\theta_s^{*m}/\partial u$

Next we derive the Slutsky equation for the effect on the investment in the bond.

$$\frac{\partial \theta_b^{*m}}{\partial u} = \frac{\partial \theta_b^{*h}}{\partial u} - \frac{\partial \theta_b^{*m}}{\partial W}\frac{\partial E}{\partial u}$$
$$= \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left[\left(\frac{\rho-\alpha+(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial u} + \frac{1}{u-d}\right]$$
$$-\frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\bar{u}}{(1-\alpha)R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}\left(\frac{d\zeta_u^{\frac{1}{\alpha-1}}}{p_u} + \frac{u\zeta_d^{\frac{1}{\alpha-1}}}{p_d}\right)\frac{\partial p}{\partial u} + \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_d^{\frac{1}{\alpha-1}}\bar{u}}{R(u-d)(1+\mathcal{A})^{\frac{1}{\rho}}}$$
$$-\frac{\left(\zeta_d^{\frac{1}{\alpha-1}} - \zeta_u^{\frac{1}{\alpha-1}}\right)\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{2\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{2(\rho-\alpha)}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})^2}\frac{\partial p}{\partial u}.$$

The first three terms show the substitution effect and the fourth term displays the income effect. The sign of each effect is ambiguous.

We can derive the total effect on the investment in the stock as follows

$$\frac{\partial \theta_b^{*m}}{\partial u} = \frac{\left(u\zeta_d^{\frac{1}{\alpha-1}} - d\zeta_u^{\frac{1}{\alpha-1}}\right)\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{R(u-d)(1+\mathcal{A})}\left[\left(\frac{\rho-\alpha+\alpha(\rho-1)\mathcal{A}}{\rho(1-\alpha)\mathcal{A}(1+\mathcal{A})}\right)\frac{\partial \mathcal{A}}{\partial u} + \frac{1}{u-d}\right]$$
$$-\frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}W}{(1-\alpha)R(u-d)(1+\mathcal{A})}\left(\frac{d\zeta_u^{\frac{1}{\alpha-1}}}{p_u} + \frac{u\zeta_d^{\frac{1}{\alpha-1}}}{p_d}\right)\frac{\partial p}{\partial u} + \frac{\beta^{\frac{\alpha}{\rho(1-\alpha)}}\mathcal{A}^{\frac{\rho-\alpha}{\rho(1-\alpha)}}\zeta_d^{\frac{1}{\alpha-1}}W}{R(u-d)(1+\mathcal{A})}.$$

The sign of the total effect is again ambiguous.

Case	Sub. effect	Income effect	Total effect
EIS < 1	+/–	+/–	+/–
EIS > 1	+/–	+/–	+/–

TABLE 12. The signs of the substitution and income effects for $\partial\theta_b^{*m}/\partial u$

The effect of a change in d on the agent's portfolio choice is similar to that of u.

5. CONCLUSION

We have investigated the effects of changes in the interest rate and the return on the risky asset on an economic agent's current and future consumptions and portfolio choice when the agent has a utility function of the Epstein and Zin type. We have derived the Slutsky equations and decomposed the total effects of changes into the substitution effects and the income effects. We have identified the role of the elasticity of intertemporal substitution and the coefficient of relative risk aversion.

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EFFECT OF POROSITY ON THE TRANSIENT MHD GENERALIZED COUETTE FLOW WITH HEAT TRANSFER IN THE PRESENCE OF HEAT SOURCE AND UNIFORM SUCTION AND INJECTION

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ABSTRACT. The transient magnetohydrodynamic (MHD) generalized Couette flow with heat transfer through a porous medium of an electrically conducting, viscous, incompressible fluid bounded by two parallel insulating porous plates is studied in the presence of uniform suction and injection and a heat source considering the Hall effect. A uniform and constant pressure gradient is imposed in the axial direction and an externally applied uniform magnetic field as well as a uniform suction and injection are applied in the direction perpendicular to the plates. The two plates are kept at different but constant temperatures while the Joule and viscous dissipations are included in the energy equation. The effect of the Hall current, the porosity of the medium and the uniform suction and injection on both the velocity and temperature distributions is investigated.

1. INTRODUCTION

The magnetohydrodynamic (MHD) flow between two parallel plates, known as Hartmann flow, is a classical problem that has many applications in MHD power generators, MHD pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and sprays. Hartmann and Lazarus [1] studied the influence of a transverse uniform magnetic field on the flow of a conducting fluid between two infinite parallel, stationary, and insulated

+plates. Then, a lot of research work concerning the Hartmann flow has been obtained under different physical effects [2-10]. In most cases the Hall and ion slip terms were

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ignored in applying Ohm's law as they have no marked effect for small and moderate values of the magnetic field. However, the current trend for the application of MHD is towards a strong magnetic field, so that the influence of electromagnetic force is noticeable [6-7]. Under these conditions, the Hall current and ion slip are important and they have a marked effect on the magnitude and direction of the current density and consequently on the magnetic force term. Soudalgekar et al. [8-9] studied the effect of the Hall currents on the steady MHD Couette flow with heat transfer. The temperatures of the two plates were assumed either to be constant [8] or to vary linearly along the plates in the direction of the flow [9]. Attia [10-11] extended the problem to the unsteady state with heat transfer, with constant pressure gradient applied.

In the present work, the transient generalized Couette flow and heat transfer through a porous medium of an incompressible, viscous, electrically conducting fluid between two infinite insulating horizontal porous plates are studied with the consideration of the Hall current and in the presence of a heat source. The upper plate is moving with a uniform velocity while the lower plate remains stationary. The fluid is acted upon by a constant pressure gradient, a uniform suction and injection and a uniform magnetic field perpendicular to the plates. The flow through a porous medium deals with the analysis in which the differential equation governing the fluid motion is based on the Darcy's law which accounts for the drag exerted by the porous medium [12-14]. The two plates are maintained at two different but constant temperatures. The governing equations including the Joule and viscous dissipations are solved numerically using the method of finite difference. The effect of the magnetic field, the Hall current, the porosity of the medium and the suction and injection on both the velocity and temperature distributions is reported.

2. Description of the Problem

The two insulating plates are located at the $y=\pm h$ planes and extend from $x=-\infty$ to ∞ and $z=-\infty$ to ∞. The upper plate is moving with a uniform velocity U_o while the lower plate remains fixed. The lower and upper plates are kept at the two constant temperatures T_1 and T_2, respectively, where $T_2>T_1$ and a heat source is included. The fluid flows between the two plates under the effect of a constant pressure gradient dP/dx in the axial x-direction, and a uniform suction from above and injection from below which are applied at $t=0$ with velocity v_o. The whole system is subjected to a uniform magnetic field B_o in the positive y-direction. This is the total magnetic field acting on the fluid since the induced magnetic field is neglected. The fluid flows between the two plates in a porous medium where the Darcy model is assumed [12-14]. From the geometry of the problem, it is evident that all quantities are independent of x and z-coordinates apart from the pressure gradient dP/dx. The existence of the Hall term results in a z-component of the velocity. Thus, the velocity vector of the fluid is $v(y,t) = u(y,t)i + v_o j + w(y,t)k$.

The initial and boundary conditions are: u=w=0 at t≤0, u=w=0 at y=-h for t>0 and u=U_o and w=0 at y=h for t>0. The temperature $T(y,t)$ at any point in the fluid satisfies both the initial and boundary conditions T=T_1 at t≤0, T=T_2 at y=+h, and T=T_1 at y=-h for t>0. The fluid flow is governed by the momentum equation

$$\rho \frac{Dv}{Dt} = \mu \nabla^2 v - \nabla P + J \wedge B_o \tag{1}$$

where ρ and μ are, respectively, the density and the coefficient of viscosity of the fluid. If the Hall term is retained, the current density $\boldsymbol{J}$ is given by

$$J = \sigma(v \wedge B_o - \beta(J \wedge B_o))$$

where σ is the electric conductivity of the fluid, and β is the Hall factor [8,9]. This equation may be solved in $\boldsymbol{J}$ yielding

$$J \wedge B_o = -\frac{\sigma B_o^2}{1+m^2}((u+mw)i + (w-mu)k) \tag{2}$$

where $m = \sigma\beta B_o$, is the Hall parameter [8,9]. Thus, in terms of Eq. (2), the two components of Eq. (1) read [15]

$$\rho \frac{\partial u}{\partial t} + \rho v_o \frac{\partial u}{\partial y} = -\frac{dP}{dx} + \mu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{1+m^2}(u+mw) - \frac{\mu}{K_p} u, \tag{3}$$

$$\rho \frac{\partial w}{\partial t} + \rho v_o \frac{\partial w}{\partial y} = \mu \frac{\partial^2 w}{\partial y^2} - \frac{\sigma B_o^2}{1+m^2}(w-mu) - \frac{\mu}{K_p} w, \tag{4}$$

The temperature distribution is governed by the energy equation [15]

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p v_o \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial y^2} + Q(T - T_1) + \mu((\frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y})^2) + \frac{\sigma B_o^2}{1+m^2}(u^2 + w^2), \tag{5}$$

where c_p and k are, respectively, the specific heat capacity and the thermal conductivity of the fluid, K_p is the Darcy's permeability and Q is the heat generation coefficient. The last terms in the right side of Eqs. (3) and (4) represent the porosity force. The second and third terms in the right side represent the viscous and Joule dissipations, respectively. Introducing the following non-dimensional quantities

$$\hat{x} = \frac{x}{h}, \hat{y} = \frac{y}{h}, \hat{z} = \frac{z}{h}, \hat{u} = \frac{u}{U_o}, \hat{w} = \frac{w}{U_o}, \hat{P} = \frac{P}{\rho U_o^2}, \hat{t} = \frac{tU_o}{h},$$

$Re = \rho h U_O / \mu$, is the Reynolds number,

$S = v_O / U_O$, is the suction parameter,

$Pr = \mu c_p / k$, is the Prandtl number,

$Ec = U_o^2 / c_p (T_2 - T_1)$, is the Eckert number,

$Ha^2 = \sigma B_O^2 h^2 / \mu$, where Ha is the Hartmann number,

$M = h\mu / (\rho U_0 Kp)$ is the porosity parameter,

$\hat{Q} = QU_0 / (\rho h c_p)$ is the dimensionless heat generation coefficient the basic Eqs. (3)-(5) are written as (the hats are dropped for convenience)

$$\frac{\partial u}{\partial t}+\frac{S}{\mathrm{Re}}\frac{\partial u}{\partial y}=-\frac{dP}{dx}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}-\frac{Ha^2}{\mathrm{Re}(1+m^2)}(u+mw)-Mu, \tag{6}$$

$$\frac{\partial w}{\partial t}+\frac{S}{\mathrm{Re}}\frac{\partial w}{\partial y}=\frac{1}{\mathrm{Re}}\frac{\partial^2 w}{\partial y^2}-\frac{Ha^2}{\mathrm{Re}(1+m^2)}(w-mu)-Mw, \tag{7}$$

$$\frac{\partial T}{\partial t}+\frac{S}{\mathrm{Re}}\frac{\partial T}{\partial y}=\frac{1}{\mathrm{Pr}}\frac{\partial^2 T}{\partial y^2}+QT+Ec((\frac{\partial u}{\partial y})^2+(\frac{\partial w}{\partial y})^2)+\frac{EcHa^2}{(1+m^2)}(u^2+w^2), \tag{8}$$

The initial and boundary conditions for the velocity become

$$t\le 0: u=w=0, t>0: u=w=0, y=-1, u=1, w=0, y=1, \tag{9}$$

and the initial and boundary conditions for the temperature are given by

$$t\le 0: T=0, t>0: T=1, y=+1, T=0, y=-1. \tag{10}$$

3. NUMERICAL SOLUTION OF THE GOVERNING EQUATIONS

Equations (6)-(8) are solved numerically using the method of finite difference [16] under the initial and boundary conditions (9) and (10) to determine the velocity and temperature distributions for different values of the parameters *Ha*, *m*, *M*, *S* and *Q*. The Crank-Nicolson implicit method is applied and the finite difference equations are written at the mid-point of the computational cell and the different terms are replaced by their second-order central difference approximations in the *y*-direction. The diffusion term is replaced by the average of the central differences at two successive time levels. The viscous and Joule dissipation terms are evaluated using the velocity components and their derivatives in the *y*-direction which are obtained from the exact solution. Finally, the block tri-diagonal system is solved using Thomas' algorithm. All computations are carried out for dP/dx=5, Re=1, Pr=1 and Ec=0.2.

4. RESULTS AND DISCUSSION

Figure 1 presents the profiles of the velocity components *u* and *w* and the temperature *T* for different values of time *t* and for *Ha*=1, *m*=3, *M*=2, *S*=1 and *Q*=0.4. It is clear from the figure that the velocity components and temperature reaches the steady state monotonically with time. Also the velocity component *u* reaches the steady state faster than *w* which, in turn, reaches the steady state faster than *T* because *u* is the source of *w*, while both *u* and *w* act as sources for the temperature.

Figure 2 indicates that the time progression of *u* and *w* at the centre of the channel *y*=0 for different values of the Hall parameter *m* and for *Ha*=1, *M*=2, *S*=0 and *Q*=0.4. It is clear from Fig. 2a that increasing the parameter *m* increases *u* because the effective conductivity ($\sigma/(1+m^2)$) decreases with increasing *m* which reduces the magnetic resistive force on *u*. In Fig. 2b, the velocity component *w* increases with increasing the parameter *m* slightly (*m*=0 to 1), since increasing *m* increases the driving force term ($mHa^2u/(1+m^2)$) in Eq. (7) which affects the flow in the z-direction. However, increasing *m* more decreases the effective conductivity that results in a reduced driving force and then, decreases *w*. It is clear from Fig. 2c that increasing *m* decreases *T* for all *t* due to decreasing the effect of the Joule dissipation.

Figure 3 presents the time progression of *u*, *w* and *T* at the centre of the channel for different values of the Hartmann number *Ha* and for *m*=3, *M*=0, *S*=0 and *Q*=0.4. Figure 3a indicates that increasing *Ha* decreases *u* as a result of increasing the damping force on *u*. Figure 3b indicates that increasing *Ha* increases *w* since it increases the driving force on *w*. Figure 3c depicts that for small *t*, increasing *Ha* increases *T* due to the increment in the Joule dissipation. But, for large *t*, increasing *Ha* decreases *T* as a result of decreasing the velocities *u* and *w* and consequently decreases the viscous and Joule dissipations.

Figure 4 presents the time progression of *u*, *w* and *T* at the centre of the channel for different values of the suction parameter *S* and for *Ha*=1, *M*=2, *m*=3 and *Q*=0. Figures 4a and 4b indicate that increasing the suction decreases both *u* and *w* due to the convection of the fluid from regions in the lower half to the centre which has higher fluid speed. Figure 4c shows that increasing *S* decreases the temperature at the centre of the channel due to the influence of convection in pumping the fluid from the cold lower half towards the centre of the channel.

Figure 5 presents the time progression of *u*, *w* and *T* at the centre of the channel for different values of the porosity parameter *M* and for *Ha*=1, *m*=3, *S*=0 and *Q*=0. Figure 5a and 5b indicate that increasing *M* decreases *u* and *w* as a result of increasing the damping force. Figure 5c depicts that increasing *M* decreases *T* due to the decrement in the Joule and viscous dissipations. Figure 6 presents the time progression of *T* at the centre of the channel for different values of the parameter *Q* and for *Ha*=1, *m*=3, *M*=2 and *S*=1. The figure indicates that increasing *Q* increases the temperature at the centre of the channel and its steady state time.

5. CONCLUSION

The transient MHD generalized Couette flow with heat transfer through a porous medium of an electrically conducting fluid under the influence of an applied uniform magnetic field has been studied considering the Hall effect in the presence of uniform suction and injection and a heat source. Introducing the Hall term gives rise to a velocity component *w* in the *z*-direction which affects the main velocity *u* in the *x*-direction. The effect of the magnetic field, the Hall parameter, the porosity parameter and the suction and injection velocity on the velocity and temperature distributions has been investigated. Both the magnetic field and the porosity of the medium have a damping effect on the velocity and temperature fields whereas the Hall parameter *m* increases the main velocity component *u*. On the other hand, increasing *m* increases the velocity component *w* for small *m* but decreases it for large *m*.

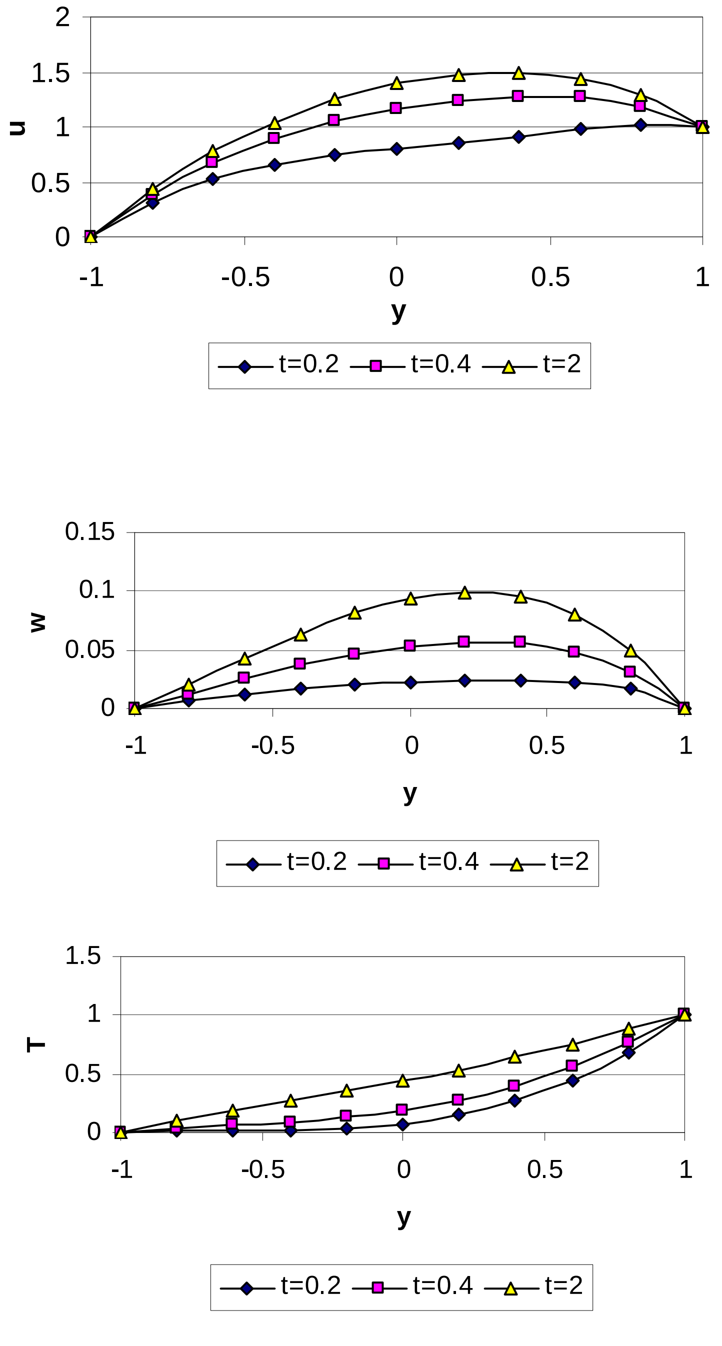

FIGURE 1. Time development of the profile of: (a) u; (b) w; and (c) T (Ha=1, m=3, M=2 and S=1, Q=0.4)

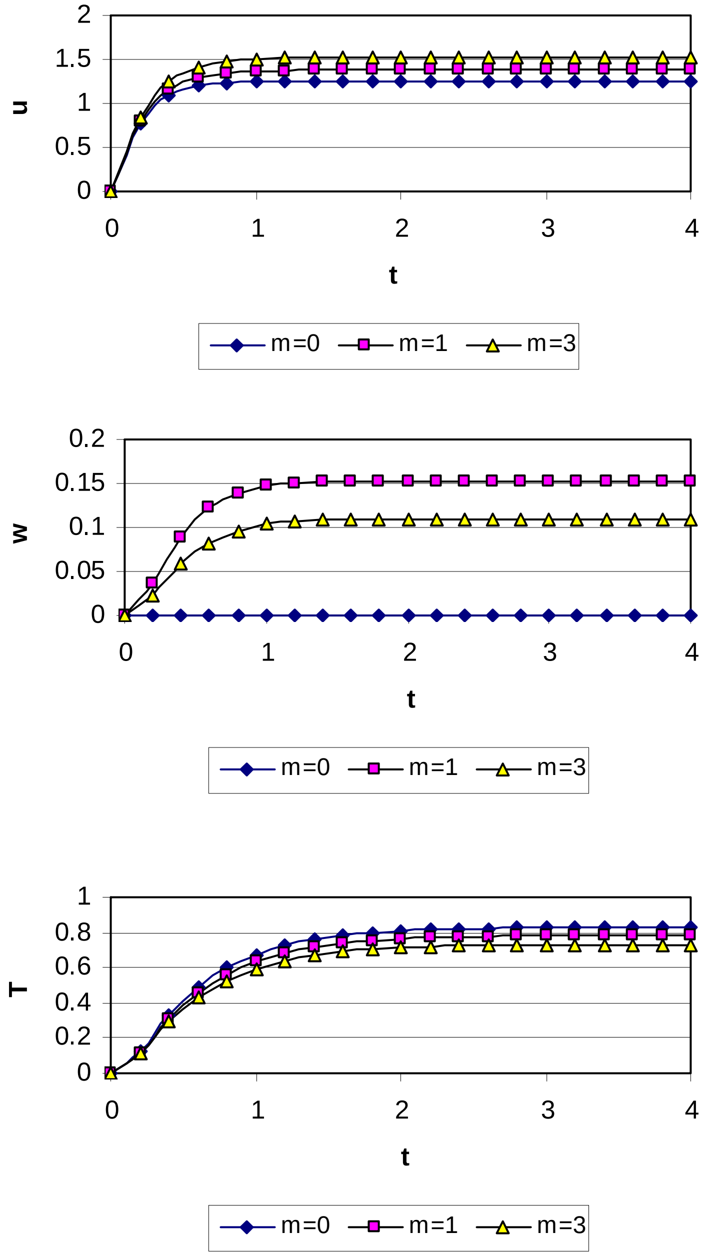

FIGURE 2. Effect of *m* on the time variation of: (a) *u* at *y*=0; (b) *w* at *y*=0 and (c) *T* at *y*=0.
(*Ha*=1, *M*=2, *S*=0, and *Q*=0.4)

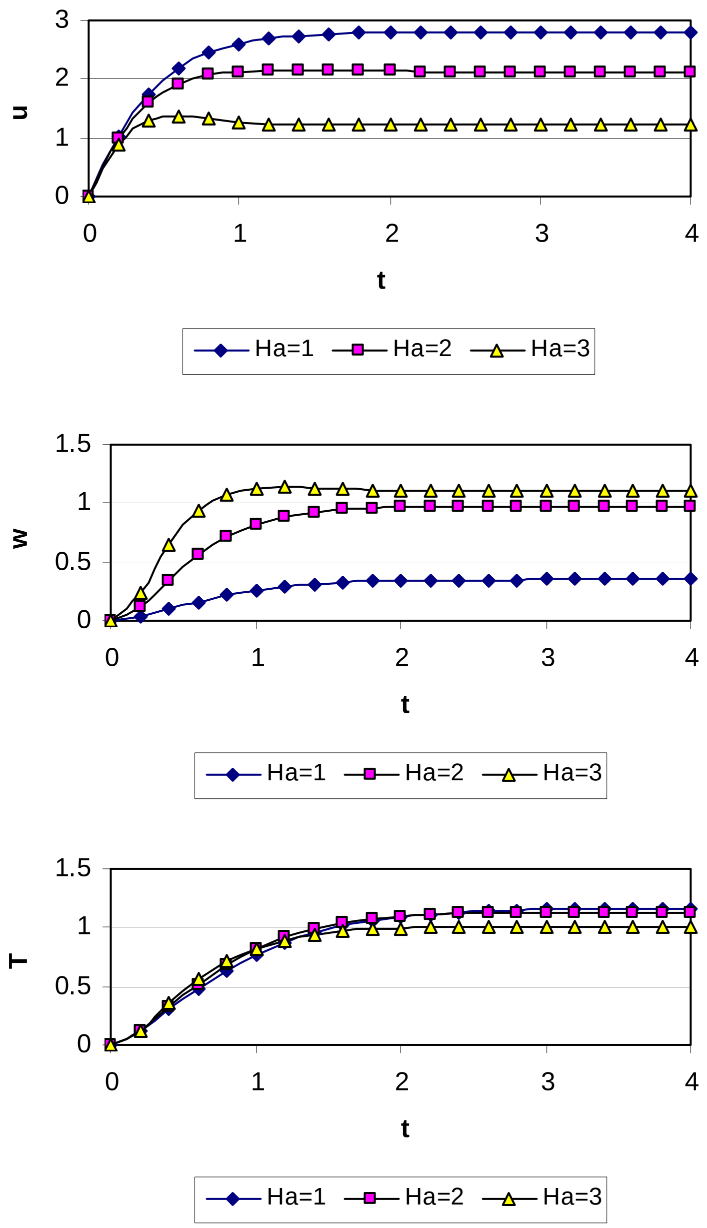

FIGURE 3. Effect of *Ha* on the time variation of: (a) *u* at *y*=0; (b) *w* at *y*=0 and (c) *T* at *y*=0.
(*m*=3, *M*=0, *S*=0 and *Q*=0.4)

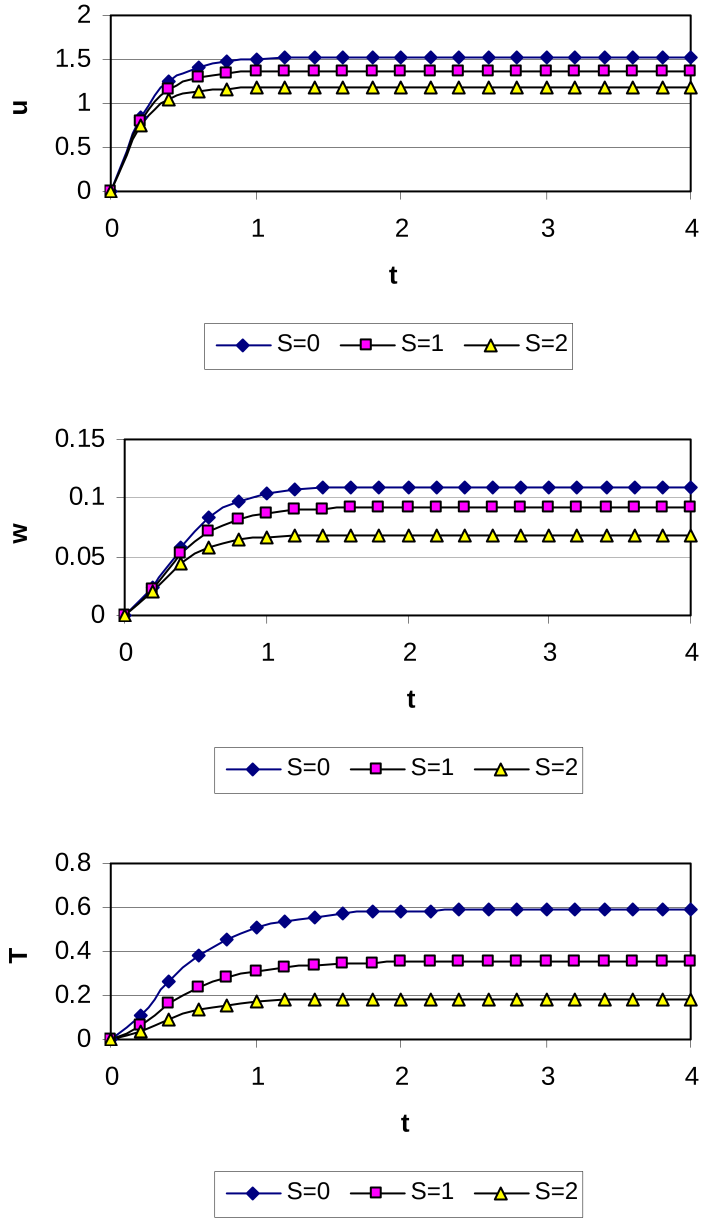

FIGURE 4. Effect of S on the time variation of: (a) u at y=0; (b) w at y=0; and (c) T at y=0. (Ha=1, M=2, m=3, and Q=0)

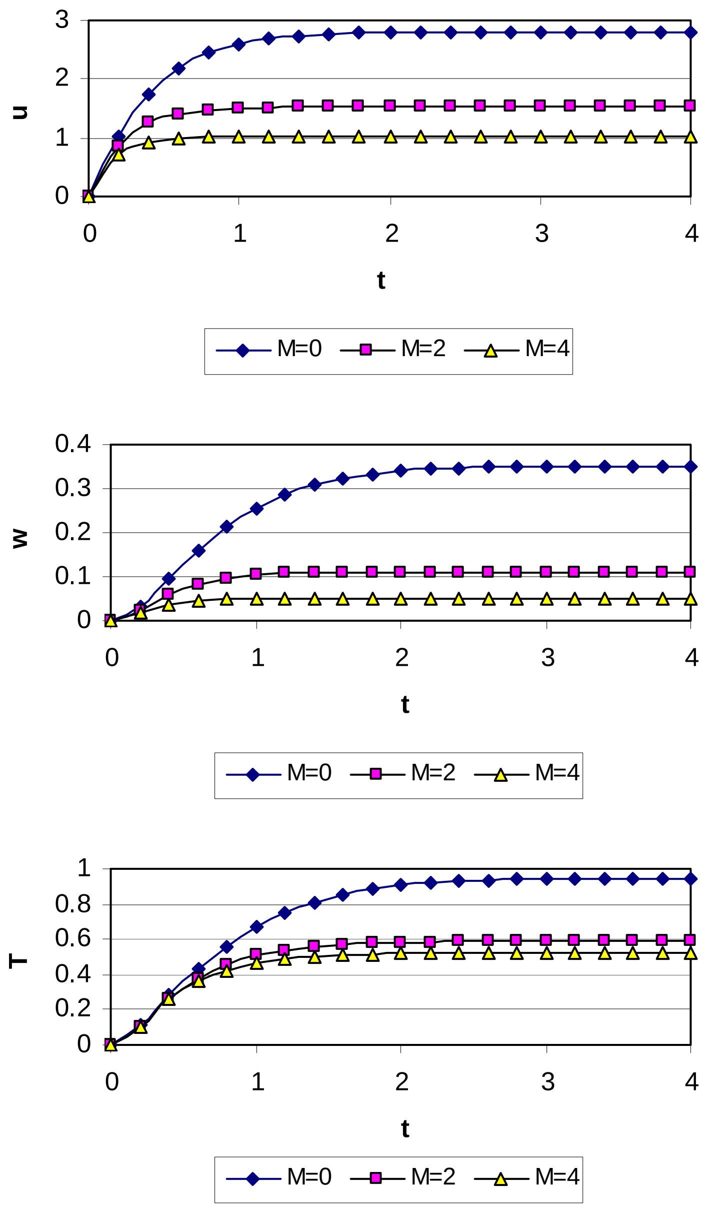

FIGURE 5. Effect of M on the time variation of: (a) u at y=0; (b) w at y=0; and (c) T at y=0. (Ha=1, m=3 and Q=0, S=0)

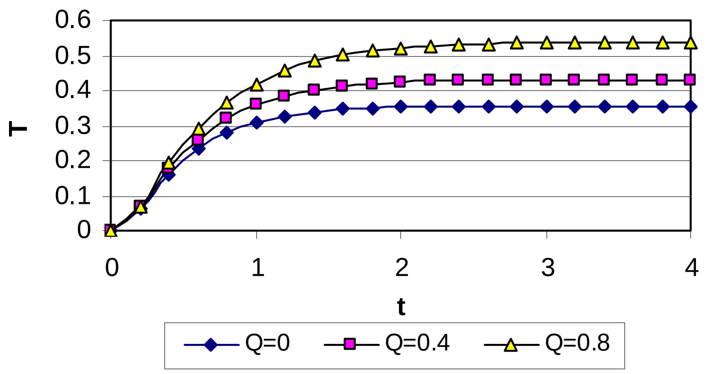

FIGURE 6. Effect of Q on the time variation of: T at y=0.
(Ha=1, m=3 and M=2, S=1)

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An efficient algorithm to measure the insurance risk of casuality insurance company using VaR methodology

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ABSTRACT. We propose an efficient method to measure the insurance risk of causality insurance companies by using the CreditRisk+ methodology. This method is superior to previous methods in several aspects. Its computation speed is very fast and the input data form is simple. It is able to aggregate both credit risk and insurance risk, so the insurance company can manage the risk in combined manner. In this paper, we propose a mathematical method to obtain the aggregate loss distribution of portfolios having correlation among products or business lines as a general case, and then suggest its implementation algorithm. Finally we apply this method to the real data from Korea Insurance Development Institute (KIDI) and discuss its availability to real applications.

1. INTRODUCTION

The insurance company has several kinds of risks which result from the type of insurance business transacted. The International Association of Insurance Supervisors (IAIS) classifies risks of insurance company into Insurance risk, Asset risk and other risks. According to IAIS, “the insurance risks (equivalently, technical risks) represent the various kinds of risk that are directly or indirectly associated with the technical or actuarial bases of calculation for premiums and technical provisions in both life and non-life insurance, as well as risks associated with operating expenses and excessive or uncoordinated growth. … They differ depending on the class of insurance. Insurance risks exist partly due to factors outside the company's area of business activities, and the company often may have little influence over these factors. The effect of such risks - if they materialise - is that the company may no longer be able to fully meet the guaranteed obligations using the funds established for this purpose, because either the claims frequency, the claims amounts, or the expenses for administration and settlement are higher than expected.”

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The insurance risk is a kind of intrinsic risk which is more difficult to measure than other risks. The development and implementation of efficient mathematical algorithms to measure exact insurance risks are necessary and urgent for introducing RBC (risk based capital) solvency margin system to Korean insurance industry. In order to measure insurance risk in some manner and apply it to solvency margin system for insurance company we need to estimate a loss distribution for each product or each business line firstly, to aggregate the loss distributions considering correlations among them secondly, so to have an aggregate loss distribution finally. We propose an efficient methodology to estimate an aggregate loss distribution for measuring the insurance risk in this paper.

The estimation of the aggregate loss distribution, the first and essential step for measuring the risks of insurance company, enables the insurance company to calculate VaRs(Value at Risk) according to each business line, each asset class, and each liability, and finally the firm-wide aggregated VaR which results in measuring the economic capital or the RBC solvency margin. And we can calculate the appropriate risks which will be prepared for future, by generating the future loss distribution using the aggregated current loss distribution combined with scenario analysis or stress test or DFA (Dynamic Financial Analysis).

The estimation of loss distribution has been studied by several researchers in traditional actuarial science. Heckman and Meyers (1983) estimated the severity and the frequency of each business lines, exactly the characteristic functions of them using the collective risk theory, and got the aggregated loss distribution by convolution. Robertson (1992) adapted the same manner and proposed a more efficient convolution method, discrete Fourier transform, DFT). The convolution methods have been done under the assumption of independent business lines, but there exists correlations among them in real cases. You can see how to estimate the correlated aggregate loss distribution in Wang (1998).

Such traditional methods, however, have several shortcomings. The estimation of loss distribution is made either by parametric methods or by nonparametric method. It is well-known that the parametric method cannot capture the rare event probability, especially for loss distribution and the nonparametric demands too much times. Even the worse, those traditional methods cannot meet practical needs in measuring the aggregated risks of market risks, credit risks, and insurance risks and so on. The usual measuring methodologies of market, and credit risk have been developed in terms of VaR, and the insurance risk measured by those traditional ones cannot be aggregated into VaR framework.

In this paper, we suggest a methodology to overcome those shortcomings and to help the insurance company manage the insurance risk in view of aggregated risk management with market, and credit risks because it belongs into VaR framework. Its algorithm is modified from that in CreditRisk+. Furthermore, our algorithm is simple to implement and fast to estimate the loss distribution.

CreditRisk+ is a credit risk measuring methodology to estimate the loss distribution of credit portfolio which consists of loans, default able bonds and so on. It is being used by many financial institutions all over the world for measuring credit VaRs and managing credit risks since its algorithm has been offered firstly by CSFB (Credit Suiss First Boston). And it adapts a nonparametric and analytic approximation to the whole probability distribution so as to be fast in estimating a loss distribution. This is why various kinds of financial institutions have adopted it as a method to measure credit risks. As results it has been well developed for practical applications of creditVaR in capital allocation and Risk-Adjusted Performance Measurement (RAPM). We study how to apply the CreditRisk+ methodology to measure the insurance risk of causality insurance company in this paper. And we sample a real loss data set classified according to business lines from Korea Insurance Development Institute to apply our method and to estimate the loss distribution coming from insurance risks.

Our method has good points when insurance companies manage the firm-wide risk in view of the aggregated risk management.

First, they can get the aggregated VaR when they use our method to measure their insurance risks and CreditRisk+ to measure their credit risks. It is of consequence to the insurance company which has individual loan portfolios because only CreditRisk+ is applicable for those portfolios.

Second, our algorithm is so fast even though it is nonparametric. As we mentioned above, our algorithm is a kind of nonparametric and analytic approximation which overcomes the shortcoming of nonparametric methods-the slow converging rate. Another nonparametric methods demand us too much computing times to get a desirable error bound.

Third, the input data format to our risk-measuring model is very simple. Our model assumes the multivariate normal distribution for severity and we estimate correlations between each business lines using serial data of severities to plug them into our model in simple manner.

Fourth, our model loosens the assumption of independence among business lines which is taken in CreditRisk+. The independent sectors assumption in CreditRisk+ is too strict for practitioners to meet, so that they may hesitate to apply the estimating VaR to capital allocation and RAPM because they ignore the assumption of independent sectors to a certain extent.

This paper is composed as the followings; we will explain CreditRisk+ methodology briefly in Chapter 2. In Chapter 3 we will explain in detail how to apply CreditRisk+ to estimate the loss distribution combining severity and frequency in each case of independent business lines and correlated business lines, and in Chapter 4, two mathematical algorithms to estimate the loss distribution: the recursive method and the FFT method. In Chapter 5 we will apply our algorithm to real data set from Korea Insurance Development Institute to get the aggregated loss distribution. We will conclude in Chapter 6.

2. CREDIT RISK+ METHODOLOGY

In this chapter we explain the CreditRisk+ methodology (CSFP, 1997) briefly. We consider a portfolio exposed to credit risks by K obligors. We assume that all obligors are affected by N risk factors $S=(S_1,S_2,...,S_N)$. Let p_A and τ_A be the expectation and the standard deviation of default probability for obligor A and let w_{Ak} be the weight of risk factor S_k for obligor A. Such weights have to satisfy $0 \le w_{Ak} \le 1$, $\sum_{k=1}^{N} w_{Ak} \le 1$. Thus

$\sum_{k=1}^{N} w_{Ak}$ specifies the systematic risk for obligor A, while $w_{A0} = 1 - \sum_{k=1}^{N} w_{Ak}$ represents the weight of idiosyncratic risk, which is independent of risk factors $S = (S_1, S_2, ..., S_N)$, of obligor A.

Consider the potential loss ε_A for obligor A. It is one of the features of CreditRisk+ to work with discretized losses. For this purpose we fix a loss unit L_0 and choose a positive integer ν_A as a rounded version of ε_A / L_0. Then the aggregate portfolio loss in CreditRisk+ as a multiple of the basic loss unit L_0 is given by

$$X = \sum_A \nu_A D_A$$

with D_A being Poisson distributed random variables with stochastic intensities

$$p_A^S = p_A \left(w_{A0} + \sum_{k=1}^{N} w_{Ak} S_k \right)$$

conditional on independent gamma distributed random variables $S = (S_1, S_2, ..., S_N)$. Then we can obtain the expectation μ_k and the standard deviation σ_k for S_k, $k = 1, 2, \cdots, N$

$$\mu_k = \sum_A w_{Ak} p_A, \quad \sigma_k = \sum_A w_{Ak} \tau_A .$$

We introduce the portfolio polynomial of the k-th factor to be

$$Q_k(z) = \sum_A w_{Ak} p_A z^{\nu_A}, \quad k \in \{0, 1, 2, \cdots, N\}.$$

Then the PGF(probability generating function) of the CreditRisk+ model can be expressed in closed analytic form

$$G(z) = \exp\left[(Q_0(z) - Q_0(1)) - \sum_{k=1}^{N} \frac{\mu_k^2}{\sigma_k^2} \ln\left\{ 1 - \frac{\sigma_k^2}{\mu_k^2} (Q_k(z) - Q_k(1)) \right\} \right].$$

On the other hand, from the definition of the PGF of a discrete, integer-valued random variable, we know that $G(z)$ may also be represented as

$$G(z) = \sum_{j=0}^{\infty} P(X = j) z^j .$$

Thus the central problem is the efficient and numerically stable computation of the probabilities $P(X = j)$. It is known that the algorithm advocated in the original CreditRisk+ document in order to the probabilities, the Panjer recursion scheme, is numerically unstable. Giese(2003) has suggested the method to calculate the probabilities directly by applying standard algorithms for logarithm and exponential of power series. However, the numerical stability was analyzed by Haaf, Reiβ and Schoenmakers(2004). An alternative algorithm using FFT(fast Fourier transform) is established by Reiβ(2004), which is described in terms of characteristic function instead of PGF. This algorithm is easy to implement and is numerically stable, so it may also be applied for large portfolios.

While the above researches have assumed independent risk factors, there are several researches considering the correlation between risk factors. Bürgisser, Kurth, Wagner and Wolf(1999) have proposed a model with a single factor approach to correlation, and Giese(2004) has discussed two multivariate factor distributions, multivariate gamma distribution and compound gamma distribution), which include risk factor correlations. However, using these methods for dealing with any correlations there is a limit.

3. THE METHODOLOGY TO ESTIMATE THE INSURANCE LOSS DISTRIBUTION

In this chapter we explain the algorithm to get the PGF for the loss distribution using CreditRisk+ in each cases of one business line, independent business lines, and correlated business lines.

3.1. A case of only one business line

First, for a case of only one business line, the method for estimating the loss distribution of severity for given frequency are described. For this we have to calculate the PGF $F_m(z)$ of severity for m events.

Suppose the severity distribution for i-th event of m events is given as follows:

Loss Level	Loss Probability
$\nu_1 L_0$	p_1^i
$\nu_2 L_0$	p_2^i
$\vdots$	$\vdots$
$\nu_M L_0$	p_M^i

Loss levels $\nu_1, \nu_2, \cdots, \nu_M$ are positive integers and $\nu_1 < \nu_2 < \cdots < \nu_M$. If each event is independent, the PGF of severity for m events is

$$F_m(z) = \prod_{i=1}^{m} \left\{ 1 - \sum_{j=1}^{M} p_j^i + \sum_{j=1}^{M} p_j^i z^{\nu_j} \right\} = \prod_{i=1}^{m} \left\{ 1 + \sum_{j=1}^{M} p_j^i \left(z^{\nu_j} - 1 \right) \right\}$$

Also, if we assume $p_j^1 = p_j^2 = \cdots = p_j^m = p_j$ for $j = 1, 2, \cdots, M$, the PGF is represented by

$$F_m(z) = \left\{ 1 + \sum_{j=1}^{M} p_j \left(z^{\nu_j} - 1 \right) \right\}^m .$$

Now, given frequency distribution, we have to estimate the loss distribution combined with severity distribution. Assume that frequency distribution is Poisson distribution with parameter λ. Then the PGF $G(z)$ of loss distribution is

(1)
$$G(z)=\sum_{m=0}^{\infty}F_m(z)\frac{\lambda^m}{m!}e^{-\lambda}=\exp\left[\lambda\left\{\sum_{j=1}^{M}p_j\left(z^{\nu_j}-1\right)\right\}\right]$$

3.2. A case of independent business lines

Assume that N business lines are independent. Let λ_k be the Poisson parameter of severity distribution for k-th business line and the severity distribution for k-th business line is given as follows:

Loss Level	Loss Probability
$\nu_1^k L_0$	p_1^k
$\nu_2^k L_0$	p_2^k
$\vdots$	$\vdots$
$\nu_{M_k}^k L_0$	$p_{M_k}^k$

Loss levels $\nu_1^k,\nu_2^k,\cdots,\nu_{M_k}^k$ are positive integers and $\nu_1^k<\nu_2^k<\cdots<\nu_{M_k}^k$. Because the business lines are independent, using equation (1) the PGF $G(z)$ of N business lines is represented by

$$G(z)=\prod_{k=1}^{N}\exp\left[\lambda_k\left\{\sum_{j=1}^{M_k}p_j^k\left(z^{\nu_j^k}-1\right)\right\}\right]=\exp\left[\sum_{k=1}^{N}\lambda_k\left\{\sum_{j=1}^{M_k}p_j^k\left(z^{\nu_j^k}-1\right)\right\}\right].$$

We can assume that $M_1=M_2=\cdots=M_N=M$ without loss of generality, and if we assume that $\nu_j^1=\nu_j^2=\cdots=\nu_j^N=\nu_j$ for $j=1,2,\cdots,M$, then the PGF $G(z)$ of portfolio is briefly

(2)
$$G(z)=\exp\left[\sum_{k=1}^{N}\lambda_k\left\{\sum_{j=1}^{M}p_j^k\left(z^{\nu_j}-1\right)\right\}\right].$$

3.3. A case of correlated business lines

In general, the events in each business line may correlate with each other. The correlation between business lines is explained by the correlation between the parameters $\lambda_k, k=1,2,\cdots,N$ of the frequency distribution:

$$C_{ij} = Corr(\lambda_i, \lambda_j),\ i = 1, 2, \cdots, N,\ j = 1, 2, \cdots, N \ .$$

We assume that this correlation matrix $C = (C_{ij})$ is positive definite.

The correlation matrix $C = (C_{ij})$ is decomposed into lower triangular matrix $L = (L_{ij})$ and its transpose matrix by Cholesky decomposition $C = LL^T$, and there exist random variables $Y_k, k = 1, 2, \cdots, N$ independent of each other such that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{pmatrix} = L \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{pmatrix} .$$

Also, let $\mu_k = E(\lambda_k),\ \sigma_k = \sigma(\lambda_k)$ be the expectation and standard deviation of λ_k, and let $L^{-1} = (D_{ij})$ be the inverse matrix of $L = (L_{ij})$, then

$$E(Y_k) = \sum_{i=1}^{N} D_{ki}\mu_i,\ \sigma^2(Y_k) = \sum_{i,j=1}^{N} D_{ki}D_{kj}\sigma_i\sigma_j C_{ij} \ .$$

Let $f_k(z) = \sum_{j=1}^{M} p_j^k \left(z^{v_j} - 1\right)$ and $\alpha_k = \dfrac{\{E(Y_k)\}^2}{\sigma^2(Y_k)}, \beta_k = \dfrac{\sigma^2(Y_k)}{E(Y_k)}$.

If the random variable Y_k is gamma-distributed with parameters α_k, β_k and its probability density function is g_{α_k,β_k}, then the PGF $G(z)$ of portfolio is calculated by

$$\begin{aligned} G(z) &= E\left[\exp\left\{\sum_{k=1}^{N} \lambda_k f_k(z)\right\}\right] \\ &= \int_0^\infty \cdots \int_0^\infty \exp\left[\sum_{k=1}^{N}\left\{\sum_{l=1}^{N} L_{kl}Y_l\right\} f_k(z)\right] g_{\alpha_1,\beta_1}(Y_1)\cdots g_{\alpha_N,\beta_N}(Y_N)dY_1\cdots dY_N \\ &= \prod_{l=1}^{N}\int_0^\infty \exp\left[Y_l \sum_{k=1}^{N} L_{kl} f_k(z)\right] g_{\alpha_l,\beta_l}(Y_l)dY_l. \end{aligned}$$

The since $g_{\alpha,\beta}(y) = \dfrac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} y^{\alpha-1}$ and $\int_0^\infty e^{xy} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} y^{\alpha-1} dy = (1-\beta x)^{-\alpha}$, then the PGF of portfolio is

$$G(z) = \prod_{l=1}^{N}\left[1 - \beta_l \sum_{k=1}^{N} L_{kl} f_k(z)\right]^{-\alpha_l} = \exp\left[-\sum_{l=1}^{N} \alpha_l \ln\left\{1 - \beta_l \sum_{k=1}^{N} L_{kl} f_k(z)\right\}\right] \tag{3}$$

4. THE COMPUTING ALGORITHM OF LOSS DISTRIBUTION

The methodology to compute the loss distribution from PGF functions has been well developed since CreditRisk+ has been offered CSFP. The recursive algorithms by Panjer(1997) and Giese(2003), and the FFT method are mostly used and implemented by practitioners. The Panjer's method was firstly adopted and published in CreditRisk+ by CSFP, but its instability is well known among credit risk managers so that we will explain Giese's algorithm(2003) and the FFT method briefly in this Chapter.

4.1. Giese's Recursive Method

Rewrite Equation (3) as follows.

$$G(z) = \exp\left[-\sum_{l=1}^{N} \alpha_l \ln\left\{ 1 + \beta_l \sum_{k=1}^{N} L_{kl} \sum_{i=1}^{M} p_i^k - \beta_l \sum_{k=1}^{N} L_{kl} \sum_{i=1}^{M} p_i^k z^{v_i} \right\} \right] \tag{4}$$

Let B be a pre-specified integer. Then, we can calculate the probability $p(Loss \le BL_0)$ that loss is equal or less than BL_0. The process consists of four steps as follows.

Step 1: Define $l = 1, 2, \cdots, N$,

$$a_0^{(l)} = 1 + \beta_l \sum_{k=1}^{N} L_{kl} \sum_{i=1}^{M} p_i^k,\ a_j^{(l)} = \beta_l \sum_{k=1}^{N} L_{kl} \sum_{i=1}^{M} p_i^k 1_{\{v_i = j\}},\ j = 1, 2, \cdots, B,$$

hence

$$1 + \beta_l \sum_{k=1}^{N} L_{kl} \sum_{i=1}^{M} p_i^k - \beta_l \sum_{k=1}^{N} L_{kl} \sum_{i=1}^{M} p_i^k z^{v_i} = a_0^{(l)} - \sum_{j=1}^{B} a_j^{(l)} z^j + O(z^{B+1}).$$

Thus

$$G(z) = \exp\left[-\sum_{l=1}^{N} \alpha_l \ln\left\{ a_0^{(l)} - \sum_{j=1}^{B} a_j^{(l)} z^j \right\} \right] + O(z^{B+1}), \tag{5}$$

and the time required for this step is $O(N^2 MB)$.

Step 2: Define $l = 1, 2, \cdots, N$,

$$b_0^{(l)} = -\ln a_0^{(l)},\ b_j^{(l)} = \frac{1}{a_0^{(l)}} \left\{ a_j^{(l)} + \frac{1}{j} \sum_{k=1}^{j-1} k b_k^{(l)} a_{j-k}^{(l)} \right\},\ j = 1, 2, \cdots, B,$$

hence

$$-\ln\left\{ a_0^{(l)} - \sum_{j=1}^{B} a_j^{(l)} z^j \right\} = \sum_{j=0}^{B} b_j^{(l)} z^j + O(z^{B+1}).$$

Thus

$$G(z) = \exp\left[\sum_{l=1}^{N} \alpha_l \sum_{j=0}^{B} b_j^{(l)} z^j \right] + O(z^{B+1}) \tag{6}$$

and the time required for this step is $O(NB^2)$.

Step 3: Define $l = 1, 2, \cdots, N$,

$$c_0 = \sum_{l=1}^{N} \alpha_l b_0^{(l)}, \ c_j = \sum_{l=1}^{N} \alpha_l b_j^{(l)}, \ j = 1, 2, \cdots, B,$$

hence

$$\sum_{l=1}^{N} \alpha_l \sum_{j=0}^{B} b_j^{(l)} z^j = \sum_{j=0}^{B} c_j z^j + O(z^{B+1}).$$

Thus

$$G(z) = \exp\left[\sum_{j=0}^{B} c_j z^j\right] + O(z^{B+1}) \tag{7}$$

and the time required for this step is $O(NB)$.

Step 4: Define

$$d_j = \exp(c_0), \ d_j = \sum_{i=1}^{j} \frac{i}{j} c_i d_{j-i}, \ j = 1, 2, \cdots, B.$$

Hence

$$\exp\left[\sum_{j=0}^{B} c_j z^j\right] = \sum_{j=0}^{B} d_j z^j + O(z^{B+1}).$$

Thus

$$G(z) = \sum_{j=0}^{B} d_j z^j + O(z^{B+1}) \tag{8}$$

and the time required for this step is $O(B^2)$.

Therefore, for $j = 0, 1, 2, \cdots, B$ the probability that loss equals jL_0 is d_j, i.e.

$$p(Loss = jL_0) = d_j.$$

4.2. Fast Fourier Transform Method

Using equation (3), the characteristic function of portfolio is

$$\Phi(z) = G(e^{iz}) = \exp\left[-\sum_{l=1}^{N} \alpha_l \ln\left\{1 - \beta_l \sum_{k=1}^{N} L_{kl} f_k(e^{iz})\right\}\right].$$

Choose a number $B = 2^n$ for some integer n. Let Λ_k be the maximum loss of k–th business line and let

$$z_j = (j - B/2)\Delta z, \ x_j = j\Delta x, \ j = 0, 1, 2, \cdots, B-1 \quad ,$$

where

$$\Delta x = \sum_{k=1}^{N} \frac{\Lambda_k}{B-1}, \ \Delta z = \frac{2\pi}{B\Delta x}.$$

Then by Fourier inversion theorem the probability $p(loss = x_j)$ is

$$p(loss = x_j) = \frac{1}{B}\sum_{k=0}^{B-1} e^{-2\pi jk/B}\Phi(z_k) .$$

The loss probability is obtained by the FFT algorithm and time required for this algorithm is $O(B\log_2 B)$. For more information see Reiβ(2004). Compared with the recursive method, this algorithm is very simple and fast.

5. DATA AND COMPUTATION RESULTS

The sample data set is a causality insurance one provided by Korea Insurance Development Institute, which consists of monthly time series data for the time period between 2000, April and 2003, April collected according to insurance products. For fixed unit $L_0 X 2048$ we rounded severities and frequencies data to display in Table 1 and 2 respectively.

In Table 2 we calculated the correlation matrix between each business line using monthly frequency time series data for the same time period. The "(Loss, Prob.) = (14, 0.00009)" in the first two columns of Table 1 means that the probability of loss event occurring by amount of 14X $L_0 X 2048$ is 0.00009. The second column in Table 2 means that the loss event of Fire business line occurs 3710 times in monthly average, and its standard deviation is 2338, and that its correlations between each other business lines are 0.9212, 0.9142 and 0.8520. Figure 1 represents the loss distribution estimated from the data in Table 1 and 2 using equation (3).

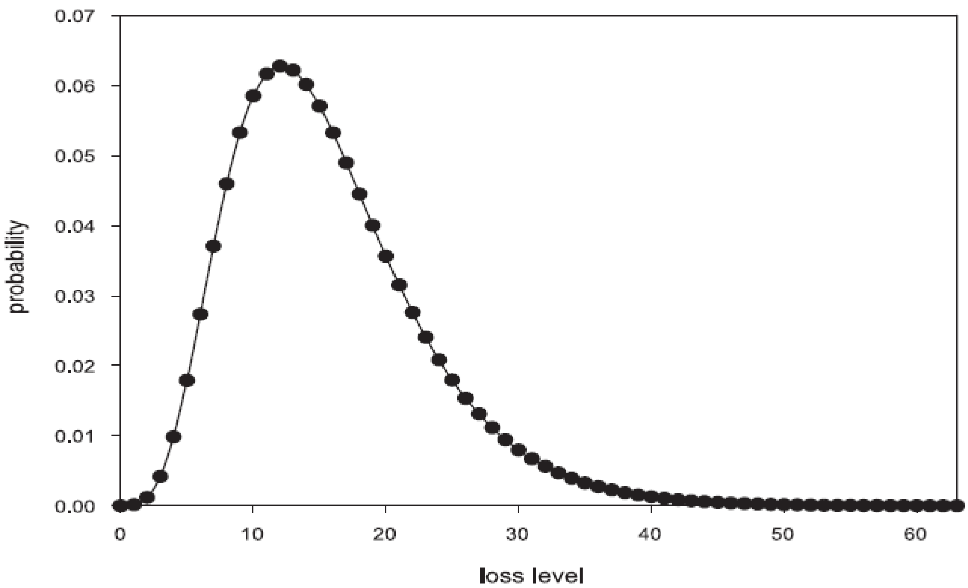
Table 1. Severity Data

Fire		Marine		Liability		Comprehensive	
Loss	Prob.	Loss	Prob.	Loss	Prob.	Loss	Prob.
14	0.00009	8	0.00013	2	0.00055	2	0.00216
17	0.00013	9	0.00016	3	0.00401	3	0.00323
23	0.00117	10	0.00041	4	0.07564	4	0.01866
24	0.00198	11	0.00005	5	0.02706	5	0.02570
25	0.00072	12	0.00069			6	0.07943
26	0.00198	13	0.00058			7	0.06626
27	0.00189	14	0.00067			8	0.02789
28	0.00117	15	0.00119			9	0.00401
29	0.00072	16	0.00192			11	0.00265
30	0.00037	17	0.00028				
31	0.00118	23	0.00003				
32	0.00004						

Table 2. Frequency Data

		Fire	Marine	Liability	Comp.
Average		3710	6226	13890	19283
St. Dev.		2338	3654	8230	11837
Corr.	Fire	1.0000	0.9212	0.9142	0.8520
	Marine	0.9212	1.0000	0.9775	0.9795
	Liability	0.9142	0.9775	1.0000	0.9774
	Comp.	0.8520	0.9795	0.9774	1.0000

Figure 1. Loss Distribution



It takes much time to calculate the loss distribution by the recursive method of Giese (2003) because we should do it by $B = 2^{17}$. On the contrary, the FFT method demands $O(B \log_2 B)$ in computation time so we adopt the FFT method. We also used the recursive method of Panjer (1997, CSFP) to estimate the loss distribution but had the very unstable results. The loss level ν_i in Figure 1 is an approximation to the actual exposed loss ε_i, *i.e.,*

$$\varepsilon_i \approx 2048 \times \nu_i L_0 .$$

In other words it is different from the loss level ν_i defined in Chapter 3 by 2048.

And we see that

$$VaR_q = 37.22 \quad \text{when } q\text{=99\%},$$

and

$$VaR_q = 48.38 \quad \text{when } q\text{=99.9\%}.$$

We note that the loss distribution has shape similar to a typical credit loss distribution with long-tailed implying the loss event being rare one.

6. CONCLUSION

In this paper we offered an efficient algorithm for causality insurance company to measure an insurance risk using CreditRisk+ methodology. In particular we proposed a mathematical method to get the loss distribution in case of correlated business lines, explained an algorithm to implement it effectively, and estimated the loss distribution and the VaR with real data.

We could compute the CreditVaR to get speedy and robust results. It is because we adopted an analytic approximation to generate the probability generating function. Even though it is a nonparametric method its input data type is so simple. For example, we estimated correlations among severities of business lines with serial data and put them in the model directly. In other words, the model allows the simplest manner of estimating correlations. The CreditRisk+ assumes independent sectors which is too strong for practitioners to accommodate for practical applications such as capital allocation and RAPM. Our model avoids the assumptions of independence among business lines. As we see in Figure 1 we know that the aggregated loss distribution estimated with our algorithm displays a typical shape of a loss distribution.

The insurance company can get the aggregated VaR in terms of credit risks and insurance risks with our method, which enables the risk manager to use the VaR as the firm-wide total risk.

Our future research direction will be to find a way to apply the estimated loss distribution to DFA, and to get the actual aggregated loss distribution of the portfolio containing loans and bonds with credit risks and insurance products with insurance risks.

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