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AN EDITORIAL NOTE ON THIS SPECIAL ISSUE

LETTER FROM THE EDITOR

This issue of Journal of KSIAM is specially organized to commemorate the 2013 KSIAM-KUMKOK award and its recipient, Professor Jin Keun Seo, at the department of computational science and engineering of Yonsei University. This special issue is composed of eight contributed papers dealing with recent progress and challenge in multi-physics problems. I believe this issue would make a meaningful contribution in advancing the relevant fields of industrial and applied mathematics as well as upgrading the overall quality of Journal of KSIAM.

On behalf of the managing editors and editorial board, I am very thankful to Professor Jin Keun Seo and all contributors who put generous efforts in completing this special issue.

Chongam Kim, Chief Editor, Journal of KSIAM

PREFACE

Mathematical techniques in science and engineering have evolved to expand our ability to visualize various physical phenomena of interest and their characteristics in detail. Many significant applied and basic research questions today are interdisciplinary in nature, involving mathematics, physics, engineering and biomedical science. A large variety of natural phenomena occurring in real-life applications from fluid flows to biology and medical imaging fields are described by means of partial differential equations (PDEs). Developing mathematical models with practical significance and value requires the fusing of the knowledge and techniques of traditional engineering fields with pure and applied mathematics. Many problems are intrinsically nonlinear. Finding solutions with practical significance and value requires an in-depth understanding of the underlying physical phenomena with data acquisition systems as well as the implementation details of algorithms. Experiences over the last three decades have shown that symbiotic interplay among theoretical mathematics, computational mathematics, and experiments is crucial to understand and solve these realistic model problems.

Significant progress in PDEs has been made in recent years with the introduction of numerical methods that allow the visualization of solutions of PDEs. Technical advances have been followed by theoretical progress aimed at understanding the solution’s structure. The numerical analysis approach is to gain insight and understanding mainly through an analysis of mathematical models implemented on computers. The problem-solving process in numerical analysis includes algorithm development, analysis, software implementation, program execution, visualization of the results and validation.
This special issue discuss recent progress, emerging issues and challenging problems in multi-physics problems such as spectroscopic electrical tissue property imaging, process tomography in measuring ground contamination, motion estimation of left ventricle from ultrasound, dual energy CT to characterize material properties, recent image processing techniques using total variation, image segmentation, and sparsity. Two papers are concerned theoretical issue regarding on convergence and asymptotic analysis, which play important roles in making a mathematical model and analyzing behavior of physical solutions. Educational issues regarding on rigor vs intuition in mathematical studies are discussed for young mathematicians in the hope of having broader view with realistic models.

We hope that this special issue gives an opportunity to this scientific community to take consolidation and identify new challenges and the most promising directions for progress.

*Jin Keun Seo, Prof. Dept. of CSE, Yonsei University*
SPECTROSCOPIC ADMITTIVITY IMAGING OF BIOLOGICAL TISSUES: CHALLENGES AND FUTURE DIRECTIONS

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ABSTRACT. Medical imaging techniques have evolved to expand our ability to visualize new contrast information of electrical, optical, and mechanical properties of tissues in the human body using noninvasive measurement methods. In particular, electrical tissue property imaging techniques have received considerable attention for the last few decades since electrical properties of biological tissues and organs change with their physiological functions and pathological states. We can express the electrical tissue properties as the frequency-dependent admittivity, which can be measured in a macroscopic scale by assessing the relation between the time-harmonic electric field and current density. The main issue is to reconstruct spectroscopic admittivity images from 10 Hz to 1 MHz, for example, with reasonably high spatial and temporal resolutions. It requires a solution of a nonlinear inverse problem involving Maxwell’s equations. To solve the inverse problem with practical significance, we need deep knowledge on its mathematical formulation of underlying physical phenomena, implementation of image reconstruction algorithms, and practical limitations associated with the measurement sensitivity, specificity, noise, and data acquisition time. This paper discusses a number of issues in electrical tissue property imaging modalities and their future directions.

1. INTRODUCTION

We may consider a biological tissue as a three-dimensional array of cells embedded in an extracellular matrix. A cell is composed of intracellular fluid and organelles surrounded by a cell membrane having a complex structure of phospholipid bilayer with protein channels. Electrical conductivity and permittivity values of a biological tissue or organ are affected by its composition and structure of extracellular matrix, cellular structure, amounts of intra- and extra-cellular fluids, concentration and mobility of ions in those fluids, temperature and other factors.

Biological tissues and organs exhibit distinct electrical properties depending on their physiological functions and pathological states [21, 18, 19, 57]. Physiological functions including

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breathing, blood flow and perfusion, neural activity, gastric motility and so on alter the conductivity and permittivity values of associated tissues or organs. They also differ for various pathological states such as ischemia, bleeding, inflammation, tumor, necrosis and others.

In this paper, we denote the admittivity \( \gamma = \sigma + i\omega\epsilon \) where \( \sigma \) is the conductivity, \( \epsilon \) is the permittivity, and \( \omega \) is the angular frequency. There have been numerous studies to measure or image the admittivity distributions inside the human body since they provide valuable diagnostic information [115, 35, 102, 103]. Considering that they are passive material properties, we should probe them in such a way that the probing produces some measurable physical quantities that provide quantitative information about the admittivity. One method is to inject dc or ac current through surface electrodes and measure the induced voltage. The other is to use an external coil to produce a time-harmonic electromagnetic field inside the body and measure the effects of the induced eddy current by using sensing electrodes and/or coils. In both cases, an electric field is produced inside the body and the admittivity value determines the relation, that is, Ohm’s law, between the electric field and the resulting current density. Without generating electric current, it is not possible to sense the electrical tissue property.

Noninvasive methods to measure the admittivity should rely on voltage or current measurements on the surface or magnetic field measurements inside or outside the human body. Since these measurements are strongly affected by the geometry as well as the material property, we end up measuring the impedance (or admittance) with resistance (or conductance) and reactance (or susceptance) terms including the overall effects of the material property, the boundary geometry and the electrode or coil configuration. Under the influence of a time-harmonic electric field, the intracellular and extracellular fluids behave as resistors, whereas the membranes appear as leaky capacitors to produce reactance terms. It is often desirable to measure an impedance spectrum over a wide range of frequency instead of one impedance value at a fixed frequency since the admittivity of a biological tissue changes with frequency.

Once we measure the impedance or trans-impedance values, we have to carefully analyze and interpret them since they reflect how the electrical tissue property has been seen by a chosen measurement method at a given scale. It is very important to understand internal current pathways since they are closely related with the sensitivity of the measurement. To extract any useful diagnostic information from the measured data or reconstructed images of the admittivity distribution, we need to understand a mapping between the physiology or pathology and the admittivity for a given application study.

To maximize the amount of extracted information, we may perform the spectroscopic admittance imaging. This requires current injections at multiple frequencies and voltage measurements at those chosen frequencies. To visualize fast changes of a physiological function, it is desirable to use a parallel data acquisition system for a high temporal resolution and adopt a time-difference imaging approach. To detect pathological states of tissues or organs, we may use a frequency-difference approach or a recently developed vibration-difference approach [3] since a previously acquired time-referenced data is not usually available in these cases. In any application study, there must be a clear link between a voxel value and its physiological or pathological origin.
To understand a voxel value produced by a spectroscopic admittivity imaging method, we need to analyze the effects of the adopted measurement method on the value. We will consider four different definitions of the admittivity: pointwise admittivity, effective admittivity, apparent admittivity, and equivalent admittivity [104, 62]. Using electrodes or coils of finite dimensions, the reconstructed or measured admittivity value becomes an ensemble average of all pointwise admittivity values within a voxel or a local region. Therefore, in this paper, we will consider the effective admittivity of a voxel, which depends on a chosen measurement scale. It changes with frequency because the voxel can be viewed as a mixture of a three-dimensional array of cells with conductive fluids and capacitive membranes.

To find the admittivity value which is a passive material property, we probe the domain and measure the relation between current, voltage and/or magnetic flux density. Since they include the effects of the domain geometry and electrode or coil configuration as well as the admittivity distribution, we obtain an impedance or admittance value from the measured relation seen by the probing and sensing methods. To extract the admittivity information from the obtained admittance value, we need to remove the effects of the domain geometry and sensor configuration. This requires an accurate computational model of the domain and sensor configuration, but its construction is hindered in most practical applications primarily due to lack of the accurate geometrical information. Therefore, there remains some effects of the probing and sensing methods in the extracted admittivity value, which we should call the apparent admittivity.

With the concepts of the effective admittivity and the apparent admittivity, we develop a spectroscopic admittivity imaging method, where the mapping between the effective admittivity in a region with a finite size and the reconstructed value in a corresponding voxel is clearly defined. Once we have established such a mapping for a designed admittivity imaging system, we have to consider the second mapping, which is between the effective admittivity value and its physiological or pathological origin. This second mapping is highly dependent on a chosen application and one can find a large amounts of literatures for numerous applications.

Since we focus on the first mapping in this paper, we briefly mention the second mapping through a simple example. Considering the oranges in figure 1, we can expect different admittivity spectra for the ripe and rotten ones. As the orange is rotten, there occur changes in its cells and compositions, which should be expressed as different effective admittivity values. In the orange juice, the orange vesicles are all crushed and their membranes are destroyed. We can expect that the admittivity spectrum of the orange juice will reveal less frequency dependent behavior compared with that of the fresh orange vesicles. Similar but a lot more complicated phenomena occur in biological tissues and organs. In this paper, we simply refer to a few histological images of biological tissues in figure 1. We can infer from these simple and intuitive examples that spectroscopic admittivity imaging provides valuable diagnostic information in its biomedical applications.

In this paper, we will first briefly review how current flows through a biological tissue. To explain the frequency-dependent changes of the tissue admittivity called the dispersion, we will introduce a few biophysical phenomena such as polarization and relaxation. We will clarify the concepts of the admittivity to describe how to understand reconstructed and measured values. We will deal with three different types of systems. The first is to measure
impedance values or spectra of tissue samples, body segments, or whole body such as bio-electrical impedance analysis (BIA)[51, 80], electrical impedance spectroscopy (EIS)[59, 52], electrical impedance plethysmography (IPG)[34, 72], impedance cardiography (ICG)[23, 7], and electrical impedance myography (IMG)[87, 88]. The second is to produce cross-sectional images of the admittivity distribution inside the human body from boundary electrical measurements and it is called electrical impedance tomography (EIT)[115, 35, 102]. The third type reconstructs cross-sectional images of the admittivity distribution from internal magnetic field measurements. Magnetic resonance electrical impedance tomography (MREIT) [119, 101, 103, 78] and magnetic resonance electrical property imaging (MREPT)[43] belong to this type.

2. Fundamentals of admittivity

If a sinusoidally varying current $I_0 \sin(\omega t)$ of the amplitude $I_0$ mA at the angular frequency $\omega$ is applied to a sample using a pair of electrodes $\mathcal{E}^+ \text{ and } \mathcal{E}^-$, the resulting current density $\mathbf{J}$ is dictated by

$$\nabla \cdot \mathbf{J} = 0 \text{ in } \Omega, \quad \frac{1}{|\mathcal{E}^\pm|} \int_{\mathcal{E}^\pm} \mathbf{J} \cdot \mathbf{n} ds = I_0 \sin(\omega t), \quad \mathbf{J} \cdot \mathbf{n}|_{\partial \Omega \setminus (\mathcal{E}^+ \cup \mathcal{E}^-)} = 0 \quad (2.1)$$

where $\partial \Omega$ denotes the boundary of the sample $\Omega$, $\mathbf{n}$ the unit outward normal vector and $\mathcal{E}^\pm \text{ the electrodes}$. Since the time-varying field varies periodically and sinusoidally with time, it is convenient to work with the phaser notation. For the time-varying electric field $\mathbf{E}$, current density $\mathbf{J}$, magnetic field $\mathbf{H}$, and magnetic flux density $\mathbf{B}$, we denote their phasors as $\mathbf{E}$, $\mathbf{J}$, $\mathbf{H}$,
and \( B \):
\[
\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\mathbf{E}(\mathbf{r}) \ e^{i\omega t}\} \quad \mathbf{J}(\mathbf{r}, t) = \text{Re}\{\mathbf{J}(\mathbf{r}) \ e^{i\omega t}\} \\
\mathbf{H}(\mathbf{r}, t) = \text{Re}\{\mathbf{H}(\mathbf{r}) \ e^{i\omega t}\} \quad \mathbf{B}(\mathbf{r}, t) = \text{Re}\{\mathbf{B}(\mathbf{r}) \ e^{i\omega t}\}
\]

We summarize Maxwell’s equations for the time-varying and time-harmonic fields:

<table>
<thead>
<tr>
<th>Name</th>
<th>Time-varying Field</th>
<th>Time-harmonic Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’s law</td>
<td>( \nabla \cdot \mathbf{E} = \rho / \epsilon )</td>
<td>( \nabla \cdot \mathbf{E} = \rho / \epsilon )</td>
</tr>
<tr>
<td>Gauss’s law for magnetism</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
<td>( \nabla \cdot \mathbf{B} = 0 )</td>
</tr>
<tr>
<td>Faraday’s law of induction</td>
<td>( \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t )</td>
<td>( \nabla \times \mathbf{E} = -i\omega \mathbf{B} )</td>
</tr>
<tr>
<td>Ampére’s circuital law</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t )</td>
<td>( \nabla \times \mathbf{H} = \mathbf{J} + i\omega \mathbf{D} )</td>
</tr>
</tbody>
</table>

\* In the free space, \( \epsilon = \epsilon_0 = 8.85 \times 10^{-12} \) and \( \mu = \mu_0 = 4\pi \times 10^{-7} \).

Writing \( \mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \) via Helmholtz decomposition, Faraday’s law of induction gives
\[
\nabla \times (\mathbf{E} + i\omega \mathbf{A}) = 0 \quad \text{in} \quad \Omega.
\]

Hence, there exist a scalar potential \( u_\omega \) satisfying \( -\nabla u_\omega = \mathbf{E}(\mathbf{r}) + i\omega \mathbf{A} \) which is dictated by
\[
\nabla \cdot (\gamma_\omega \nabla u_\omega) = -i\omega \nabla \gamma_\omega \cdot \mathbf{A} \quad \text{in} \quad \Omega. \tag{2.2}
\]

with the boundary conditions
\[
(u_\omega + z_\pm \gamma_\omega \partial u_\omega / \partial n)|_{\mathcal{E}_\pm} = \text{constant}, \quad \gamma_\omega \partial u_\omega / \partial n|_{\partial \Omega \setminus (\mathcal{E}_+ \cup \mathcal{E}_-)} = 0 \\
\int_{\mathcal{E}_+} \gamma_\omega \partial u_\omega / \partial n \, ds = I = -\int_{\mathcal{E}_-} \gamma_\omega \partial u_\omega / \partial n \, ds \tag{2.3}
\]

where \( z_\pm \) is the contact impedance of \( \mathcal{E}_\pm \).

2.1. **Conduction.** We consider a homogeneous electrolyte with positive and negative ions. Under the influence of an externally applied electric field, the mobile charges in the electrolyte experience the Coulomb forces acting on them. The forces make them move in the same and opposite directions, respectively, of the applied electric field. The movements of these charges constitute an electrical current and the induced internal current density subject to the ion conduction is proportional to the applied electric field as
\[
\mathbf{J}(\mathbf{r}) = \sigma(\mathbf{r}) \mathbf{E}(\mathbf{r})
\]

where \( \sigma \) is the pointwise conductivity, which is determined by the concentrations and mobilities of the ions at the infinitesimal point \( \mathbf{r} \). For the homogeneous electrolyte, the pointwise conductivity should be same everywhere in the solution.

We now consider a heterogenous domain such as a biological tissue with cells, extracellular matrix, intra- and extra-cellular fluids. Theoretically, we may express the pointwise conductivity as a scalar function of the position. If we consider a local region with a finite dimension of micrometer, millimeter, or centimeter, then we should use the concept of the effective conductivity, which will be more rigourously treated later in this paper. We can intuitively understand that the effective conductivity will be affected by the tissue structure. For example, at low
frequency, cell membranes are insulators and current can flow around them. As cell swelling occurs, the extra-cellular space within the region decreases and this results in a reduce effective conductivity. If cells are densely packed, the effective conductivity becomes smaller for the same reason. If cells are tightly packed along the horizontal direction and the gaps between cells in the vertical direction are bigger, then the effective conductivity will be different depending on the direction of the applied electric field. This means that we should consider the anisotropy as well as the inhomogeneity in the effective conductivity.

2.2. **Polarization, relaxation, and dispersion.** Biological tissues contain a lot of water molecules, which are electric dipoles. There exist other molecules with zero net charge but with polar structures. A medium with this kind of molecules is called the dielectric. Under an externally applied electric field, they experience no translational force to make them move. However, they may experience torques to make them rotate. Rotations of such molecules in the dielectric under the applied electric field is called the polarization.

The dielectric response or polarization of polar molecules inside a biological tissue depends on the frequency of the applied electric field.

The electric charge movement inside the biological material in response to an externally applied electric field is influenced by the frequency dependent dielectric properties of the materials. Under an external field applied to the dielectric material, the movement of the free charges inside the material is controlled by its conductivity ($\sigma$). Under an alternating field excitation, the biological tissue display extremely high dielectric constants at low frequencies, and as the excitation frequency increases, the dielectric constants of the tissues decreases [104, 56]. The frequency dependent electric and dielectric properties of biological tissue such as conductivity, permittivity and admittivity all are highly influenced by the the dielectric phenomena of the materials such as dielectric polarization, dielectric relaxation and dielectric dispersion [104, 56].

The polar molecules in a dielectric medium without any electric field applied on it, generally, are in a random orientation and as soon as an external electric field is applied the electric field polarizes the material by orienting the dipole moments of polar molecules. The orientation of the polar molecules in a dielectric medium due to an external electric field is called the dielectric polarization. Thus the dielectric polarization may be defined as the electric field induced disturbance (shift from average equilibrium positions) of the charge distribution in a region [104, 56].

Due to the so-called interfacial polarization, a heterogeneous mixture composed of two phases which differ from each other in dielectric constant and electrical conductivity shows a dielectric dispersion [27] which is the dependence of the permittivity of a dielectric material on the frequency of the external electric field applied on the medium. Thus the dielectric dispersion in biological tissues can be assumed to depend upon the permittivity of tissue material with applied electric field frequency [104, 56]. Hence, in other words, a significant change in dielectric properties over a frequency range, is called a dielectric dispersion [104, 50].
In a dielectric media under an external electric field, the rate of change of the dielectric properties does not match with the rate of change of the applied electric field [104]. Therefore, as there is always a lag between the variation in an applied electric field and changes in polarization, the electric permittivity of the biological media is found as a complex-valued function of the frequency of the applied electric field. The term dielectric relaxation in a biological medium, means the delay or lag in its response to create the dielectric polarization following the application of electric field across the tissue sample [104, 111]. In other words, the dielectric relaxation of a biological material, can be defined as the lag (momentary delay) in the dielectric constant which is usually caused by the delay in molecular polarization with respect to a change in applied electric field [104].

Schwan [92, 93] conducted a number of studies on the properties of biological tissue and cell suspensions over a large frequency range, and observed that the dielectric properties of biological tissues are characterized by three major dispersions:

- $\alpha$ dispersion: The $\alpha$ dispersion is found between the frequency range of $10 \, \text{Hz} \leq \frac{\omega}{2\pi} \leq 10 \, \text{kHz}$ and this dispersion is associated with tissue interfaces such as membranes [92].
- $\beta$ dispersion: The $\beta$ dispersion is observed between $10 \, \text{kHz} \leq \frac{\omega}{2\pi} \leq 10 \, \text{MHz}$. The $\beta$ dispersion arises, principally, from interfacial polarization (Maxwell-Wagner effect) [30] of cell membranes [50] and it is caused by the polarization of cellular membranes and polarization of protein and other organic macromolecules [92].
- $\gamma$ dispersion: The $\gamma$ dispersion in biological tissues is observed [110] between $\frac{\omega}{2\pi} \geq 10 \, \text{GHz}$. It is caused by the reorientation of water molecules.

Figure 2 shows each of these dispersions.

2.3. Effective admittivity. As a most simple case, we consider a cylindrical object filled with a homogeneous material including both mobile and immobile charges. The cylinder has a cross-sectional area $S$ and a vertical height $L$. When we apply a sinusoidal current $I(t) = \Re\{I e^{i\omega t}\}$ between the top and bottom surfaces, the resulting potential difference can be expressed as $V(t) = \Re\{V e^{i\theta} e^{i\omega t}\}$. Then, the effective admittivity is expressed as

$$\gamma_\omega = \sigma_\omega + i\omega\epsilon_\omega = \frac{L}{S} \frac{I}{V e^{i\theta}}$$

Most biological tissues are resistive at low frequencies of less than 10 kHz but the capacitive term (imaginary part) is not negligible beyond 10 kHz [20].

For a three-dimensional heterogeneous body, the effective admittivity is defined by Ohm’s law in the sense of the ensemble average of the fields. The effective admittivity at point $\mathbf{r}$ describes the linear relationship between the ensemble mean current density and the ensemble mean electrical field over the voxel $\diamond_{\mathbf{r}}$ containing $\mathbf{r}$:

$$\int_{\diamond_{\mathbf{r}}} \mathbf{J}(\mathbf{r}')d\mathbf{r}' = -\gamma_\omega(\mathbf{r}) \int_{\diamond_{\mathbf{r}}} \mathbf{E}(\mathbf{r}')d\mathbf{r}' \quad \text{for all pairs of the fields } (\mathbf{E}, \mathbf{J}).$$

Hence, the effective admittivity $\gamma_\omega(\mathbf{r})$ depends on the size of the voxel.
We defined the effective admittance at a macroscopic scale as an ensemble average of the pointwise admittance values via homogenization. The pointwise conductivity and permittivity are assumed to be isotropic and independent of $\omega$, whereas the effective conductivity and permittivity depend on the frequency $\omega$, and it can be approximately represented by a $3 \times 3$ symmetric matrix. Obtaining most of the frequency-dependent behavior of the effective admittance, we may increase the amount of the measurable information and, therefore, the distinguishability among different functions and states. We may measure the effective admittance spectra of biological tissues or organs for their tissue characterizations.

2.4. Homogenization and effective admittance. For a given voxel $\Diamond_r$, the effective admittance $\gamma_{\omega}(x)$ is an ensemble average of the pointwise admittance $\gamma_{\omega}^{pt} = \sigma_{\omega}^{pt} + i\omega\epsilon_{\omega}^{pt}$ via homogenization. We express it as a tensor-valued function, which satisfies roughly

$$\int_{\Diamond_r} \gamma_{\omega}^{pt}(x) \nabla u(x) \, dx \approx \gamma_{\omega}(x) \int_{\Diamond_r} \nabla u(x) \, dx$$

(2.4)

for all $u \in \{ u \in H^1(\Diamond_r) : \nabla \cdot (\gamma_{\omega}^{pt} \nabla u) = 0 \text{ in } \Diamond_r \}$. 

**Figure 2.** Dielectric properties of biological tissues and the frequency variation of bioelectric impedance with tissue health
There have been numerous studies on the effective conductivity of a cell suspension both analytically and numerically [84, 58, 113, 17, 62, 104]. The main issue is to understand the role of the thin membrane on the frequency-dependent behavior of the complex potential. Numerical approaches using the finite element method (FEM) have suffered from a large amount of computations due to the presence of the thin membrane.

![Figure 3](image)

**Figure 3.** Effective admittivity measurement of a rectangular tissue sample in two dimension. Electrical current is injected through the red lines $I_{xx}$ and $I_{yy}$, while voltages are measured between two opposite surfaces.

Although the concept of effective admittivity has been used widely, it seems that its precise definition has been overlooked. How can we determine the effective admittivity of a given cubic region? The effective conductivity for the special case of a strongly dilute suspension of spherical particles was studied by Maxwell [58] in 1873, Wagner, Poisson[84] in 1826, and Faraday in 1827. Fricke [17] in 1924 and several scientist studied an expression for effective conductivity of cell suspensions by considering the capacity due to a polarization at the interphases or the presence in the interphases of thin insulating membranes. All these expression are based on very special geometry such as sphere or ellipse. In [104], Seo et al provide a definition of effective admittivity in a way to measure it from boundary current-voltage data. Let us briefly explain it in two dimensional case. Consider a two-dimensional domain $\Omega$ which can be viewed as a union of voxels $V_{oxel}$ as shown in figure 3. The effective admittivity $\gamma^{ef}$ for a square domain can be given by

$$[\gamma_{\omega}]^{-1} = \begin{pmatrix} \gamma_{xx}^{\omega} & \gamma_{xy}^{\omega} \\ \gamma_{xy}^{\omega} & \gamma_{yy}^{\omega} \end{pmatrix}^{-1} := \frac{1}{I_0} \begin{pmatrix} V_{xx}^{\omega} \omega & V_{xy}^{\omega} \omega \\ V_{xy}^{\omega} \omega & V_{yy}^{\omega} \omega \end{pmatrix}$$

(2.5)

where the meanings of $V_{xx}, V_{xy}, V_{yy}$ are give in figure 3. In the presence of thin insulating membranes in the domain as shown in figure 4, the internal current streamlines vary widely from a low frequency to a hight frequency. Since the voltage difference between two opposite surfaces will change with frequency, this will in turn affect the effective admittivity. Figure 4 shows how $E$ and $J$ change with frequency.
Recently, Ammari et al. [5] provided a rigorous mathematical analysis on the frequency-dependent effective admittivity based on the homogenization theory.

Consider a periodic array of membranes $\Gamma$ in a two-dimensional domain and denote $\gamma^m = \sigma^m + i\omega \epsilon^m$ on $\Gamma$ and $\gamma^0 = \sigma^0 + i\omega \epsilon^0$ in $\Gamma^+ \cup \Gamma^-$. See figure 5 for the notations. Under the assumption that the fraction $\nu = \frac{|\Gamma^-|}{|\Gamma^+ \cup \Gamma^-|}$ is small, the frequency-dependent behavior of the effective admittivity of the medium [5] is given by

$$
\gamma_\omega = (\sigma^0 + i\omega \epsilon^0) \left( I + \nu M \left( I - \frac{\nu}{2} M \right)^{-1} \right) + o(\nu^2) \quad (2.6)
$$

with the two-dimensional tensor operator

$$
M_{ij} = \left( -\frac{d(\sigma^0 + i\omega \epsilon^0)}{(\sigma^m + i\omega \epsilon^m)} \right) \int_{\frac{1}{\sqrt{\nu}} \Gamma} \mathbf{n}_j \left( I + \frac{d(\sigma^0 + i\omega \epsilon^0)}{(\sigma^m + i\omega \epsilon^m)} \mathcal{L} \frac{1}{\sqrt{\nu}} \Gamma \right)^{-1} [\mathbf{n}_i] \quad (2.7)
$$
Figure 5. Homogenization (a local average) for the effective admittivity in the presence of thin membranes: (a) membrane admittivity $\gamma^m = \sigma^m + i\omega\epsilon^m$ and background admittivity $\gamma^0 = \sigma^0 + i\omega\epsilon^0$ and (b) periodic array of membranes.

where $\frac{1}{\sqrt{\eta}}\Gamma$ is the scaled domain by the scaling factor $\eta^{-1/2}$ and

$$\mathcal{L}_\Gamma[\phi](x) = \frac{1}{2\pi} \int_{\Gamma} \frac{\partial^2 \ln |x - y|}{\partial n(x) \partial n(y)} \phi(y) ds(y), \quad x \in \Gamma.$$  \hspace{1cm} (2.8)

In the special case when $\Gamma$ is a circle with radius $r$ and thickness $d$, (2.7) gives the Maxwell-Wagner-Fricke formula:

$$M = \frac{2\pi r^3 d\omega (\epsilon^m \sigma^0 - \epsilon^0 \sigma_m)}{(2r \sigma^m + \sigma^0 d)^2 + \omega^2 (2r \epsilon^m + \epsilon^0 d)^2} I$$

which gives the Debye relaxation time

$$\tau = (2r \sigma^m + \sigma^0 d) / (2r \epsilon^m + \epsilon_0 d).$$

To derive the approximation of (2.6), we used the two-scale convergence and homogenization by assuming that $\eta \approx 0$ in figure 5. Writing the potential $u^\eta$ as a function of the scaling factor $\eta$, we have the following two-scale convergence roughly:

$$u^\eta(x) = u^0(x) + u^1(x; \frac{x}{\eta}) + o(\eta) \Rightarrow u_0 \quad \text{as} \quad \eta \to 0$$

$$\nabla \cdot \left( (\sigma_\omega + i\omega\epsilon_\omega) \nabla u^0 \right) = 0$$

where $\gamma_\omega = \sigma_\omega + i\omega\epsilon_\omega$ is the effective admittivity distribution described in (2.6).
2.5. **Apparent and equivalent admittivity.** We define the apparent admittivity as an admittivity value measured or seen by a given measurement method. If we could get rid of all the effects of the domain geometry and sensor configuration from a measured admittance or impedance value, the apparent admittivity should equal to the effective admittivity. If not, it may differ from the effective admittivity. For a given measurement method, there may exist such cases even though the current and voltage data are noise-free. Two objects with different structures and compositions may have the same effective admittivity values and we call one of them the equivalent admittivity of the other.

In figure 6, the pointwise conductivity is \( \sigma_{pt} = \sigma_0 (1 + (10^{-8} - 1) \chi_C) \) where \( C \) represents the thin insulating OHP film excluding the small holes. When we inject dc current, the corresponding potential \( u_{pt} \) satisfies
\[
\nabla \cdot ((1 + (10^{-8} - 1) \chi_C) \nabla u_{pt}) = 0 \quad \text{in} \ \Omega.
\]
When the film is too thin to be visualized by a given impedance imaging system, we can perceive only its apparent conductivity. If we use an MREIT system with dc injection current, we may reconstruct a following conductivity image:
\[
\sigma_{apprent} = \begin{cases} 
\sigma_1 & \text{inside the film} \\
\sigma_2 & \text{outside the film}.
\end{cases}
\]
If there is no hole on the insulating film, the entire region inside the film will appear as an insulator. Figure 7 shows the results of MREIT experiments using the setup in figure 6. We can see that the reconstructed conductivity images should be interpreted as the apparent conductivity images, of which image contrast of the region inside the film changes with the diameter of the holes.

2.6. **Anisotropic admittivity.** The effective admittivity of a biological tissue may be anisotropic. In particular, biological tissues such as muscles and nerves show strong anisotropic behavior.
The ratio of anisotropy depends on the type of tissue, and the human skeletal muscle may show the anisotropy of up to one to ten between the longitudinal and transversal direction.

For ease of explanation, we consider the effective conductivity at low frequency $\omega \epsilon \approx 0$. The following conductivity tensor is a simplified model to explain the anisotropic behavior:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \quad \text{in two dimension} \quad \& \quad \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \quad \text{in three dimension.} \quad (2.9)$$

Under the approximations $\gamma_{\omega} \approx \sigma$ and $\omega \nabla \gamma_{\omega} \cdot \mathbf{A} \approx 0$, the identity (2.2) becomes

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega. \quad (2.10)$$

Figure 8 shows numerical simulations for the conductivity phantom containing a thin membrane with two holes. In this case, when dc current is injected horizontally between two electrodes on the left and right, some currents can pass through the holes. However, the vertically injected current is blocked by the membrane. This means that each current will see a different apparent conductivity value. Figure 8 shows how the current streamlines and equipotential lines vary with frequency. Currents at low frequency can not penetrate the insulating membrane while they can at high frequency. Therefore, anisotropy disappears at high frequency.

Recently, Lee et al [54] carried out MREIT experiments and theoretical analyses to identify the conductivity anisotropy in a macroscopic scale. To perceive the anisotropic conductivity (three unknowns in two dimension and seven unknowns in three dimension), we should inject currents in several directions. For any two-dimensional current densities $\mathbf{J}^j, j = 1, \cdots, n$, there exist the corresponding equivalent scalar conductivities $\sigma^j$ and potentials $u^j$ for $j =
1, \ldots, n such that
\[ J^j(r) = -\sigma^j(r)\nabla u^j(r) \quad \text{for } r \in \Omega. \]
To see the connection between the tensor \( \sigma \) in (2.9) and \( \sigma^j \) in the above identity, we consider that \( \nabla u^1 = (1, 0, 0), J^1 \times (1, 0, 0) = 0, \nabla u^2 = (0, 1, 0), \) and \( J^2 \times (0, 1, 0) = 0. \) Then, it must be \( \sigma_{11} = \sigma^1 \) and \( \sigma_{22} = \sigma^2. \)

3. **Spectroscopic Bioimpedance Measurement**

The electrical impedance of a body tissue has been an interest to the biomedical scientists and researchers to noninvasively assess the tissue health. Bioelectrical impedance analysis (BIA), electrical impedance spectroscopy (EIS), impedance plethysmography (IPG), and Impedance cardiography (ICG) use four electrode array to measure the surface potentials to estimate the small changes in electrical impedance of the body part. In these methods, the current signal is injected through two driving electrodes (outer electrodes) called current electrodes and the voltages are measured on two sensing electrodes (inner electrodes) named as voltage electrodes (see the Figure 9).

IPG and ICG measure the electrical impedance of a limb or body part is measured in a lumped form to assess the blood volume changes to detect the thrombosis and to evaluate the hemodynamic parameters to diagnose the transthoracic health, respectively. BIA and EIS have been studied by several research groups to measure the lumped electrical impedance of the biological tissues.
Recently, numerous bioimpedance analyzers have been commercialized by several companies such as RJL Systems Inc., Biospace Inc., Tanita Corporation, Jawon Medical, Omron Healthcare, and so on. The impedance analyzers manufactured by different companies are basically the same working principle, but the applications may be different due to their own specifications.

In this section, we only discuss the commercial system of InBody made by Biospace, Korea. The InBody is a body composition analyzer which performs the direct segmental multi-frequency bioelectrical impedance analysis separately and measures the impedance of the trunk, arms, and legs of our body. The InBody instruments calculates body composition by considering the human body (see Figure 10) as a structure of five cylinders: right arm, left arm, thorax, right leg and left leg as shown in the figure which are assumed as the homogeneous. Assuming the impedance of these five body parts as \( Z_1, Z_2, Z_3, Z_4 \) and \( Z_5 \), a constant amplitude sinusoidal current is injected to the six combinations of these five body parts and the six voltage data \( (V_1, V_2, V_3, V_4, V_5, \text{ and } V_6) \) are collected and to calculate the body compositions. When the current is injected through the right arm and the left arm, the measured voltage data \( (V_1) \) gives the algebraic sum of the impedances of two arms \( (Z_1 + Z_2) \). The measured voltage data \( (V_2) \) obtained by injecting the current through the right leg and the left leg, yields the algebraic sum of the impedances of two legs \( (Z_4 + Z_5) \). When the current signal is injected through the right arm and the right leg, the measured voltage \( (V_3) \) calculates the algebraic sum of the impedances of right arm, thorax and the right leg \( (Z_1 + Z_3 + Z_4) \). Similarly, the current injection through the left arm and the left leg produces a violate data \( (V_4) \) which provides the algebraic sum of the impedances of right arm, thorax and the right leg \( (Z_1 + Z_3 + Z_4) \). As shown in the Figure 10, the voltages measured from the current injection through the right arm and the left leg \( (V_5) \) and through the through left arm and the right leg \( (V_6) \) will help us to estimate the sum of the impedances of right arm, thorax and the left leg \( (Z_1 + Z_3 + Z_5) \) and the sum of the impedance of left arm, thorax and the right leg \( (Z_2 + Z_3 + Z_4) \) respectively. Thus the entire experiments will provide a system of six independent equations of five unknown variables and hence the system of equation can be easily solved to calculate all the impedance parameters of five body parts.
All these impedances are measured at two different frequencies and the extracellular water and intracellular water are calculated. Using standard formula the body compositions are calculated from the impedance values measured.

4. SPECTROSCOPIC ADMITTIVITY IMAGING USING EIT

All the methods described in the previous section can be viewed as bioimpedance measurement techniques instead of bioimpedance imaging. Electrical impedance tomography (EIT) is designed for bioimpedance imaging which aims to produce tomographic images of an admittivity distribution inside an electrically conducting object.

In EIT, we attach electrodes $E_1, E_2, \ldots, E_E$ on the boundary of an imaging domain $\Omega$. It is equipped with current sources and voltmeters to measure trans-impedances or equivalent current-voltage data sets. EIT systems can be classified into two types. The first is characterized as one current source with switching networks. In this case, current is sequentially injected between a chosen pair of electrodes and there always exists only one active current source. The second type uses multiple current sources without any switching for current injection. With this type, one may inject a pattern of current through multiple electrodes using multiple active current sources. In most EIT systems belonging to both types, voltages between many electrode pairs are simultaneously measured using multiple voltmeters. Typical examples of the first and second types are Mk3.5 from Sheffield [116] and ACT3 from RPI [15], respectively. Figure 11 shows examples of EIT system and its use for chest imaging [75, 76, 77].

The typical Electrical Impedance Tomography system takes use of 16 or 32 electrical electrodes $\mathcal{E}_k, (k = 1, 2, \ldots, E)$ which are uniformly attached on the surface of human body. When a sinusoidal current $I \sin(\omega t)$ is applied between two adjacent electrodes $\mathcal{E}_j$ and $\mathcal{E}_{j+1}$,
then the induced time-harmonic potential $u^j$ satisfies

$$\begin{align*}
\nabla \cdot (\gamma \nabla u^j) &= 0 \quad \text{in } \Omega \\
(u + z_k \gamma \frac{\partial u^j}{\partial n})|_{\epsilon_k} &= U^{j,k} \quad k = 1, 2, \cdots, E \\
\gamma \frac{\partial u^j}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus \cup_1^E \epsilon_k \\
\int_{\epsilon_k} \gamma \frac{\partial u^j}{\partial n} ds &= 0, \\
\int_{\epsilon_j} \gamma \left. \frac{\partial u^j}{\partial n} \right| ds = I = -\int_{\epsilon_{j+1}} \gamma \left. \frac{\partial u^j}{\partial n} \right| ds & \quad \text{for } k \in \{1..E\} \setminus \{j, j + 1\}
\end{align*}$$

(4.1)

where $z_k$ is contact impedance, $U^{j,k}$ is the measured voltage on electrode $\epsilon_k$ and $n$ is the unit normal derivative with respect to boundary $\Omega$. Note that the above boundary value problem has a unique solution up to a constant [106]. EIT inverse problem is to reconstruct the admittivity distribution $\gamma$ from the discrete measured voltage data $F(\gamma)$:

$$F(\gamma) = \begin{bmatrix}
V^{1,1} & V^{1,2} & \cdots & V^{1,k} & \cdots & V^{1,E} \\
V^{2,1} & V^{2,2} & \cdots & V^{2,k} & \cdots & V^{2,E} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
V^{j,1} & V^{j,2} & \cdots & V^{j,k} & \cdots & V^{j,E} \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
V^{E,1} & V^{E,2} & \cdots & V^{E,k} & \cdots & V^{E,E}
\end{bmatrix}_{E \times E}$$

(4.2)

where $F(\gamma)$ is a symmetric matrix according to reciprocity theorem and $V^{j,k} := U^{j,k} - U^{j,k+1}$ is the measured voltage between adjacent electrodes $\epsilon_k$ and $\epsilon_{k+1}$ when electrical current is injected between electrodes $\epsilon_j$ and $\epsilon_{j+1}$. 
Calderon’s problem is to recover $\gamma$ from Neumann-to-Dirichlet data \( \{ u^P|_{\partial \Omega} : P^\pm \in \partial \Omega \} \). Note that $t$ and $\tilde{t}$ are different. The (slow) time $t$ is used for the time change of the impedance, while the (fast) time $\tilde{t}$ is related to the angular frequency $\omega$ in the time harmonic expression.

For the ease of explanation of EIT structure, we consider the simplified version of (4.1) by identifying each electrodes $\mathcal{E}_j$ as points $P_j$:

\[
\begin{align*}
\nabla \cdot (\gamma(\mathbf{r}) \nabla u^P_j(\mathbf{r})) &= 0 \quad \text{in } \Omega \\
\sigma(\mathbf{r}) \frac{\partial}{\partial n} u^P_j(\mathbf{r}) &= I(\delta(\mathbf{r} - P_j) - \delta(\mathbf{r} - P_{j+1})) \quad \text{on } \partial \Omega
\end{align*}
\]

Then, the measure Dirichlet Data can be expressed as the set of vectors \( (\mathbf{V}_1, \ldots, \mathbf{V}_{n_E}) \):

\[
k - \text{th comp. of } \mathbf{V}_j = V_{j,k}[\gamma] = u^P_j(P_k) - u^P_j(P_{k+1}).
\]

EIT is based on the following reciprocity principle:

\[
u^P(Q^+,\omega,t) - u^P(Q^-,\omega,t) = \frac{1}{I} \int_{\Omega} \gamma \nabla u^P \cdot \nabla u^Q \, d\mathbf{r} = u^Q(P^+,\omega,t) - u^Q(P^-,\omega,t)
\]
Static EIT data depends mainly on conductivity distribution near boundary.

FIGURE 13. The images are low vectors of the sensitivity matrix $S$. EIT has low sensitivity in any interior region, whereas it has relative high sensitivity near the driving and sensing electrodes.

Discretizing $\Omega$ into $n_p$ elements as $\Omega = \bigcup_{n=1}^{n_p} q_n$, the above identity leads to

$$u^{P_j}(P_k, \omega, t) - u^{P_j}(P_{k+1}, \omega, t) \approx \sum_{n=1}^{n_p} \left( \frac{1}{T} \int_{q_n} \nabla u^{P_j} \cdot \nabla u^{P_k} \, dq \right) \gamma|_{q_n}$$

(4.3)

Denote the corresponding sensitivity matrix by $S$ whose $n$th column vector is given by

$$s_n = (s_{1,1}^n, \cdots, s_{1,E}^n, s_{2,1}^n, \cdots, s_{2,E}^n, \cdots, s_{E,1}^n, \cdots, s_{E,E}^n)^T.$$

Then, (4.4) can be expressed as the following system

$$S \gamma = b$$

(4.4)

Hence, the inverse problem of solving $SX = b$ is to find a best linear combination of the column vectors of the sensitivity matrix $S$ which produces the data $b$. The reconstructed image relies roughly on truncated SVD. Figure 13 shows images of low vectors of the sensitivity matrix $S$ and the sensitivity values in the interior region are very close to zero when compared
with the sensitivity in the region near the driving and sensing electrodes. Hence, the following solution would be very sensitive to noise when the pixel size is small:

$$X = (S^*S)^{-1}S^*b.$$  

To extract any useful impedance information from EIT data, we need to get rid of the boundary influences. The use of difference data enable us to cancel out the boundary influences.

- **Time difference EIT (tdEIT):** $\frac{\partial}{\partial t} Data \rightarrow \frac{\partial}{\partial t} \gamma (r, \omega, t)$
- **Frequency difference EIT (fdEIT):** $\frac{\partial}{\partial \omega} Data \rightarrow \frac{\partial}{\partial \omega} \gamma (r, \omega, t)$

Now, for imaging the impedance distribution of a biological tissue with an noticeable distiguishability the tissue should have a suitable impedance changes which can produce a measurable electrical signal at the boundary say boundary potential or current. Hence, the diseases or the tissue health abnormalities which produces the measurable impedance changes can be images by EIT. As for example, the cell swelling, tumor tissue such as Lung carcinoma or other can be images by EIT as they have distinguishable impedance response compared to the normal body tissues. In the cell swelling the swelled cells occupy more spaces in the tissue and reduces the extracellular space which increases the real part of the complex impedance. The tumor tissues need more amount of blood and oxygen to continue their rapid growth which reduces the electrical impedance of the tumor.

EIT provides the spatial distribution of the tissue impedance and hence it can be used to visualize the insulating objects in saline tank as the impedance difference between the saline solution and insulator produced a measurable boundary voltages. Similarly the vegetable tissue and insulator or other high resistive materials can be distinguished by frequency difference EIT as the frequency responses of the insulator or biological tissues are different and measurable by electronic instruments. But, is it possible to distinguish two different biological tissues in a saline tank? Is it possible to distinguish the cucumber and carrot tissue samples placed in a same saline tank? Frequency difference EIT can provide the distinguishable impedance image of cucumber and carrot with a saline background but is it possible to identify the tissues from their conductivity or resistivity map in difference EIT? The questions drive us to think
about a more effective and accurate impedance imaging modality such as spectroscopic admittance imaging of biological tissue which will provide the pixel wise admittance map with the spectroscopic information to distinguish the different biological tissues.

5. DUAL-FREQUENCY ADMITTIVITY IMAGING USING MRI

Electrical tissue property imaging methods using MRI such as MREIT and MREPT can produce conductivity and permittivity images with a pixel size of a few millimeters. The contrast information from these novel imaging modalities is unique since there is currently no other method to reconstruct high-resolution conductivity and permittivity images. MREIT aims to provide conductivity images at dc or frequencies below a few kHz, whereas MREPT produces both conductivity and permittivity images at the Larmor frequency of 128 MHz at 3 T, for example. MREIT relies on measured magnetic field data at low frequencies which are influenced by the low-frequency conductivity distribution, whereas MREPT relies on acquired B1 maps which are influenced by both the conductivity and permittivity distributions at the Larmor frequency. Due to the frequency-dependent behavior of admittance values of biological tissues, MREIT and MREPT are supplementary each other providing different images of the same object in terms of its low- and high-frequency conductivity distributions, respectively. We note that an MREIT experiment may include steps needed to obtain B1 maps simultaneously. We, therefore, suggest to develop a method to perform MREIT and MREPT together by properly manipulating acquired k-space MR data.

MREIT was motivated by the well-known ill-posedness of the image-reconstruction problem of electrical impedance tomography (EIT). In 1989, a research group at Toronto University developed a current density imaging (CDI) technique using MRI to visualize the internal current density distributions of an object due to an injected current through surface electrodes. The, from the relation of $\nabla \times \mathbf{H} + \sigma \nabla \mathbf{u} = 0$, we can recover $\sigma$ via J-substitution algorithm

$$\nabla \cdot \left( \frac{\left| \nabla \times \mathbf{B} \right|}{\left| \nabla \mathbf{u} \right|} \nabla \mathbf{u} \right) = 0.$$ 

The major drawback of CDI is the requirement of object rotation inside the MRI scanner to acquire all three components of the induced magnetic flux density, as MRI can measure only the $z$-component of the magnetic flux density, where the $z$-axis is the axial magnetization direction.
of the MRI scanner. Thus, we must rotate the object in order to acquire three components of
the magnetic flux density, which leads to serious technical difficulties. Although there have
been numerous attempts to deal with the requirement of the object rotation since the 1990s,
there remain serious technical difficulties in handling this drawback, which seriously limits
the clinical applicability of this method. In order to reach the stage of animal and human
imaging, we should recover the conductivity distribution using only Bz data to avoid object
rotation. Until 2000, conductivity imaging using only Bz data seemed impossible. According
to Maxwell’s equations, the current density is directly related to the three components B =
(Bx, By, Bz) of the magnetic flux density and the conductivity σ must be computed from
the relationship between the current flux density J and the electrical field E. Hence, most
researchers consider that only Bz data is insufficient for conductivity reconstruction.

In 2002, Seo et al carefully investigated the nonlinear relationship between the conductivity
and the measured data via the Biot-Savart law, making a key observation that the Laplacian of
Bz data probes changes in the logarithm of the conductivity distribution along any equipotential
curve in each imaging slice [45, 48, 49]. In this method, two different currents are injected into
the body to generate two linearly independent current flux densities. They showed that if the
area of the parallelogram made by these two vector fields is non-zero at every position in the
imaging slice, the spatial change of the conductivity distribution can be precisely reconstructed:

\[
\nabla^2 \ln \sigma(r) = \nabla \cdot \left( A^\dagger(r) \begin{bmatrix} \nabla^2 B_{z,1}(r) \\ \nabla^2 B_{z,2}(r) \end{bmatrix} \right)
\]

where \( A^\dagger(r) := \frac{1}{\mu_0} \begin{bmatrix} \sigma \frac{\partial u_1[\sigma]}{\partial y}(r) & -\sigma \frac{\partial u_1[\sigma]}{\partial x}(r) \\ \sigma \frac{\partial u_2[\sigma]}{\partial y}(r) & -\sigma \frac{\partial u_2[\sigma]}{\partial x}(r) \end{bmatrix}^{-1} \)

Moreover, they used a geometric index theory in mathematics to prove rigorously that the area
of the parallelogram is non-zero when the two pairs of surface electrodes are appropriately
attached. Taking advantage of these mathematical observations, he found a representation for-
mula for the conductivity which leads to the development of a constructive irrotational MREIT
algorithm termed the harmonic Bz algorithm. This representation formula exists in an implicit
form due to the non-linear relationship between the conductivity and the measured data, but it
was designed to use a fixed point theory. In other words, the formula has a contraction mapping
property such that an iterative method can be used. The major drawback of EIT, ill-posedness
is mainly due to the fact that the overall flow of the current density is insensitive to local pertur-
bations in the conductivity distribution. However, the harmonic Bz method takes advantage of
this fact to make the algorithm work. Since then, imaging techniques in MREIT have been ad-
vanced rapidly and now can offer state-of-the-art conductivity imaging for animal and human
experiments. Based on this mathematical analysis, his group successfully carried out animal
and human experiments for the first time.

MREPT is a relatively new MR-based imaging modality to provide both conductivity (σ) and
permittivity (ε) images at MR Larmor frequency (about 128 MHz at 3 Tesla MRI). Using B1
mapping technique, we can measure the positive rotating magnetic field

\[
H^+ = \frac{1}{2} (H_x + iH_y)
\]
which is governed by
\[ -\nabla^2 H = \frac{\nabla \gamma}{\gamma} \times \nabla \times H - i\omega\mu_0 \gamma H \]

Unfortunately, we cannot measure \( H^- = \frac{1}{2}(H_x - iH_y) \). Hence, each components \( H_x, H_y, H_z \) are not available. The inverse problems of EPT is to reconstruct the conductivity \( \sigma \) and permittivity \( \epsilon \) (at frequency 126 MHz) from the given \( H^+ \) data and
\[
-\nabla^2 H = \frac{\nabla \gamma}{\gamma} \times \nabla \times H - i\omega\mu_0 \gamma H
\]

Hence, \( H^+ = \frac{1}{2}(H_x + iH_y) \) probes \( \gamma = \sigma + i\omega\epsilon \) through PDE
\[
-\nabla^2 H^+(r) = \frac{1}{2}((\hat{x} + i\hat{y}) \times (\nabla \times H(r))) \cdot \nabla \ln \gamma - i\omega\mu_0 \gamma H^+(r).
\]

The conventional methods ignore the refraction term using the assumption of local homogeneity of \( \gamma \) to get the following simple form:
\[
-\nabla^2 H = \frac{\nabla \gamma}{\gamma} \times \nabla \times H - i\omega\mu_0 \gamma H
\]

Wen [117] uses this local homogeneity assumption of \( \gamma \) to get
\[
\gamma(r) = \frac{i}{\omega\mu_0} \frac{\nabla^2 H^+(r)}{H^+(r)}.
\]

Katscher et al [42] performed initial experiments on a standard clinical MRI system: For any disk \( D_\delta(r_0) \subset \Omega \) where \( \nabla \gamma \approx 0 \),
\[
\gamma = \frac{\int_{D_\delta} \nabla \times H \cdot d\ell}{i\omega\mu_0 \int_{D_\delta} H \cdot dS}.
\]

However, neglecting \( \nabla \ln \gamma(r) \times [\nabla \times H(r)] \) causes serious artifacts. Seo et al perform error analysis of this direct reconstruction formula: Denoting \( \gamma^* = \frac{1}{i\omega\mu_0} \frac{\nabla^2 H^+}{H^+} \), we have
\[
\text{Error} = \gamma - \gamma^* = \left( \frac{1}{i\omega\mu_0} \frac{\nabla^2 H^+}{H^+} - \frac{\nabla^2 H^-}{i\omega\mu_0 H^-} \right) \left[ 1 - \frac{H^+ \frac{\partial}{\partial z} H^-}{H^- \frac{\partial}{\partial z} H^+} \right]^{-1}
\]

where \( H = (H^+ + H^-, -iH^+ + iH^-, H_z) \).

Hence, we should include the refraction term \( \nabla \ln \gamma(r) \times [\nabla \times H(r)] \) in the reconstruction algorithm. In 2003, Song and Seo [107] found that under the assumption of \( \frac{\partial H^+(r)}{\partial z} = 0 \) and \( \partial H^+ \neq 0 \) in \( \Omega \), \( \gamma \) satisfies the following semilinear PDE:
\[
\nabla_x^2 \ln \gamma + \nabla_x \cdot (\gamma f_{H^+}) = h_{H^+} \quad \text{on each cross-section},
\]

(5.1)
where

\[
f_{H^+} = \frac{\omega \mu_0 H^+}{2 \partial H^+} (i, 1) \quad \text{and} \quad h_{H^+} = \partial \left( \frac{\nabla^2 H^+}{\partial H^+} \right).
\]

Here, \( x = (x, y) \), \( \nabla_x = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \) \& \( \partial = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \). Based on this semilinear PDE, we can apply iterative method. Unfortunately, the sequence may not be convergent due to lack of compactness; the semi-linear elliptic equation involves the critical Sobolev exponent. This means that a simple fixed-point iteration may risk non-convergence.

At present, there exist distinct differences between MREIT and MREPT in the use of the term \( \nabla \log(\sigma + i\omega\epsilon) \times \nabla \times \mathbf{H} \). In terms of their imaging capabilities at present, MREIT successfully probes the contrast \( \nabla \sigma \), whereas MREPT has a difficulty in dealing with an inhomogeneous admittivity distribution especially along edges. On the other hand, MREPT is advantageous over MREIT in recovering an absolute value of \( \gamma \) inside a locally homogeneous region. Implementation of MREPT on a clinical MRI scanner is easier since it does not require any additional instrument while MREIT requires attachments of surface electrodes and a constant current source.

Future studies for MREIT should overcome a few technical barriers to advance the method to the stage of routine clinical uses. The biggest hurdle at present is the amount of injection current that may stimulate muscle and nerve. Reducing it down to a level that does not produce undesirable side effects is the key to the success of this new bio-imaging modality. To advance MREPT further, we need to improve the quality of acquired B1 maps in terms of both magnitude and phase and devise a way to properly handle the edges of admittivity changes. Considering the rapid progresses in these relatively new imaging methods, we expect that MREIT and MREPT will find clinically useful applications in near future and provide unique diagnostic information.

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ℓ GOES TO PLUS INFINITY : AN UPDATE

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ABSTRACT. The goal of this note is to describe the asymptotic behaviour of problems set in cylinders when the size of them is becoming infinite. This leads to consider problems in unbounded domains as well as new singular perturbations issues.

1. INTRODUCTION

It has been observed since a long time that the solution of problems set in cylinders, when the data are depending on the section only, are also almost independent of the axial direction – at least, as we will see, on a large portion of the cylinder at stake. At a time when one was trying to save some computation power this has lead many numerical analysts to be convinced that it was enough to carry out the simulations in two dimensions i.e. on the section of the cylinder. If this seems reasonable, trying to justify that mathematically turns out to be often a challenging task which has been the topic of recent research with sometimes unexpected developments in different fields like for instance singular perturbations. Since the publication of the book “ℓ goes to plus infinity”, which attacks some of the issues mentioned above (Cf. [7], [18]), a lot of progresses have been made in different directions. This is the goal of this note to advertise them and to report on the issues which are still open. We refer to [33], [34] for classical results and notation.

The simplest problem of the type “ℓ goes to plus infinity” is the following. Let \( \Omega_\ell \) be the rectangle defined as

\[
\Omega_\ell = (-\ell, \ell) \times (-1, 1).
\] (1.1)

If we denote by \((x_1, x_2)\) the points in \(\mathbb{R}^2\), let

\[
f = f(x_2) \in L^2(-1, 1).
\] (1.2)
Then by the Lax-Milgram theorem, it is well known that there exists a unique weak solution to the problem

\begin{equation}
\begin{cases}
-\Delta u_\ell = f & \text{in } \Omega_\ell, \\
u_\ell = 0 & \text{on } \partial \Omega_\ell.
\end{cases}
\end{equation}

(\(\Delta\) is the usual Laplace operator, \(\partial \Omega_\ell\) denotes the boundary of \(\Omega_\ell\)). The issue is then to determine what happens for \(u_\ell\) when \(\ell\) goes to plus \(+\infty\). A natural candidate for the limit is \(u_\infty\) the solution of a similar problem on the section of the rectangle that is to say the solution to

\begin{equation}
\begin{cases}
-\partial_{x_2}^2 u_\infty = f & \text{in } (-1, 1), \\
u_\infty(\pm 1) = 0.
\end{cases}
\end{equation}

One can indeed show (see [29], [8], [47]):

**Theorem 1.1.** There exists positive constants \(C, \alpha\) such that

\[ ||\nabla(u_\ell - u_\infty)||_{2,\Omega_\ell}^2 \leq Ce^{-\alpha \ell}. \]

(\(||\cdot||_{2,A}\) denotes the usual \(L^2(A)\)-norm, \(|\cdot|\) the euclidean norm in \(\mathbb{R}^2\)).

**Remark 1.1.** The above theorem states in particular that, on any fixed subdomain of the strip \(\Omega_\infty = \mathbb{R} \times (-1, 1)\), \(u_\ell\) converges toward \(u_\infty\) at an exponential rate. Note that \(u_\ell\) itself is not independent of \(x_1\) unless it is identically equal to 0 and \(f\) as well.

**Open question:** We do not know if the exponential rate of convergence is preserved when one replaces in (1.3) the Laplace operator by the \(p\)-Laplace operator. For some convergence results we refer to [24], [28], [45].
2. THE GENERAL FRAMEWORK

We denote by $\omega_1, \omega_2$ two bounded open subsets of $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively and we suppose that

$$0 \in \omega_1 \text{ and } \omega_1 \text{ is star shaped with respect to } 0.$$  \hspace{1cm} (2.1)

For $\ell > 0$ we set

$$\Omega_\ell = \ell \omega_1 \times \omega_2.$$  \hspace{1cm} (2.2)

The points $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ are split into two components $X_1, X_2$ where

$$X_1 = x_1, ..., x_m, \quad X_2 = x_{m+1}, ..., x_n$$  \hspace{1cm} (2.3)

i.e. $X = (X_1, X_2)$.

Let us denote by $f$ a function independent of $X_1$, i.e.

$$f = f(X_2)$$  \hspace{1cm} (2.4)

and consider $u_\ell$ the weak solution to

$$\begin{cases}
-\partial_{x_i}(|\partial_{x_i} u_\ell|^{p-2}\partial_{x_i} u_\ell) = f & \text{in } \Omega_\ell, \\
u_\ell = 0 & \text{on } \partial \Omega_\ell,
\end{cases}$$  \hspace{1cm} (2.5)

(with the summation convention on $i = 1, ..., n$). When $p = 2$ the above operator is the Laplace operator. We will assume in what follows that

$$p \geq 2.$$  \hspace{1cm} (2.6)

Then, if for instance

$$f \in L^q(\omega_2), \quad \frac{1}{p} + \frac{1}{q} = 1,$$  \hspace{1cm} (2.7)

the problem above admits a unique weak solution in $W^{1,p}_0(\Omega_\ell)$ -i.e. there exists a unique $u_\ell$ satisfying (see [8])

$$\begin{cases}
u_\ell \in W^{1,p}_0(\Omega_\ell), \\
\int_{\Omega_\ell} |\partial_{x_i} u_\ell|^{p-2}\partial_{x_i} u_\ell \partial_{x_i} v \, dx = \int_{\Omega_\ell} f v \, dx \forall v \in W^{1,p}_0(\Omega_\ell).
\end{cases}$$  \hspace{1cm} (2.8)

We would like to find out the limit behaviour of $u_\ell$ when $\ell \to +\infty$. For that we will assume that $\omega_1$ in addition to (2.1) possesses the following property. For every $a > 0$ there exists a function $\rho_a = \rho_a(X_1)$ such that

$$0 \leq \rho_a \leq 1, \quad \rho_a = 1 \text{ on } a\omega_1, \quad \rho_a = 0 \text{ on } \partial\{(a+1)\omega_1\}, \quad |\nabla \rho_a| \leq C$$  \hspace{1cm} (2.9)

for some constant $C$.

**Remark 2.1.** If $\omega_1$ is convex the property above holds true.
Then, if \( u_\infty \) denotes the solution to
\[
\begin{cases}
  u_\infty \in W^{1,p}_{0}(\Omega_2),
  \\
  \int_{\Omega_2} |\partial_x u_\infty|^{p-2} \partial_x u_\infty \partial_x \varphi \, dx = \int_{\Omega_2} \varphi \, dx \quad \forall \varphi \in W^{1,p}_{0}(\Omega_2),
\end{cases}
\]  
(2.10)
(note that in the integral above the summation convention is made for \( i = m + 1, \ldots, n \), \( dX_2 = dx_{m+1} \ldots dx_n \), we have :

**Theorem 2.1.** There exists positive constants \( C, \alpha \) such that
\[
\|\nabla (u_\ell - u_\infty)\|_{p,\Omega_\ell} \leq Ce^{-\alpha \ell}.
\]  
(2.11)

**Proof.** Let us denote by \( \ell_1 \) a real number such that \( 0 < \ell_1 \leq \ell - 1 \). It is clear that
\[
\rho_{\ell_1}(X_1)(u_\ell - u_\infty) \in W^{1,p}_0(\Omega_\ell)
\]  
(2.12)
and thus it is a suitable test function for (2.8). Moreover for a.e. \( X_1 \) (Cf. [7])
\[
\rho_{\ell_1}(X_1)(u_\ell - u_\infty)(X_1, \cdot) \in W^{1,p}_0(\Omega_2).
\]  
(2.13)
and thus is a suitable test function for (2.10). Thus, for a.e. \( X_1 \) it holds
\[
\int_{\Omega_2} |\partial_x u_\infty|^{p-2} \partial_x u_\infty \partial_x \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx = \int_{\Omega_2} \varphi \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx.
\]  
(2.14)
(Note that in the first integral above \( i \) can run from 1 to \( n \) since \( u_\infty \) is independent of \( X_1 \)). Integrating (2.14) in \( X_1 \) we get
\[
\int_{\Omega_\ell} |\partial_x u_\infty|^{p-2} \partial_x u_\infty \partial_x \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx = \int_{\Omega_\ell} \varphi \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx.
\]  
(2.15)
Taking \( \varphi = \rho_{\ell_1}(u_\ell - u_\infty) \) in (2.8) and subtracting the equality above we obtain
\[
\int_{\Omega_\ell} \{\partial_x u_\ell|^{p-2} \partial_x u_\ell - |\partial_x u_\infty|^{p-2} \partial_x u_\infty\} \partial_x \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx = 0.
\]  
(2.16)
Since \( \rho_{\ell_1} \) vanishes outside \( \Omega_\ell_1+1 \) it is enough in the integral above to integrate on \( \Omega_\ell_1+1 \) and we obtain easily
\[
\int_{\Omega_\ell_1+1} \{\partial_x u_\ell|^{p-2} \partial_x u_\ell - |\partial_x u_\infty|^{p-2} \partial_x u_\infty\} \partial_x \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx
\]  
(2.17)
\[
= \int_{\Omega_\ell_1+1 \setminus \Omega_\ell_1} \{\partial_x u_\ell|^{p-2} \partial_x u_\ell - |\partial_x u_\infty|^{p-2} \partial_x u_\infty\} \partial_x \{\rho_{\ell_1}(u_\ell - u_\infty)\} \, dx.
\]  
(We used the fact that \( \rho_{\ell_1} = 1 \) on \( \Omega_\ell_1 \). In the last integral \( i \) runs from 1 to \( m \) since \( \rho_{\ell_1} \) is independent of \( X_2 \)). It is well known that for some constant \( C_p \) one has
\[
C_p|a - b|^p \leq (|a|^{p-2}a - |b|^{p-2}b)(a - b) \quad \forall a, b \in \mathbb{R}
\]  
(2.18)
and thus from (2.17) we derive easily
\[
C_p \int_{\Omega_{\ell+1}} |\partial_{x_i}(u_{\ell} - u_{\infty})|^p \rho_{\ell} \ dx \leq \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\partial_{x_i} u_{\ell}|^{p-1} |\partial_{x_i} \rho_{\ell}| |u_{\ell} - u_{\infty}| \ dx. \tag{2.19}
\]
Since \(\rho_{\ell} = 1\) on \(\Omega_{\ell}\) and \(u_{\infty}\) is independent of \(X_1\) it implies that
\[
C_p \int_{\Omega_{\ell}} |\partial_{x_i}(u_{\ell} - u_{\infty})|^p \ dx \leq \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\partial_{x_i}(u_{\ell} - u_{\infty})|^{p-1} |\partial_{x_i} \rho_{\ell}| |u_{\ell} - u_{\infty}| \ dx. \tag{2.20}
\]
(Note that in the last integral \(i\) is running from 1 to \(m\)). Using the equivalence of norms in \(\mathbb{R}^n\) and (2.9) we derive easily
\[
\int_{\Omega_{\ell}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx \leq C \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\nabla X_1(u_{\ell} - u_{\infty})|^{p-1} |u_{\ell} - u_{\infty}| \ dx \tag{2.21}
\]
where \(C = C(p,m,n)\) is a constant depending on \(p, m\) and \(n\), \(|\cdot|\) is the euclidean norm, \(\nabla X_1 = (\partial_{x_1}, \ldots, \partial_{x_m})\) is the gradient in \(X_1\). Using then the Young inequality
\[
ab \leq \frac{a^q}{q} + \frac{b^p}{p}
\]
with \(q = \frac{p}{p-1}\) we arrive to
\[
\int_{\Omega_{\ell}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx \leq C \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\nabla X_1(u_{\ell} - u_{\infty})|^p + |u_{\ell} - u_{\infty}|^p \ dx. \tag{2.22}
\]
By the Poincaré inequality we have for a.e. \(X_1 \in \ell\omega_1\)
\[
\int_{\omega_2} |(u_{\ell} - u_{\infty})|^p(X_1, X_2) dX_2 \leq C(\omega_2) \int_{\omega_2} |\nabla X_2(u_{\ell} - u_{\infty})|^p(X_1, X_2) dX_2 \tag{2.23}
\]
for some constant \(C(\omega_2)\), \(\nabla X_2 = (\partial_{x_{m+1}}, \ldots, \partial_{x_n})\) is the gradient in \(X_2\). Integrating in \(X_1\) we derive
\[
\int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |(u_{\ell} - u_{\infty})|^p \ dx \leq C(\omega_2) \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\nabla X_2(u_{\ell} - u_{\infty})|^p \ dx. \tag{2.24}
\]
Thus going back to (2.22) we get for some constant \(C\)
\[
\int_{\Omega_{\ell}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx \leq C \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx \tag{2.25}
\]
which leads to
\[
\int_{\Omega_{\ell}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx \leq \frac{C}{C + 1} \int_{\Omega_{\ell+1}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx. \tag{2.26}
\]
Starting from \(\ell_1 = \frac{\ell}{2}\) and iterating this inequality \([\frac{\ell}{2}]\)-times where \([\ ]\) denotes the integer part of a real we get
\[
\int_{\Omega_{\ell}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx \leq \left(\frac{C}{C + 1}\right)^{\left[\frac{\ell}{2}\right]} \int_{\Omega_{\ell+\left[\frac{\ell}{2}\right]}} |\nabla(u_{\ell} - u_{\infty})|^p \ dx. \tag{2.27}
\]
Setting \( \gamma = \frac{C}{C+1} < 1 \) and noticing that
\[
\frac{\ell}{2} - 1 < \left\lfloor \frac{\ell}{2} \right\rfloor \leq \frac{\ell}{2}
\]  
we obtain
\[
\int_{\Omega_{\ell}} |\nabla (u_\ell - u_\infty)|^p \, dx \leq \gamma^{\frac{\ell}{2} - 1} \int_{\Omega_{\ell}} |\nabla (u_\ell - u_\infty)|^p \, dx. \tag{2.29}
\]

We try now to evaluate the right hand side integral above. Taking \( v = u_\ell \) in (2.8) we get easily using in particular the Poincaré inequality:
\[
\int_{\Omega_{\ell}} |\partial_{x_i} u_\ell|^p \, dx = \int_{\Omega_{\ell}} f u_\ell \, dx 
\leq |f|_{q, \Omega_{\ell}}^m |u_\ell|_{p, \Omega_{\ell}} 
\leq |f|_{q, \Omega_{\ell}} C(\omega_2) |\nabla_{X_2} u_\ell|_{p, \Omega_{\ell}} 
\leq C(\omega_2) |f|_{q, \Omega_{\ell}} |\nabla u_\ell|_{p, \Omega_{\ell}} \tag{2.30}
\]
and thus, by equivalence of the norms in \( \mathbb{R}^n \)
\[
||\nabla u_\ell||_{p, \Omega_{\ell}}^p \leq C |f|_{q, \Omega_{\ell}} ||\nabla u_\ell||_{p, \Omega_{\ell}} 
\Rightarrow ||\nabla u_\ell||_{p, \Omega_{\ell}}^p \leq C |f|_{q, \Omega_{\ell}} \frac{p}{p-1} = C \left( \int_{\omega_1} \int_{\omega_2} |f(X_2)|^q \, dx \right) \tag{2.31}
\]
where \( C = C(\omega_1, \omega_2, m, n) \). Similarly taking \( v = u_\infty \) in (2.10) we derive
\[
||\nabla u_\infty||_{p, \omega_2}^p \leq C |f|_{q, \omega_2}^m |f|_{q, \omega_2}^p \tag{2.32}
\]
and thus
\[
||\nabla u_\infty||_{p, \Omega_{\ell}}^p \leq C \ell^m |f|_{q, \omega_2}^p. 
\]
Going back to (2.29) we get
\[
\left\{ \int_{\Omega_{\ell}} |\nabla (u_\ell - u_\infty)|^p \, dx \right\}^\frac{1}{p} \leq \frac{e^{\frac{\ell}{2} \ln \gamma}}{\gamma} \left\{ \int_{\Omega_{\ell}} |\nabla (u_\ell - u_\infty)|^p \, dx \right\}^\frac{1}{p} 
\leq C |f|_{q, \omega_2}^\frac{1}{p} \ell^\frac{m}{p} e^{\frac{\ell}{2p} \ln \gamma} 
\leq Ce^{-\alpha \ell},
\]
for any \( \alpha < -\frac{\ln \gamma}{2p} \). This completes the proof of the theorem.

\( \Box \)
Remark 2.2. In the case $p = 2$ we have obtained at the same time the proof of Theorem 1.1. Note that in the case of the $p$-Laplace operator i.e. when

$$-\Delta_p u = -\partial_{x_i}\{|\nabla u|^{p-2}\partial_{x_i}\}$$

the technique above does not work since all the derivatives are involved in the operator (see (2.20)).

3. Periodic Problems

In the two sections above we considered a function $f$ constant in some directions. A natural generalization is to consider a $f$ which is periodic – in one direction or several. For instance if we consider (1.3) with $f$ periodic in $x_1$ does it force $u_\ell$ to converge toward a periodic function? The answer is yes and we refer the reader to [8] for a proof and some generalizations. What we would like to do in this section is to prove that this works also in a parabolic framework. We consider again the simple case of $\Omega_\ell$ given by (1.1) and a function

$$f = f(x_1, x_2)$$

periodic in $x_1$ with period $P$ -i.e. we assume

$$f(x_1 + P, x_2) = f(x_1, x_2) \text{ a.e. } x.$$

We denote by $C$ the period cell

$$C = (0, P) \times (-1, 1)$$

where it is enough to define $f$ and we assume that

$$f \in L^2(C).$$

We define

$$C_{0,per}^1(C) = \{v \in C^1(\overline{C}) : v(., \pm 1) = 0, \ v(0, .) = v(P, .)\},$$

$$V = H_{0,per}^1(C) = \text{ the closure of } C_{0,per}^1(C) \text{ in } H^1(C),$$

i.e. $V = H_{0,per}^1(C)$ is the set of $H^1(C)$-functions periodic in $x_1$ and vanishing on the part of the boundary of $C$ contained in the straight lines $x_2 = \pm 1$. It is clear that $H_{0,per}^1(C)$ is a Hilbert space when equipped with the Dirichlet norm

$$\left(\int_C |\nabla u|^2 \, dx\right)^{\frac{1}{2}}.$$  \hfill (3.1)
Clearly \( f \in V' \) the dual of \( V \) and for \( u^0 \in L^2(C) \) there exists a unique \( u_\infty \) (this notation will be clear later) solution to
\[
\begin{cases}
u_\infty \in L^2(0,T;V), & \partial_t u_\infty \in L^2(0,T;V'), \\
\partial_t(u_\infty,v) + \int_C \nabla u_\infty \cdot \nabla v \, dx = \int_C f v \, dx \quad \forall v \in V \text{ in } D'(0,T), \\
u_\infty(0,.) = u^0.
\end{cases}
\]
(\( T \) is a positive number). We suppose from now on that \( u_\infty \) and \( u^0 \) are extended by periodicity \( P \) on the whole strip \( \mathbb{R} \times (-1,1) \). For \( \ell > 0 \) we set
\[
\Omega_\ell = (-\ell P, \ell P) \times (-1,1), \quad V_\ell = H^1_0(\Omega_\ell).
\]
We suppose \( V_\ell \) equipped with the Dirichlet norm - i.e. defined by (3.1) with \( C \) replaced by \( \Omega_\ell \). Then \( f \in V'_\ell \) and there exists a unique \( u_\ell \) solution to
\[
\begin{cases}
u_\ell \in L^2(0,T;V_\ell), & \partial_t u_\ell \in L^2(0,T;V'_\ell), \\
\partial_t(u_\ell,v) + \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \quad \forall v \in V_\ell \text{ in } D'(0,T), \\
u_\ell(0,.) = u^0.
\end{cases}
\]
(3.3)
Then we have :

**Theorem 3.1.** When \( \ell \to +\infty \), \( u_\ell \to u_\infty \) with an exponential rate of convergence. (i.e. the periodicity of \( f \) forces \( u_\ell \) to be periodic at the limit).

We will need the following lemma:

**Lemma 3.1.** Let \( u_\infty \) be the periodic extension of the solution to (3.2). Then for every \( \Omega \subset \mathbb{R} \times (-1,1) \) bounded, it holds
\[
\partial_t(u_\infty,v) + \int_\Omega \nabla u_\infty \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1_0(\Omega) \text{ in } D'(0,T).
\]

**Proof.** For \( z \in \mathbb{Z} \) we denote by \( C_z \) the translated of \( C = C_0 \) the reference cell i.e.
\[
C_z = C_0 + (zp,0).
\]
We suppose that \( v \) is extended by 0 outside \( \Omega \). Then the support of \( v \) intersects only a finite number of cells \( C_z \) -see the figure below.

In \( D'(0,T) \) one considers the distribution
\[
A = \partial_t(u_\infty,v) + \int_\Omega \nabla u_\infty \cdot \nabla v \, dx = \partial_t\left( \sum_z (u_\infty,v)_{2,C_z} \right) + \sum_z \int_{C_z} \nabla u_\infty \cdot \nabla v \, dx.
\]
\( \ell \) GOES TO PLUS INFINITY

**Figure 3.2**

\((,)_2, D\) denotes the usual \(L^2(D)\)-scalar product. Since \(C_z = C_0 + (zP, 0)\) one has

\[
A = \partial_t \left\{ \sum_z (u_\infty(t, x_1 + zP, x_2), v(x_1 + zP, x_2))_{2,C} \right\}
+ \sum_z \int_C \nabla u_\infty(t, x_1 + zP, x_2) \cdot \nabla v(x_1 + zP, x_2) dx.
\]

Since \(u_\infty\) has no jump through the different cells, \(u_\infty\) and \(\nabla u_\infty\) are periodic of period \(P\) in the \(x_1\) direction and thus

\[
A = \partial_t (u_\infty, \sum_z v(x_1 + zP, x_2))_{2,C} + \int_C \nabla u_\infty \cdot \nabla \left\{ \sum_z v(x_1 + zP, x_2) \right\} dx.
\]

It is clear that the function

\[
\sum_z v(x_1 + zP, x_2)
\]

is \(P\)-periodic in \(x_1\) and vanishes on \(x_2 = \pm 1\) (note that only a finite number of terms in the sum are different of 0). Thus by (3.2) we have in \(D'(0, T)\)

\[
A = \int_C f(x_1, x_2) \sum_z v(x_1 + zP, x_2) dx
= \sum_z \int_C f(x_1 + zP, x_2) v(x_1 + zP, x_2) dx = \sum_z \int_{C_z} f v dx = \int_\Omega f v dx
\]  

(3.5)

since \(f\) is periodic in the direction \(x_1\). This completes the proof of the Lemma.

\[\Box\]

**Proof of the Theorem** For \(\ell_1 \leq \ell - 1\) we denote by \(\rho = \rho_{\ell_1}\) the function which graph is described below (compare to (2.9)).

Clearly one has

\[
(u_\ell - u_\infty)\rho(x_1) \in V_\ell
\]
\begin{align*}
< \partial_t (u_\ell - u_\infty), (u_\ell - u_\infty) \rho > + & \int_{\Omega_{\ell+1}} \nabla (u_\ell - u_\infty) \cdot \nabla \{(u_\ell - u_\infty) \rho \} dx = 0 \\
\text{i.e.} \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega_{\ell+1}} (u_\ell - u_\infty)^2 \rho dx + \int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_\infty)|^2 \rho dx \quad & \quad = \int_{D_{\ell+1}} \partial x_1 (u_\ell - u_\infty) (\partial x_1 \rho) (u_\ell - u_\infty) dx \\
\text{where we have set} \quad D_{\ell+1} = \Omega_{\ell+1} \setminus \Omega_\ell. \\
\text{Since } |\partial x_1 \rho| \leq 1 \text{ we derive by Young's inequality} \quad & \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega_{\ell+1}} (u_\ell - u_\infty)^2 \rho dx + \int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_\infty)|^2 dx \quad \leq \frac{1}{2} \int_{D_{\ell+1}} \partial x_1 (u_\ell - u_\infty)^2 + (u_\ell - u_\infty)^2 dx. \\
\text{By the Poincaré inequality one has} \quad & \quad \int_{D_{\ell+1}} (u_\ell - u_\infty)^2 dx \leq 2 \int_{D_{\ell+1}} \partial x_2 (u_\ell - u_\infty)^2 \quad \text{and thus it holds} \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega_{\ell+1}} (u_\ell - u_\infty)^2 \rho dx + \int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_\infty)|^2 dx \leq \int_{D_{\ell+1}} |\nabla (u_\ell - u_\infty)|^2 dx. \\
\text{This can be written as} \quad & \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega_{\ell+1}} (u_\ell - u_\infty)^2 \rho dx + 2 \int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_\infty)|^2 dx \leq \int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_\infty)|^2 dx. \\
\text{Integrating between } 0 \text{ and } T \text{ we obtain (recall that } u_\ell = u_\infty = u^0 \text{ for } t = 0) 
\end{align*}
\[
\frac{1}{2} \int_{\Omega_{\ell_1}} (u_\ell - u_\infty)^2(T, x) dx + 2 \int_0^T \int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 dx dt \leq \int_0^T \int_{\Omega_{\ell_1+1}} |\nabla (u_\ell - u_\infty)|^2 dx dt
\]

which implies

\[
\frac{1}{2} \int_{\Omega_{\ell_1}} (u_\ell - u_\infty)^2(T, x) dx + 2 \int_0^T \int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 dx dt \\
\leq \frac{1}{2} \left( \frac{1}{2} \int_{\Omega_{\ell_1+1}} (u_\ell - u_\infty)^2(T, x) dx + 2 \int_0^T \int_{\Omega_{\ell_1+1}} |\nabla (u_\ell - u_\infty)|^2 dx dt \right).
\]

Setting

\[
F(\ell_1) = \frac{1}{2} \int_{\Omega_{\ell_1}} (u_\ell - u_\infty)^2(T, x) dx + 2 \int_0^T \int_{\Omega_{\ell_1}} |\nabla (u_\ell - u_\infty)|^2 dx dt
\]

we have obtained

\[
F(\ell_1) \leq \frac{1}{2} F(\ell_1 + 1), \forall \ell_1 \leq \ell - 1.
\]

Starting from \( \ell_1 = \frac{\ell}{2} \) and iterating this inequality \( \lfloor \frac{\ell}{2} \rfloor \)-times one derives

\[
F\left( \frac{\ell}{2} \right) \leq \left( \frac{1}{2} \right)^{\lfloor \ell/2 \rfloor} F\left( \frac{\ell}{2} + \left\lfloor \frac{\ell}{2} \right\rfloor \right).
\]

Using again (2.28) we get

\[
F\left( \frac{\ell}{2} \right) \leq 2\left( \frac{1}{2} \right)^{\frac{\ell}{2}} F(\ell) = 2e^{-\frac{\ell}{2}\ln2} F(\ell).
\]

In order to conclude we need to estimate \( F(\ell) \). Taking \( v = u_\ell \) in (3.3) and integrating in \( t \) we get

\[
\frac{1}{2} |u_\ell|_{2,\Omega_\ell}^2(T) - \frac{1}{2} |u^0|_{2,\Omega_\ell}^2 + \int_0^T \int_{\Omega_\ell} |\nabla u_\ell|^2 dx dt = \int_0^T \int_{\Omega_\ell} f u_\ell dx dt
\]

\[
\leq \int_0^T |f|_{2,\Omega_\ell}^2 dt + \frac{1}{4} \int_0^T |u_\ell|_{2,\Omega_\ell}^2
\]

\[
\leq \int_0^T |f|_{2,\Omega_\ell}^2 dt + \frac{1}{2} \int_0^T \int_{\Omega_\ell} |\nabla u_\ell|^2 dx dt.
\]

This leads to

\[
\frac{1}{2} |u_\ell|_{2,\Omega_\ell}^2(T) + \frac{1}{2} \int_0^T \int_{\Omega_\ell} |\nabla u_\ell|^2 dx dt \leq \frac{1}{2} |u^0|_{2,\Omega_\ell}^2 + \int_0^T \int_{\Omega_\ell} f^2 dx dt \leq C(u^0, f, T, \ell).
\]

The same estimate holds for \( u_\infty \) so that

\[
F(\ell) \leq C \ell
\]
for some constant independent of $\ell$. It follows that
\[
\int_{\Omega_{\ell}} (u_\ell - u_\infty)^2(T, x) dx + \int_0^T \int_{\Omega_{\ell}} |\nabla (u_\ell - u_\infty)|^2 dx dt \leq C e^{-\frac{\ell}{2}\ln 2} \leq C e^{-\alpha \ell}
\]
for every $\alpha < \frac{\ln 2}{2}$. This completes the proof of the theorem.

\[\square\]

4. Existence results in unbounded domains

When considering (1.3) there is no reason to assume $f$ independent of $x_1$. Indeed assuming for instance $f \in L^2(\Omega_{\ell})$ for every $\ell$ then $u_\ell$ is perfectly defined. The question is then to find out if $u_\ell$ possesses a limit when $\ell \to +\infty$. Of course a natural candidate is $u_\infty$ solution to
\[
\begin{cases}
-\Delta u_\infty = f(x_1, x_2) & \text{in } \Omega_\infty = \mathbb{R} \times (-1, 1), \\
u_\infty(., \pm 1) = 0.
\end{cases}
\]

Unfortunately the solution to (4.1), if it exists, is not unique. Indeed, for instance the functions
\[
u_\infty = Ce^{\frac{\pi}{2}x_2}cos\left(\frac{\pi}{2}x_2\right)
\]
all satisfy (4.1) for $f = 0$. Now the only bounded function of the type (4.2) is obtained when $C = 0$. Besides non uniqueness it is also not clear that existence of a solution to (4.1) does exist. Indeed when
\[
f \in H^{-1}(\Omega_\infty)
\]
then certainly the Lax-Milgram theorem provides existence and uniqueness of a solution in $H^1_0(\Omega_{\infty})$ to (4.1) (this solution being 0 when $f = 0$). But they are many simple functions which are not in $H^{-1}(\Omega_\infty)$ (see [31]). For instance this is the case for a function $f = f(x_2) \neq 0$. Indeed consider such a function. If $f \in H^{-1}(\Omega_\infty)$ there exists a unique solution to
\[
u \in H^1_0(\Omega_\infty) : \int_{\Omega_\infty} \nabla \nu \cdot \nabla v dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega_\infty).
\]

By uniqueness it is easy to show that $\nu$ is translation invariant in the $x_1$ direction. This implies that $\nu = u_\infty(x_2)$. But this function cannot be in $L^2(\Omega_\infty)$ unless $u = f = 0$. So in order to establish existence of a solution to (4.1) some new arguments have to be developped. The idea is that if $f$ is not growing too fast at infinity $u_\ell$ defined by (1.3) is a Cauchy sequence (see also [43]). We will consider the situation described by the figure below -i.e. $\Omega$ is an unbounded domain contained in the strip $\mathbb{R} \times (-a, a)$, $a > 0$. We set
\[
\Omega_{\ell} = (-\ell, \ell) \times (-a, a) \cap \Omega,
\]
\[
V_{\ell} = \{v \in H^1(\Omega_{\ell}) : v = 0 \text{ on } \partial \Omega_{\ell}\}
\]
Let us denote by $V'_\ell$ the dual of $V_\ell$ and suppose that $f \in V'_\ell$ for every $\ell > 0$. Then there exists a unique $u_\ell$ solution to
\[
 u_\ell \in V_\ell, \quad \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v dx = <f, v> \quad \forall v \in V_\ell. \tag{4.3}
\]
(< > denotes the $V'_\ell, V_\ell$ duality). Then we have:

**Theorem 4.1.** Suppose that for some $\gamma > 0$ we have
\[
 |f|_{V'_\ell} = O(\ell^\gamma), \tag{4.4}
\]
where $|\cdot|_{V'_\ell}$ is the strong dual norm in $V'_\ell$. Then $u_\ell$ converges toward the unique solution to
\[
 \begin{cases}
 u_\infty \in H^1_{\text{loc}}(\Omega), & u_\infty = 0 \text{ on } \partial\Omega, \\
 -\Delta u_\infty = f \text{ in } \Omega, & ||\nabla u_\infty||_{2,\Omega_\ell} = O(\ell^\gamma). 
\end{cases} \tag{4.5}
\]
Moreover
\[
 ||\nabla (u_\ell - u_\infty)||_{2,\Omega_\ell} \leq Ce^{-\beta \ell} \tag{4.6}
\]
for some positive constants $C$ and $\beta$.

**Proof.** For $\ell_1 \leq \ell - 1$ consider the function $\rho = \rho_{\ell_1}$ defined in the preceding section.

**a) Estimate of $u_\ell - u_{\ell+r}$ for $0 \leq r \leq 1$**

One has clearly
\[
 (u_\ell - u_{\ell+r})\rho(x_1) \in V_\ell \text{ and } V_{\ell+r}.
\]
Thus from (4.3) one deduces
\[
 \int_{\Omega_{\ell+1}} \nabla (u_\ell - u_{\ell+r}) \cdot \nabla \{(u_\ell - u_{\ell+r})\rho\} dx = 0.
\]
This implies
\[
\int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_{\ell+r})|^2 \rho \, dx
\]
\[
= \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} \partial_{x_1} (u_\ell - u_{\ell+r}) \partial_{x_1} \rho (u_\ell - u_{\ell+r}) \, dx
\]
\[
\leq \int_{D_{\ell+1}} \partial_{x_1} (u_\ell - u_{\ell+r}) \|(u_\ell - u_{\ell+r})\| \, dx
\]
\[
\leq \frac{\epsilon}{2} \int_{D_{\ell+1}} \{\partial_{x_1} (u_\ell - u_{\ell+r})\}^2 \, dx + \frac{1}{2\epsilon} \int_{D_{\ell+1}} (u_\ell - u_{\ell+r})^2 \, dx
\]  
(4.7)

\(D_{\ell+1}\) has been defined previously, the last inequality follows from Young’s inequality. One has the Poincaré inequality (without choosing here the best constant)
\[
\int_{D_{\ell+1}} (u_\ell - u_{\ell+r})^2 \, dx \leq 2a^2 \int_{D_{\ell+1}} \{\partial_{x_2} (u_\ell - u_{\ell+r})\}^2 \, dx
\]

and thus
\[
\int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx \leq \frac{\epsilon}{2} \int_{D_{\ell+1}} \{\partial_{x_1} (u_\ell - u_{\ell+r})\}^2 \, dx + \frac{2a^2}{2\epsilon} \int_{D_{\ell+1}} \{\partial_{x_2} (u_\ell - u_{\ell+r})\}^2 \, dx.
\]

Choosing \(\epsilon = a\) we obtain
\[
\int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx \leq a \int_{\Omega_{\ell+1} \setminus \Omega_{\ell}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx
\]
i.e.
\[
\int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx \leq \frac{a}{a+1} \int_{\Omega_{\ell+1}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx.
\]

Starting from \(\ell/2\) and iterating \([\ell/2]\) times this inequality we get (see (2.28))
\[
\int_{\Omega_{\ell/2}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx \leq a + \frac{1}{a} \left( \frac{a}{a+1} \right)^{\ell/2} \int_{\Omega_{\ell}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx
\]
\[
\leq Ce^{-\alpha \ell} \int_{\Omega_{\ell}} |\nabla (u_\ell - u_{\ell+r})|^2 \, dx
\]  
(4.8)

with \(C = \frac{a+1}{a}\), \(\alpha = -\frac{1}{2} \ln \left( \frac{a+1}{a} \right)\).
b) Estimate of $u_\ell$

We just take $v = u_\ell$ in (4.3) to get

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx = <f, u_\ell> \leq |f|_{V_\ell}^2 \|\nabla u_\ell\|_{2, \Omega_\ell}.$$

(We suppose $V_\ell$ normed with the Dirichlet norm $|u|_{V_\ell}^2 = \int_{\Omega_\ell} |\nabla u|^2 dx$). It follows that

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \leq |f|_{V_\ell}^2 = O(\ell^{2\gamma}). \tag{4.9}$$

c) $u_\ell$ is a Cauchy sequence

Combining (4.8) and (4.9) we have

$$\int_{\Omega_\ell} |\nabla (u_\ell - u_{\ell+T})|^2 dx \leq C e^{-\alpha\ell} \{ |\nabla u_\ell|^2_{2, \Omega_\ell} + |\nabla u_{\ell+T}|^2_{2, \Omega_{\ell+T}} \}$$

$$\leq C' e^{-\alpha\ell} \{ (\ell^{2\gamma} + (\ell + r)^{2\gamma}) \}$$

$$\leq C' e^{\ell^{2\gamma}} e^{-\alpha\ell} \{ 1 + (1 + \frac{r}{\ell})^{2\gamma} \} \leq C e^{-2\beta\ell} \tag{4.10}$$

for some constant $C$ when $\ell > 1$ -recall that $0 \leq r \leq 1$. This can be written as

$$|u_\ell - u_{\ell+T}|_{V_\ell}^2 \leq C e^{-\beta\ell}.$$

Thus for $t$ arbitrary we derive then

$$|u_\ell - u_{\ell+T}|_{V_\ell}^2 \leq |u_\ell - u_{\ell+T}|_{V_\ell}^2$$

$$+ |u_{\ell+T} - u_{\ell+T+1}|_{V_{\ell+T+1}} + \cdots + |u_{\ell+T+[t]} - u_{\ell+T}|_{V_{\ell+T+[t]}}$$

$$\leq C e^{-\beta\ell} + C e^{-\beta(\ell+1)} + \cdots + C e^{-\beta(\ell+[t])}$$

$$\leq C e^{-\beta\ell} \{ 1 + e^{-\beta} + \cdots + e^{-\beta[t]} \} \leq C \frac{1}{1 - e^{-\beta}} e^{-\beta\ell} = C' e^{-\beta\ell} \tag{4.11}$$

independently of $t$.

Suppose that we choose $\ell_0 \leq \ell$ then it follows from above that $u_\ell$ is a Cauchy sequence in $H^1(\Omega_{\ell_0})$. It converges toward $u_\infty$ such that $u_\infty = 0$ on $\partial\Omega_{\ell_0} \cap \partial\Omega$. Moreover, passing to the limit in $t$ in (4.11) provides (4.6).

d) Limit problem

From above we have

$$\int_{\Omega_{\ell_0}} \nabla u_\ell \cdot \nabla v dx = <f, v> \quad \forall v \in H^1_0(\Omega_{\ell_0}).$$
and passing to the limit in $\ell$ we get
\[ \int_{\Omega_{\ell_0}} \nabla u_\infty \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega_{\ell_0}). \]

**e) Estimate of $|\nabla u_\infty|_{2,\Omega_\ell}$**

From (4.11) one has
\[ |u_\ell - u_{\ell + t}|_{V_\ell} \leq C' e^{-\beta \ell} \]
i.e.
\[ |u_{2\ell} - u_{2\ell + t}|_{V_\ell} \leq C' e^{-2\beta \ell}. \]
Passing to the limit in $t$ we obtain
\[ |u_{2\ell} - u_\infty|_{V_\ell} \leq C' e^{-2\beta \ell} \]
which implies by (4.9)
\[ |u_\infty|_{V_\ell} \leq C' e^{-2\beta \ell} + |u_{2\ell}|_{V_\ell} \leq C' e^{-2\beta \ell} + |u_{2\ell}|_{V_\ell} \]
\[ \leq C' e^{-2\beta \ell} + C''(2\ell) \gamma = O(\ell) \gamma. \] (4.12)

**f) Uniqueness**

Suppose that $u_\infty$, $u'_\infty$ are two solutions to (4.5). Then one has
\[ \int_{\Omega_\ell} \nabla (u_\infty - u'_\infty) \nabla v \, dx = 0 \quad \forall v \in H^1_0(\Omega_\ell). \]
Taking $v = (u_\infty - u'_\infty) \rho_{\ell_1}$ and arguing as above will lead to
\[ \int_{\Omega_{\ell/2}} |\nabla (u_\infty - u'_\infty)|^2 \, dx \leq C^2 e^{-2\beta \ell}. \]
Letting $\ell \to \infty$ leads to $u_\infty = u'_\infty$. This completes the proof of the theorem.

**Remark 4.1.** One can replace the assumption $|f|_{V_\ell'} = O(\ell')$ by $|f|_{V_\ell'} = O(e^{\gamma \ell})$ for some small enough $\gamma$. Note that we are also recovering here the theorem 1.1.

### 5. Anisotropic Singular Perturbations

Let $u_\ell$ the solution to (1.3). Set
\[ v_\ell(x_1, x_2) = u_\ell(\ell x_1, x_2). \]
Clearly $v_\ell$ is defined on $\Omega_1 = (-1, 1)^2$. Moreover one has
\[ \partial_{x_1}^2 v_\ell(x_1, x_2) = \ell^2 \partial_{x_1}^2 u_\ell(\ell x_1, x_2), \quad \partial_{x_2}^2 v_\ell(x_1, x_2) = \ell^2 \partial_{x_2}^2 u_\ell(\ell x_1, x_2). \]
Thus $v_\ell$ is solution to
\[
\begin{cases}
-\frac{1}{\ell^2} \frac{\partial^2}{\partial x_1^2} v_\ell - \frac{\partial^2}{\partial x_2^2} v_\ell = f \quad \text{in } \Omega_1, \\
v_\ell = 0 \quad \text{on } \partial \Omega_1.
\end{cases}
\tag{5.1}
\]
Setting $\epsilon = \frac{1}{\ell}$, $v_\ell = w_\epsilon$ we see that $w_\epsilon$ is solution of a problem of the type
\[
\begin{cases}
-\epsilon^2 \frac{\partial^2}{\partial x_1^2} w_\epsilon - \frac{\partial^2}{\partial x_2^2} w_\epsilon = f \quad \text{in } \Omega_1, \\
w_\epsilon = 0 \quad \text{on } \partial \Omega_1.
\end{cases}
\tag{5.2}
\]
This is an anisotropic singular perturbation problem i.e. the diffusion in the direction $x_1$ is much smaller when $\epsilon \to 0$ than in the direction $x_2$. The techniques developed for studying $u_\ell$ allow to investigate such problems, we refer the interested reader to [9], [10], [11], [13], [14], [15].

In a more general framework like in our section 2 we have
\[
\Omega_1 = \omega_1 \times \omega_2.
\]
Consider then a matrix $A$ defined as
\[
A(x) = \begin{pmatrix} A_{11}(x) & A_{12}(X_2) \\ A_{21}(x) & A_{22}(X_2) \end{pmatrix},
\]
where $A_{11}$ is a $m \times m$ matrix and $A_{22}$ is a $(n-m) \times (n-m)$ one. Suppose that $A$ is elliptic and in particular satisfies for some constants $\lambda$, $\Lambda$
\[
\lambda |\xi|^2 \leq A(x)\xi \cdot \xi, \quad |A(x)\xi| \leq \Lambda |\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e } x \in \Omega_1.
\]
Then a generalization of (5.2) is obtained by setting
\[
A_\epsilon(x) = \begin{pmatrix} \epsilon^2 A_{11}(x) & \epsilon A_{12}(X_2) \\ \epsilon A_{21}(x) & A_{22}(X_2) \end{pmatrix}.
\]
For $f = f(X_2) \in L^2(X_2)$ there exists a unique $w_\epsilon$ solution to
\[
\begin{cases}
w_\epsilon \in H^1_0(\Omega_1), \\
\int_{\Omega_1} A_\epsilon(x) \nabla w_\epsilon \cdot \nabla v dx = \int_{\Omega_1} f v dx \quad \forall v \in H^1_0(\Omega_1)
\end{cases}
\tag{5.3}
\]
which converges in a certain sense when $\epsilon \to 0$ toward $w_0$ solution to
\[
\begin{cases}
w_0 \in H^1_0(\omega_2), \\
\int_{\omega_2} A_{22}(X_2) \nabla X_2 w_0 \cdot \nabla X_2 v dX_2 = \int_{\omega_2} f v dX_2 \quad \forall v \in H^1_0(\omega_2),
\end{cases}
\tag{5.4}
\]
(see [8], [11]).

**Remark 5.1.** Correctors for such problems can be found in [12]. Also these anisotropic perturbations issues allow to get existence results for exotic problems (see [13], [14]).
6. SOME CONCLUDING REMARKS

We presented above only a few problems which were addressed in the recent years. Many other developments occur. The most important achievement being to reach, with relatively simple techniques, an exponential rate of convergence for the solution of problems set in a domain of the type $\Omega_\ell$ toward a solution set in $\Omega_\infty$. This gives credit to the numerical analysts' pioneering vision. They indeed did not wait for the theoretical results to reduce in their computations the dimension of the problem.

What can be done for the Dirichlet problem can be extended to the stokes problem in the case of forces independent of one direction as well as in the periodic case (see [1], [16], [17]). It would be interesting to extend this to the two dimensional time dependent Navier-Stokes problem in the spirit of our section 3. Extension in the case of stationary Navier-Stokes problem is more delicate since uniqueness might fail – the same issue occurs for the higher dimensional evolution Navier-Stokes problem. Some higher order problems are investigated in [4], [38]. Hyperbolic issues are considered in [5], [35], [40], [41]. Regarding singular perturbations a self contained study is available in [15] - see also [10], [11], [36], [38], [40].

If $\lambda^k_\ell$ denotes the $k$-th eigenvalue of the problem

$$\begin{cases}
-\Delta u_\ell = \lambda^k_\ell u_\ell & \text{in } \Omega_\ell, \\
u_\ell = 0 & \text{on } \partial \Omega_\ell,
\end{cases}$$

(6.1)

where $\Omega_\ell = \ell \omega_1 \times \omega_2$ it can be shown that $\lambda^k_\ell$ converges when $\ell \to +\infty$ towards the first eigenvalue of the problem

$$\begin{cases}
-\Delta X_2 u_\infty = \lambda^1_\infty u_\infty & \text{in } \omega_2, \\
u_\infty = 0 & \text{on } \partial \omega_2,
\end{cases}$$

(6.2)

($\Delta X_2$ is the usual Laplace operator in $\omega_2$) more precisely one has

$$0 \leq \lambda^k_\ell - \lambda^1 = O\left(\frac{1}{\ell^2}\right),$$

when $\ell \to +\infty$. This is easily extended to general elliptic operators (see [22]) and one has also convergence of the eigenfunctions. The passage to the Neumann case could appear straightforward. It is not so, for instance if one considers for $\Omega_\ell$ given by (1.1) the first eigenvalue problem.

$$\begin{cases}
-\nabla \cdot (A \nabla u_\ell) = \lambda^1_\ell u_\ell & \text{in } \Omega_\ell, \\
u_\ell = 0 \text{ on } x_2 = \pm 1, \ A\nabla u_\ell \cdot n = 0 \text{ on } x_1 = \pm \ell,
\end{cases}$$

(6.3)

$n$ being the outward unit normal to $\Omega_\ell$ one does not have $\lambda^k_\ell \to \lambda^1$ as above unless $A$ is a $2 \times 2$ diagonal matrix. A substantial analysis of this problem and its generalisations is done in [23], [44].
As we mentioned already in the case of the $p$-Laplace operators, exponential rates of convergence are more complicated to reach for nonlinear problems (Cf. also [42]). It can go through for obstacle problems but remains open in many situation in the framework of variational inequalities (see [30], [32], [47]). Some quasi linear problems are investigated in [7] and also various different norms are considered (i.e. not only the norm of the gradient). To this respect many results can be extended to reach an exponential rate of convergence.

Since everything extends to the periodic case in the spirit of section 3, many issues remain open there (see [1], [25], [26]).

As we might have convinced the reader, any problem set in a cylinder or in a generalised cylinder of the type (2.2) can be analyse the way we did and lead to interesting issues. In the seventies, one of the first application of the obstacle problem and of variational inequalities was the so called Dam Problem. It was set and studied widely in two dimensions and the invoked reason was that one was assuming that the dam at stake was of cylindrical shape and long enough so that the flow in this porous medium was the same in every cross section of the dam. A rigorous proof of this assertion is still pending and challenging. (See [2], [3]) for notions on the problem.

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IMAGE SEGMENTATION BASED ON THE STATISTICAL VARIATIONAL FORMULATION USING THE LOCAL REGION INFORMATION

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ABSTRACT. We propose a variational segmentation model based on statistical information of intensities in an image. The model consists of both a local region-based energy and a global region-based energy in order to handle misclassification which happens in a typical statistical variational model with an assumption that an image is a mixture of two Gaussian distributions. We find local ambiguous regions where misclassification might happen due to a small difference between two Gaussian distributions. Based on statistical information restricted to the local ambiguous regions, we design a local region-based energy in order to reduce the misclassification. We suggest an algorithm to avoid the difficulty of the Euler-Lagrange equations of the proposed variational model.

1. INTRODUCTION

Segmentation has been widely studied in the field of image processing and computer vision. It basically separates an image into several homogeneous regions according to a criterion. The quality of segmentation is crucial in other fields of image analysis such as motion tracking and image classification. Methods of segmentation are classified into two categories; one is the boundary-based segmentation which uses gradient information of an image to detect abrupt changes in the image and the other is the region-based segmentation which uses similarity information of an image to separate the image into several regions.

As basic boundary-based segmentations, there are filters by Roberts, Prewitt, Sobel [1, 2], and Canny [3]. These filters have a problem of irregularity of a curve which detects a boundary of an object in an image. Recently based on active contours models [4], variational formulations on deformable curves have been developed in order to handle the irregularity problem. These models consider energy functionals which consist of smoothness and attraction of a
curve. The energy for attraction of the curve commonly introduces an edge-detector function which makes a curve evolution converge at a boundary of an object as a minimizer of the energy functional. More advanced methods have been suggested in geometric active contours [5], geodesic active contours [6], and gradient vector flows [7]. With the help of the level set method [8, 9], these methods are able to deal with topological change which is a problem in the parametric deformable model [4]. However, there still remains a problem to capture weak boundaries changed smoothly from strong boundaries because these methods depend on edge-detector functions.

As region-based segmentations, there are thresholding, region growing [10], active contours without edges [11], and region competition [12]. Whereas the gradient-based segmentation uses local information on a deformable curve, the region-based segmentation uses global information of an image. In other words, an energy functional of the latter is formed by statistical information such as means and standard deviations of intensities on a region inside a deformable curve and a region outside the curve. Region competition [12] has been a fundamental framework in the region-based segmentation with an assumption that a given image is a mixture of two Gaussian distributions. Many region-based segmentations with the assumption have a problem of misclassification due to a small difference between two Gaussian distributions. The misclassification easily happens when an object has various intensities inside the boundary of the object or outside the boundary; see Figure 1.

Combinations of the boundary-based segmentation and the region-based segmentation have been also studied. Geodesic active regions [13, 14] combines geodesic active contours [6] and region competition [12] in order to segment a texture in an image. Region-aided geometric snake [15] adds the region force term in the geodesic active contours [6]. The region force is obtained from any region-based segmentation such as mean shift algorithm [16].

In this paper we propose a variational model based on statistical information of intensities in an image. The model consists of both a global region-based energy and a local region-based energy. The former roughly captures outlines of objects which we want to segment in an image. The latter reduces misclassification which happens in a typical statistical variational model with an assumption that the image is a mixture of two Gaussian distributions. We find local ambiguous regions where the misclassification happens due to small difference between two Gaussian distributions. Using statistical information restricted to the local ambiguous regions, we design the local region-based energy in order to reduce misclassification. We suggest an algorithm to avoid the difficulty of the Euler-Lagrange equations of the proposed energy functional.

The outline of this paper is as follows. In Section 2, we review the region competition and show what misclassification is in an image. In Section 3, we propose a statistical variational model and suggest an algorithm for segmentation. In Section 4, we show examples and comment numerical aspects of the algorithm. This paper is concluded in Section 5.
2. Region competition

Let $\Omega$ be a bounded closed subset of $\mathbb{R}^2$ and $I : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be an image. A segmentation problem is to partition the image into several subregions which are characterized by prescribed properties. We assume that the image consists of two disjoint regions, i.e., $\Omega = \mathcal{R}_A \cup \mathcal{R}_B$ and intensities in the regions follow Gaussian distributions with probability density functions (PDFs) $\mathcal{P}(I; \alpha_i), i \in \{A, B\}$, where $\alpha_i = (\mu_i, \sigma_i)$ is a Gaussian parameter of a mean $\mu_i$ and a standard deviation $\sigma_i$:

$$\mathcal{P}(I(x); \alpha_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(I(x) - \mu_i)^2}{2\sigma_i^2}\right).$$

Let $\Gamma$ be the boundary separating two regions $\mathcal{R}_A$ and $\mathcal{R}_B$ and $\alpha = (\alpha_A, \alpha_B)$ be Gaussian parameters of two regions. Many energy functionals in typical statistical variational models [12–14] contain an energy term

$$E(\Gamma, \alpha) = -\sum_{i \in \{A, B\}} \int_{\mathcal{R}_i} \log \mathcal{P}(I(x); \alpha_i) dx,$$

which generates a region competition as the energy is decreased.

In order to find a minimizer of the energy functional (2.1), an algorithm which consists of two alternating steps was used in [12]. In the first step, when a boundary $\Gamma$ is fixed, the functional has a minimum if $\alpha_i = (\mu_i, \sigma_i)$ is the Gaussian parameter of the region $\mathcal{R}_i$:

$$\mu_i = \frac{1}{|\mathcal{R}_i|} \int_{\mathcal{R}_i} I(x) dx \quad \text{and} \quad \sigma_i^2 = \frac{1}{|\mathcal{R}_i|} \int_{\mathcal{R}_i} (I(x) - \mu_i)^2 dx$$

for $i \in \{A, B\}$. In the second step, when $\alpha$ is fixed, the Euler-Lagrange equation with respect to $\Gamma$ is deduced and the gradient descent method is used. That is, the energy has a minimum if $\Gamma$ is a curve $C$ that is a steady state solution of

$$\frac{\partial C(x, t)}{\partial t} = F(x)\vec{n}, \quad F(x) = \log \mathcal{P}(I(x); \alpha_A) - \log \mathcal{P}(I(x); \alpha_B),$$

where $\vec{n}$ is the outward normal vector to the curve (we assume that $\mathcal{R}_A$ is inside the curve $C$).

The evolution of the curve is generated by the sign of the force (2.2). The sign indicates that a point on the curve is more likely included into one of two regions $\mathcal{R}_A$ and $\mathcal{R}_B$. If $F > 0$ at a point, then the curve at the point moves along $\vec{n}$. It means that the point belongs to the region $\mathcal{R}_A$. If $F < 0$ at a point, on the other hand, the curve at the point moves along $-\vec{n}$ and the point belongs to the region $\mathcal{R}_B$. In this way, all points on the boundary belong to one of two regions. This process is called “region competition”.

Unfortunately, the region competition process may cause a problem which we call “misclassification”. In Figure 1-(a), a minimizer of the energy functional (2.1) clearly captures the cross shape in the image. However, if an illumination is changed inside and outside the object in Figure 1-(c), a minimizer of the functional does not capture the cross shape. The reason is that some points whose intensities are near an intersection of two Gaussian PDFs are classified into a wrong region. That is, points whose intensities are near the intersection $I^*$ in Figure 1-(d)
are possibly misclassified in segmentation. We call the region which consists of such points as “local ambiguous region”. In [12], in order to reduce misclassification, an energy functional was constructed using an average probability inside a window around points in an image. The algorithm to minimize the energy functional has a drawback that it is difficult to measure an optimal size of an window to obtain proper segmentation. It is also hard to decide how many seed points are needed in order to obtain computational efficiency. In this paper, we propose a variational model which overcome those drawbacks. The main purpose of the model is to handle misclassification in the local ambiguous regions.

3. Statistical Variational Formulation

3.1. Modeling of energy functional. Let $\Gamma$ be a boundary separating two regions $\mathcal{R}_A$ and $\mathcal{R}_B$ and $\alpha = (\alpha_A, \alpha_B)$ be Gaussian parameters of two regions. Figure 1 tells us that if a difference of two Gaussian PDFs is large at a point, there is no misclassification when we classify the point according to the difference of Gaussian PDFs. However if the difference is small at a point, the classification of the point could be wrong. We define a set of such points as a local ambiguous region depending on Gaussian parameter $\alpha$:

$$\mathcal{D}(\alpha) = \{ x \mid H(f_{\alpha}(x)) = 1 \},$$

(3.1)
Figure 2. Two graphs in (a) are Gaussian PDFs on $\mathcal{R}_A$ and $\mathcal{R}_B$. There are three intervals $[I_{\min}, I_0]$, $[I_1, I_2]$, and $[I_3, I_{\max}]$, where the difference of two PDFs is less than $\zeta(\alpha)$ in (3.2). The local ambiguous region $\mathcal{D}(\alpha)$ in (b) is a set of points $x$ whose intensities $I(x)$ lie in the intervals. Each connected component in $\mathcal{D}(\alpha)$ is painted with green color.

where $H$ is the Heaviside function,

$$f_\alpha(x) \equiv \zeta(\alpha) - |\mathcal{P}(I(x); \alpha_A) - \mathcal{P}(I(x); \alpha_B)|,$$

and $\zeta(\alpha)$ will be determined in Section 4.1. In Figure 2, (a) shows two Gaussian PDFs and three intervals $[I_{\min}, I_0]$, $[I_1, I_2]$, and $[I_3, I_{\max}]$, where the difference of PDFs is less than $\zeta(\alpha)$.

Here $I_{\min}$ and $I_{\max}$ are the smallest intensity and the largest intensity in the image. Then the pixels whose intensities are in the intervals belong to the local ambiguous region. In (b), we depict the local ambiguous regions for Figure 1-(c) with green color.

Now, we propose a variational model which reduces misclassification in the local ambiguous region $\mathcal{D}(\alpha)$:

$$E(\Gamma, \alpha, \alpha^l) = -\sum_{i \in \{A, B\}} \left[ \int_{\mathcal{R}_i \setminus \mathcal{D}(\alpha)} \log \mathcal{P}(I(x); \alpha_i) dx \right. $$

$$+ \int_{\mathcal{R}_i \cap \mathcal{D}(\alpha)} \log \mathcal{P}(I(x); \alpha_i^l) dx \right],$$

where $\alpha=(\alpha_A, \alpha_B)$ and $\alpha^l=(\alpha_A^l, \alpha_B^l)$. For a simple explanation, we assume that $\mathcal{D}(\alpha)$ is a connected subset of $\Omega$. In practice, we consider different parameters $\alpha^l$ on each connected component of $\mathcal{D}(\alpha)$. The first term in (3.3) which we call the global region-based energy is the same as one in the model (2.1) except that the energy is contributed in the region $\mathcal{R}_i \setminus \mathcal{D}(\alpha)$ where a difference of two PDFs is large. In most cases, this term plays a dominant role to segment an outline of objects in an image. The second term which we call the local region-based energy is defined on the local ambiguous region $\mathcal{R}_i \cap \mathcal{D}(\alpha)$ where the difference is small. Whereas the global Gaussian parameter $\alpha_i$ makes a misclassification in $\mathcal{D}(\alpha)$, the local
Gaussian parameter $\alpha_i^{l}$ reduces the misclassification because $\alpha_i^{l}$ is obtained on not a global region $R_A$ or $R_B$, but a local region $R_A \cap D(\alpha)$ or $R_B \cap D(\alpha)$.

In order to find Euler-Lagrange equations for each variable, we write the proposed energy functional (3.3) using the function $f_\alpha$ in (3.2):

$$E(\Gamma, \alpha, \alpha^l) = - \sum_{i \in \{A, B\}} \int_{R_i} \left[ H(f_\alpha) \log \mathcal{P}(I(x); \alpha_i^l) ight. $$

$$+ \left. (1 - H(f_\alpha)) \log \mathcal{P}(I(x); \alpha_i) \right] dx.$$

First, we consider the minimization of the energy with respect to the boundary $\Gamma$. Fixing $\alpha$ and $\alpha^l$, the Euler-Lagrange equation for $\Gamma$ becomes

$$- \left[ H(f_\alpha)(\log \mathcal{P}(I(x); \alpha_A^l) - \log \mathcal{P}(I(x); \alpha_B^l)) ight. $$

$$+ \left. (1 - H(f_\alpha))(\log \mathcal{P}(I(x); \alpha_A) - \log \mathcal{P}(I(x); \alpha_B)) \right] \vec{n} = 0,$$

where $x \in \Gamma$. When the boundary $\Gamma$ and $\alpha$ are fixed, the Euler-Lagrange equation for $\alpha^l = (\alpha_A^l, \alpha_B^l) = (\mu_A^l, \sigma_A^l, \mu_B^l, \sigma_B^l)$ produces the Gaussian parameter for intensities in regions indicated by $H(f_\alpha)$:

$$\mu_i^l = \frac{\int_{R_i} H(f_\alpha) I(x) dx}{\int_{R_i} H(f_\alpha) dx} \quad \text{and} \quad (\sigma_i^l)^2 = \frac{\int_{R_i} H(f_\alpha)(I(x) - \mu_i^l)^2 dx}{\int_{R_i} H(f_\alpha) dx}$$

for $i \in \{A, B\}$.

Finally fixing the boundary $\Gamma$ and $\alpha^l$, we consider the minimization of the energy with respect to the parameter $\alpha = (\mu_A, \sigma_A, \mu_B, \sigma_B)$. The necessary condition $\frac{\partial E}{\partial \mu_k} = 0$, $k \in \{A, B\}$ gives the equations

$$\sum_{i \in \{A, B\}} \int_{R_i} H'(f_\alpha) \frac{\partial f_\alpha}{\partial \mu_k} (\log \mathcal{P}(I(x); \alpha_i) - \log \mathcal{P}(I(x); \alpha_i^l)) dx$$

$$- \int_{R_k} (1 - H(f_\alpha)) (I(x) - \mu_k)^2 \frac{1}{\sigma_k^2} dx = 0. \quad (3.4)$$

The condition $\frac{\partial E}{\partial \sigma_k} = 0$, $k \in \{A, B\}$ gives the equations

$$\sum_{i \in \{A, B\}} \int_{R_i} H'(f_\alpha) \frac{\partial f_\alpha}{\partial \sigma_k} (\log \mathcal{P}(I(x); \alpha_i) - \log \mathcal{P}(I(x); \alpha_i^l)) dx$$

$$- \int_{R_k} (1 - H(f_\alpha)) \left( -\frac{1}{\sigma_k^2} + \frac{(I(x) - \mu_k)^2}{\sigma_k^4} \right) dx = 0. \quad (3.5)$$
Unfortunately, solving the equations (3.4) and (3.5) is not possible and what is worse, the parameter $\alpha$ is not relevant to Gaussian parameters on $\mathcal{R}_A$ and $\mathcal{R}_B$. The main problem comes from the fact that the local ambiguous region $\mathcal{D}(\alpha)$ in (3.1) depends on the parameter $\alpha$.

### 3.2. Modification of Euler-Lagrange equations

In this section, we propose an algorithm to avoid the difficulty of Euler-Lagrange equations for $\alpha$ in (3.4) and (3.5). Since the difficulty comes from the dependence of the local ambiguous region on $\alpha$ in the proposed energy functional (3.3), to remove the dependence we regard $\mathcal{D}$ as a parameter in the functional

$$
E(\Gamma, \mathcal{D}, \alpha, \alpha^l) = - \sum_{i \in \{A,B\}} \left[ \int_{\mathcal{R}_i \setminus \mathcal{D}} \log \mathcal{P}(I(x); \alpha_i) dx + \int_{\mathcal{R}_i \cap \mathcal{D}} \log \mathcal{P}(I(x); \alpha_i^l) dx \right].
$$

(3.6)

However, we do not drop the dependence of $\mathcal{D}$ on $\alpha$; we use (3.6) to find Euler-Lagrange equations with respect to $\alpha$ and $\mathcal{D}$ is determined by (3.1). Then we propose the following procedure

a) Initialize a curve $\Gamma^n$, $n = 0$.

b) Find regions $\mathcal{R}_A$ and $\mathcal{R}_B$ separated by the curve $\Gamma^n$ and compute global Gaussian parameters $\alpha_A$ and $\alpha_B$ in each region:

$$
\mu_i = \frac{1}{|\mathcal{R}_i|} \int_{\mathcal{R}_i} I(x) dx \quad \text{and} \quad \sigma_i^2 = \frac{1}{|\mathcal{R}_i|} \int_{\mathcal{R}_i} (I(x) - \mu_i)^2 dx,
$$

for $i \in \{A, B\}$.

c) Determine the local ambiguous region $\mathcal{D}(\alpha)$ in (3.1).

d) Compute local Gaussian parameters $\alpha_A^l$ and $\alpha_B^l$:

$$
\mu_i^l = \frac{1}{|\mathcal{D}_i|} \int_{\mathcal{D}_i} I(x) dx \quad \text{and} \quad (\sigma_i^l)^2 = \frac{1}{|\mathcal{D}_i|} \int_{\mathcal{D}_i} (I(x) - \mu_i^l)^2 dx,
$$

where $\mathcal{D}_i = \mathcal{R}_i \cap \mathcal{D}(\alpha)$ for $i \in \{A, B\}$.

e) Obtain the curve $\Gamma^{n+1}$ by solving the equation

$$
- \left[ H(f_\alpha) \left( \log \mathcal{P}(I(x); \alpha_A^l) - \log \mathcal{P}(I(x); \alpha_B^l) \right) + (1 - H(f_\alpha)) \left( \log \mathcal{P}(I(x); \alpha_A) - \log \mathcal{P}(I(x); \alpha_B) \right) \right] \bar{n} = 0.
$$

(3.7)

f) If $\Gamma^{n+1}$ is not changed from $\Gamma^n$, we stop the procedure. Otherwise, set $n = n + 1$ and repeat.

Note that the local ambiguous region $\mathcal{D}(\alpha)$ usually has many disjoint connected components, i.e., $\mathcal{D}(\alpha) = \bigcup_{k=1}^N \mathcal{D}^k$, $\mathcal{D}^i \cap \mathcal{D}^j = \emptyset$, $i \neq j$; see Figure 2.

Strictly speaking, the algorithm solves neither the Euler-Lagrange equations in Section 3.1 nor the variational model (2.1) on three different regions $\mathcal{R}_A$, $\mathcal{R}_B$ and $\mathcal{D}$. However, it keeps the main purpose of the proposed energy functional (3.3), which is to find the local ambiguous regions depending on the global Gaussian parameter $\alpha$ and to use the local Gaussian parameter $\alpha$. 
to reduce misclassification. In the proposed algorithm, the parameter $\alpha$ represents statistical information on regions separated by the evolving curve $\Gamma$ and the difficulty of Euler-Lagrange equations for $\alpha$ is avoided.

### 3.3. Extension to multi-dimensional Gaussian PDFs.

Our algorithm can be easily extended to multi-dimensional Gaussian PDFs which are used in a color image. The multi-dimensional Gaussian PDF is given by

$$
\mathcal{P}(I(x)) = \frac{1}{2\pi^{d}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(I(x) - \mu)^T \Sigma^{-1} (I(x) - \mu)\right),
$$

where $d$ is the number of channels and $\mu$ is the mean vector and $\Sigma$ is the covariance matrix. Basically dealing with the RGB color model, we compute three dimensional probability density functions. If the channels are linearly dependent it is not possible to compute the PDF because $|\Sigma| = 0$ and hence $\Sigma$ is not invertible in (3.8). To overcome the difficulty, we examine the correlation between channels to select independent channels. The correlation between two channels $X, Y$ is given by

$$
\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y},
$$

where $\text{Cov}(X, Y)$ is the covariance of $X, Y$ and $\sigma_X, \sigma_Y$ are the standard deviations. If $|\rho_{XY}|$ is close to 1, $X$ and $Y$ are likely to be linearly dependent and if it is close to 0, they are almost linearly independent. Therefore if the correlation is larger than a criterion, then we take the average of two channels as one channel. Likewise we make a gray image if three channels are almost linearly dependent. Then two Gaussian PDFs $\mathcal{P}(I(x); \alpha_A)$ and $\mathcal{P}(I(x); \alpha_B)$ use the same channels.

### 4. Numerical aspects and examples

In this section, we raise numerical issues to solve (3.7) and show how to take a threshold $\zeta(\alpha)$ in (3.2).

#### 4.1. Local ambiguous region.

In order to determine the local ambiguous region $\mathcal{D}(\alpha)$ in (3.1), we need to determine a criterion $\zeta(\alpha)$ in (3.2). The smaller the value $\zeta(\alpha)$ is, the smaller the region $\mathcal{D}(\alpha)$ becomes and the more dominant a global parameter $\alpha$ is in finding a boundary of an object in the proposed algorithm. Hence, the selection of $\zeta(\alpha)$ is crucial in our algorithm for segmentation.

Given a Gaussian PDF $\mathcal{P}(x)$ with a mean $\mu$ and $\eta(0 \leq \eta \leq 1)$, we first find $\beta$ such that

$$
\int_{\mu-\beta}^{\mu+\beta} \mathcal{P}(x)dx = \eta.
$$

This means that the confidence interval of $100 \times \eta\%$ is $[\mu - \beta, \mu + \beta]$. Taking $\mathcal{P}(\mu + \beta)$ as a criterion, if $\mathcal{P}(x) < \mathcal{P}(\mu + \beta)$ at $x \in \Omega$, then we may regard the point $x$ as an uncertain
point for classification. Based on this idea, we compute $\beta_A$ and $\beta_B$ for two Gaussian PDFs $\mathcal{P}(I(x); \alpha_A)$ and $\mathcal{P}(I(x); \alpha_B)$, respectively, in the proposed algorithm. Then we take

$$\zeta(\alpha_A, \alpha_B) = \min\{\mathcal{P}(\mu_A + \beta_A; \alpha_A), \mathcal{P}(\mu_B + \beta_B; \alpha_B)\}.$$  

We used $\eta = 0.7$ for all examples.

4.2. Evolution of the boundary. With the gradient descent method and the level set method, (3.7) is formulated as

$$\frac{\partial \phi(x, t)}{\partial t} = \left[ H(f_\alpha)(\log \mathcal{P}(I(x); \alpha_A) - \log \mathcal{P}(I(x); \alpha_B)) \right. + \left. (1 - H(f_\alpha))(\log \mathcal{P}(I(x); \alpha_A) - \log \mathcal{P}(I(x); \alpha_B)) \right] |\nabla \phi(x, t)|,$$

where the zero level set of $\phi(x, t)$ is the evolving curve which separates two regions. Note that $\phi$ is a signed distance function with being positive inside the curve. The equation (4.1) may be solved numerically by the nonoscillatory scheme for space and the explicit Euler scheme for time [17] with $\phi^n$ as an initial curve. The numerical scheme, however, cause a severe restriction on the stability condition because the force is not bounded. Instead we find directly a steady state solution for the equation when the Gaussian parameters $\alpha$ and $\alpha'$ are given. Since the steady state solution satisfies

$$F|\nabla \phi(x, t)| = 0$$

with $F$,

$$F = H(f_\alpha)(\log \mathcal{P}(I(x); \alpha_A') - \log \mathcal{P}(I(x); \alpha_B')) + (1 - H(f_\alpha))(\log \mathcal{P}(I(x); \alpha_A) - \log \mathcal{P}(I(x); \alpha_B))$$

we simply take a step function

$$\phi(x) = \begin{cases} 1 & \text{if } F(x) > 0, \\ -1 & \text{if } F(x) < 0, \\ 0 & \text{if } F(x) = 0, \end{cases}$$

for $x \in \Omega$

and take the reinitialization process [18]. Then it becomes the solution $\phi^{n+1}$.

4.3. Examples. In this section, we present several examples to illustrate our model and algorithm. We have two synthetic images on the top in Figure 3. These images are expected to have two modes, the object and the background. Note that two images have various intensities around boundaries of objects by illumination. We select the initial curve manually for each image as shown at the bottom in Figure 3. Figure 4 shows several iterates for two synthetic images when we apply our algorithm. For each image, local ambiguous regions are depicted by green color. In Figure 5 we compare our method with the region competition based on (2.1) and the Chan-Vese model in [11], which are most popular among the region-based segmentations. Figure 5-(a) is the results of two synthetic images when the region competition is applied, and (b) and (c) account for our model and the Chan-Vese model, respectively. Both (a) and (c) give
wrong segmentation by passing though boundaries where illumination changes and these misclassification is inevitable, while our model gives exact boundary by reducing misclassification in ambiguous region as shown in Figure 4.

We apply our method to the real image taken in a studio. Figure 6-(a) is the original image where the object has various intensities inside the object as well as near its boundary, in particular, the top. The top is illuminated and forms weak edge. Notice that both sides of the object have highly concave parts. Thus the main problem for this image is to detect the weak edge and highly concave boundary simultaneously. With the same initial curve in Figure 6-(b), three models are applied; (c) is for the region competition, (d) and (e) are for our model and the Chan-Vese model, respectively. Three models detect correctly highly concave boundaries on both sides of the object. However, the region competition and Chan-Vese model fail to detect the weak edge on the top where the curves pass through the boundary of the object. On the other hand our model detects the weak edge correctly.

In Figure 7 there are three real images. Figure 7-(a) is a gray blurred noisy image, (b) is a shaded color image, and (c) is taken from the seashore with a digital camera. For three images we apply our method. In (d) numbers in the blurred noisy image are segmented without
FIGURE 4. Several iterates are shown for two synthetic images when our method is applied. From the top to the bottom, $n = 1, 2, 3, 4$. For each image, local ambiguous regions are depicted by green color.
corruption of noise. Alphabet letters are separated from the background in (e). (f) shows segmentation between a boy and the sandy plain. The shadow near the boy is excluded.

5. CONCLUSIONS

In this paper we proposed a variational segmentation model based on statistical information of intensities in an image. Since the global region-based energy model which uses the statistical information in whole region causes misclassification, in order to reduce misclassifications, we defined the local ambiguous region, where the difference of two Gaussian PDFs is small and considered an energy using the local Gaussian parameter restricted to the local region. Then we combined the local region-based energy with the global region-based energy. This model, however, induces the difficulty in handling Euler-Lagrange equations. We thus proposed an algorithm to avoid the difficulty. It consists of several steps to solve Euler-Lagrange equations; the local ambiguous region as a parameter of the energy is determined explicitly after the global Gaussian parameters are computed. With several examples we confirmed that our model and the algorithm reduced misclassification and hence detected the boundary of the object when an illumination is changed.
(a) (b) (c) (d) (e)

**Figure 6.** (a) is an image taken from a studio. It has weak boundary on the top as well as highly concave shape on the sides of the object. An initial curve (red line) is shown in (b). With this initial curve, three segmentation models are applied. Different results are shown at the bottom. (c) is obtained from the region competition, (d) is from our model, and (e) is from the Chan-Vese model. Our method detects both weak edge and highly concave shape.

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FIGURE 7. There are three real images on the top. (a) is a gray noisy image which is blurred. (b) is a shaded color image and (c) is taken from the seashore with a digital camera. Images at the bottom are the results of our algorithm.


A REVIEW ON DENOISING

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ABSTRACT. This paper aims to give a quick view on denoising without comprehensive details. Denoising can be understood as removing unwanted parts in signals and images. Noise incorporates intrinsic random fluctuations in the data. Since noise is ubiquitous, denoising methods and models are diverse. Starting from what noise means, we briefly discuss a denoising model as maximum a posteriori estimation and relate it with a variational form or energy model. After that we present a few major branches in image and signal processing; filtering, shrinkage or thresholding, regularization and data adapted methods, although it may not be a general way of classifying denoising methods.

1. INTRODUCTION

In a broad sense, denoising is removing noise which may be unwanted parts in signals, images, measurements, data, and so on. Imaging or data acquisition devices, including diverse modalities from daily-life devices such as cellular phones and digital cameras to medical imaging devices such as CT and MRI output noisy measurements of incoming signals. Noise incorporates intrinsic random fluctuations in the data. Fundamentally, noise is ubiquitous. Usually an elementary denoising process is already installed in the device itself and further operations can be proceeded as post-process if the quality of the acquisition or reconstruction is not satisfactory.

Through various disciplines, tremendous techniques are developed and under developing; typing denoising related keywords in Google will show numerous articles in this topic. Some methods are very problem specific, since they exploit a given specific situation or incorporate physical and engineering circumstances. A general technique or principle such as total variation [20] and soft thresholding [9] still can improve the quality of results. However, finding a universal method which outperforms in overall cases is hopeless; there is no panacea at all, since the scope of denoising is too wide and diverse. Even if one narrows the focus, only on
specific sounds or images for example, they contain various situations as if zooming in fractals shows recurrent intricacies.

Although a unified theory might be a daydream of mathematician, developing common mathematical foundations and frameworks is still worthwhile to pursuit. Such efforts lead to speculating proper spaces for signals and images such as BV space and Besov space, or finding a best or sparse representation using wavelets and some orthonormal bases or a dictionary by learning.

This paper gives a short review on denoising and by no means do we intend to give a comprehensive review on it. Denoising belongs to applied science and engineering and the primal matter of concern in denoising is practicability. Theory should come in the second. Deng Xiaoping said that “It doesn’t matter whether a cat is white or black, as long as it catches mice.” The same tenet may go to denoising. Thus, classifying existing methods in a few categories might not be proper. We illustrate a few tracks of them in this paper and some important methods could be missing or misclassified by author’s carelessness or ignorance.

Here is the organization of the paper; we start from what noise means in Section 2. In Section 3, we briefly discuss a denoising model as maximum a posteriori estimation and relate it with a variational form or energy model. Also we discuss some probabilistic foundations and measures of denoising quality are given. After that we present a few major branches in image and signal denoising; filtering in Section 4, shrinkage or thresholding in Section 5, regularization in Section 6 and data adapted techniques in Section 7, although it may not be a general way of classifying denoising methods. Since those methods are usually devised for images and signals, we discuss those methods assuming the given data is an image or signal. But their applications are not limited to those fields and there are also many mutations for other fields. We close this review by giving a brief remark in Section 8.

2. WHAT IS NOISE?

Often a denoising paper starts in this fashion: “From the observation \( u_o \), we want to restore \( u \) under the assumption \( u_o = u + n \) where \( n \) is a Gaussian white noise.” Since it is already a well-established problem, possibly no more insight or explanation is necessary to justify the importance and soundness of the problem. Thus, researchers often add random numbers generated by a computer to whatever they already have, signals and images, for examples, and just enjoy games how the computed \( \tilde{u} \) is close to \( u \) which they originally have. The term ‘Gaussian’ has mathematical definition; it is the most popular distribution in the probability theory. Then how about ‘noise’ or ‘white’?

From this standpoint, we’d like to discuss noise first, before we discuss de-noising; before we want to remove ‘something’, we should understand what ‘something’ is. Merriam-Webster dictionary defines it as ‘a loud or unpleasant sound’ and ‘unwanted electronic signals that harm the quality of something’. The first definition is subjective or psychological but the second one looks objective or scientific. Wikipedia does similarly; it starts with the sentence, “Noise means any unwanted sound.” to define noise. However, ‘Noise (electronics)’ starts with that “In
electronics, noise is a random fluctuation in an electrical signal, a characteristic of all electronic circuits.” Now, what is noise? Can we define it mathematically?

Consider the following two scenarios: 1. You talk to a friend on a street and a bus passes by. 2. You wait for a bus in a bus station. Is the sound of bus noise? You may consider an opposite situation. Strangers talk next to you but you wait for a bus. Is their talk noise? Basically, one may split the heard into two part; meaningful + meaningless, wanted + unwanted, interesting + uninteresting, and important + unimportant. Noise is just the general name for the second category. Now how one can define the second category mathematically? It may be uninteresting because it is unstructured or unpredictable, i.e., random. In other words, we can define it mathematically as a random process. Then the first category should be structured or ordered; it should be regular mathematically. In short, ‘signal + noise’.

We close this section by defining a noise model formally. From the original signal \( u \), we observe noisy version \( u_0 \) of \( u \):

\[
u_0 = u + n,
\]

where \( n \) is a random process and \( u \) belongs to a certain type of regular signals. It is called the additive noise model. As you can guess, one another possible model is multiplicative such as \( u_0 = u \cdot n \). By taking the logarithm, one may derive an additive formulation. Hence we consider the former exclusively.

3. WHAT IS DENOISING?

Removing noise from signals or images is called denoising. More specifically, the goal is to reconstruct the original signal \( u \) from noisy observation \( u_0 \) [6]. If \( u \) is an 1-dimensional function, it includes signals such as audio, if 2-dimensional, it includes images, and if 3-dimensional, it includes video (It could be even higher dimensional. what would it be?). It involves designing an operator \( \mathcal{D} \) called a denoising processor: \( \tilde{u} = \mathcal{D}u_0 \). Ideally, if \( n \) is given or known, then \( u \) can be recovered by the subtraction. so, \( \tilde{u} = u_0 - n = u \). However, \( n \) is random. Thus, to remove noise, we require some prior information.

How much do we know about \( n \)? At least its distribution should be known. Usually homogeneous white noise is assumed. White noise means the randomness with a constant power spectral density. In other words, it has the same density along frequencies after Fourier transform. It is white as a light, otherwise it is color noise. For a discrete signal or image, it is used for independent random variables with zero mean and finite variance. Homogeneity implies same means and variances everywhere. Distribution doesn’t change spatially, for example.

Gaussian white noise is often assumed; noise follows a normal distribution with zero mean. It is the standard or universal noise model. If the noise property is not known or you have no idea, just assume it. However, often quantities interested are nonnegative, sounds and photos, for examples. Photography records light by means of sensing photons, which are positive. Noise should also consist of photons, so its mean cannot be zero except the case of no noise, and usually a different distribution is assumed for photons. But under the sunlight, huge amounts of photons are captured by each sensor of digital camera. Thus, the value returned by each sensor can be regarded as a number drawn from a normal distribution by the following theorem:
Theorem 3.1 (Central Limit Theorem (CLT)). Let $X_1, X_2 \ldots$ be iid random variables with $E X_i = \mu$ and $\text{var}(X_i) = \sigma^2 < \infty$. Let $S_k = X_1 + \cdots + X_k$. Then $\frac{S_k - k\mu}{\sigma \sqrt{k}}$ converges weakly to a standard normal distribution.

When the light source is about constant, the number of photons received by each sensor fluctuates around its average by CLT. One may set the mean of noise zero if it is small enough, or by mean-shifting if considerable. Hence Gaussian white noise can be understood as an ideal noise.

We consider a perfect denoising shortly; how can we guarantee $\hat{u} = D u_o = u$? Assume we can generate multiples of data as many as possible:

$$u_o^{(1)} = u + n^{(1)}, u_o^{(2)} = u + n^{(2)}, \ldots,$$

then the sample mean $1/k \sum_{i=1}^{k} u_o^{(i)}$ will recover the true $u$ in the limit sense by the following theorem:

Theorem 3.2 (Law of Large Numbers (LLN)). Let $X_1, X_2 \ldots$ be iid random variables with $E|X_i| < \infty$. Let $S_k = X_1 + \cdots + X_k$ and $E X_i = \mu$. Then,

$$S_k/k \to \mu \text{ a.s. as } k \to \infty.$$

Thus, assuming white noise or mean-zero noise, if we sufficiently collect data, i.e., $k$ is large enough,

$$\hat{u} = \frac{1}{k} \sum_{i=1}^{k} u_o^{(i)} \approx u. \quad (3.1)$$

Often in experimental situations, researchers measure multiple instances to average out errors. Humble human beings also use the same tactic through experiences in everyday life, and such strategies can be justified by Theorem 3.2 (LLN) under proper assumptions. In reality, the limitation is that the number of samples is always limited. For example, taking multiple shots of the exact same scene is fairly circumscribed, due to moving objects such as humans and cars, sudden illumination such as reflection by windows, variations by wind, etc. In some situations, obtaining multiple copies is very costly or impossible. But we note that LLN tells us the fundamental principle for denoising: taking average. We also note that mean is the Maximum Likelihood Estimator (MLE) of the normal distribution and it is closely related to least squares, which approximates the optimal solution of overdetermined systems with the most important application, data fitting.

We introduce a general denoising model by invoking statistical modeling; since noise itself is viewed as random phenomena, probabilistic/statistic interpretation and understanding are inevitable. The denoising processor $D$ can be modeled as Maximum A Posteriori (MAP) estimation [7, 6]. By Bayes' formula, the posterior probability given observation $u_o$ is

$$p(u|u_o) = \frac{p(u_o|u)p(u)}{p(u_o)} \quad (3.2)$$

The prior model specifies how images are distributed a priori, or equivalently, which images occur more frequently than others. Probabilistically, it specifies the prior probability $p(u)$. Note
that $u_o$ denotes the noised data that are observed or measured. The data model is to model how $u_o$ is generated from $u$, or to specify the conditional probability $p(u_o|u)$. Now, the denoising processor $D$ is achieved by solving the MAP problem $\max_u p(u|u_o)$, which is equivalent to maximizing the product of the prior model and the data model, since the denominator is a fixed normalization constant once $u_0$ is given. In words, we seeks 'what is the most plausible $u$ under the given observation $u_o$?'.

An variational form or energy model can be driven from MAP (3.2). Under the notice that probability distributions are often expressed by the exponential functions, by taking the logarithm on the right hand side of (3.2), we have

$$\log p(u_o|u) + \log p(u) - \log p(u_o).$$

Since $u_o$ is already observed, the last term may be dropped. By changing sign and replacing notations properly, we have the following variational form [7]:

$$\min_u E[u] + \frac{\lambda}{2} E[u_0|u],$$

where $\lambda/2$ is the Lagrange multiplier. The first term is called the image prior or the regularity term and the second term is called the data-fitting term or data-fidelity term. The Lagrange multiplier $\lambda$ expresses the balance between prior and fitting. Due to the Lagrange multiplier, the variational form is closely related to the following constrained optimization problems:

$$\begin{cases}
\min_u E[u] \\
\text{subject to } E[u_0|u] \leq C_1
\end{cases} \quad \text{and} \quad \begin{cases}
\min_u E[u_0|u] \\
\text{subject to } E[u] \leq C_2
\end{cases}. \quad (3.4)$$

Designing a proper image prior and a data-fitting is called the image modeling and deriving a suitable image model on the given situation is very crucial in image and signal processing.

We close this section by introducing methodologies of measuring the signal quality. Signal-to-noise ratio (SNR) is often adopted for such a criterion, which compares the level of a desired signal to the level of background noise [23]. Although the perception of human beings on the quality may be a little different from SNR, it provides a value-neutral quantity. More specifically, SNR is defined as the ratio between the variance of a signal and that of noise:

$$\text{SNR} = \frac{\sigma_u^2}{\sigma_n^2} = \frac{E[|u|^2]}{E[|u_0 - u|^2]},$$

where $\sigma_u$ and $\sigma_n$ represent the standard deviations of the signal and noise, respectively. If $u$ is replaced by $D u = \tilde{u}$, it can be treated as the numerical value of the risk [16]. SNR is often expressed using the logarithmic scale $\text{SNR}_{\text{dB}} = 10 \log_{10} \text{SNR}$, measured in decibels. One another available alternative is peak signal-to-noise ratio (PSNR). It measures the ratio between the maximum possible power of the signal and the power of noise. Its technical definition can be easily found in the literature, even in Wikipedia [22]. Lastly, method noise is recently introduced by Buades et al [3] to check which geometrical features or details are preserved and which are eliminated after image denoising. It is defined as the difference between an image $u$ and its denoised version $D u = \tilde{u}$:

$$n(D, u) = u - D u.$$
4. Filtering

In general, a filter is a device or process that removes some unwanted components or features from a signal [21]. This category includes the most traditional but the most widely used methods since the structure is simple, so a fast or real-time implementation is possible, although it may be inaccurate compared to other sophisticated methods.

It is reasonable to assume the local homogeneity in a neighborhood; if we assume some regularity as a function, values should be similar locally. For denoising purpose, one may take local average as a substitute or approximate for the denoising scheme (3.1) by LLN. For example, one may apply the following averaging filter or lowpass filter [13]:

$$\tilde{u}(i, j) = \frac{1}{|N|} \sum_{(i', j') \in N} u_o(i', j'),$$

where $N$ is a neighborhood of $(i, j)$. If $N = \{(i + k, j + m) \mid k, m = 0 \text{ or } \pm 1\}$ is chosen for the neighborhood, 9 point average is taken. More generally, a weighted average can be taken:

$$\tilde{u}(i, j) = \sum_{(i', j') \in N} w(i', j') u_o(i', j'),$$

where $\sum_{(i', j') \in N} \omega(i', j') = 1$. Such time or space-invariant filter can be rewritten as a convolution:

$$\tilde{u}(x, y) = \sum_{s = -a}^{a} \sum_{t = -b}^{b} w(s, t) u_o(x - s, y - t) := \omega * u_o. \quad (4.1)$$

If the weight $w$ is chosen from a normal distribution according to the distance from the origin, it can be regarded as the solution to the heat equation which performs averaging infinitesimally. If the explicit forward time method with the five-point stencil discrete Laplacian is applied to the standard heat equation, we have

$$\frac{u_{t+\triangle t}^{i,j} - u_{t}^{i,j}}{\triangle t} = u_{(i+1,j)}^{t} + u_{(i-1,j)}^{t} + u_{(i,j+1)}^{t} + u_{(i,j-1)}^{t} - 4u_{(i,j)}^{t}.\quad (4.2)$$

By a rearrangement, we derive the following iteration of the weighted mean filter:

$$u_{t+\triangle t}^{i,j} = (1 - 4\alpha)u_{t}^{i,j} + \alpha \left[ u_{(i+1,j)}^{t} + u_{(i-1,j)}^{t} + u_{(i,j+1)}^{t} + u_{(i,j-1)}^{t} \right],$$

with $\alpha = \triangle t/h^2 < 1/4$. Actually one may glimpse the relation between random walk and diffusion as a continuum limit.

If the assumption on local homogeneity is broken, it will average out inhomogeneity. Jumps or edges in an image are crashed and the image becomes blurry. Note that the heat equation is also an isotropic diffusion. That is one clear drawback of the averaging filter. To fix it, many other filters are introduced using other statistics such as median filter.

If Fourier transform is taken on the convolution in (4.1),

$$\tilde{u}(\omega) = \hat{w}(\omega) \hat{u}_o(\omega).$$
Thus, it can be viewed as the modulation by \( \hat{w} \) along the frequency spectrum of \( u_o \), attenuating high frequency bands for example. Thus, one may also modulate the magnitude along the spectrum by attenuating or strengthening, which is called frequency filtering. The former is called spatial filtering.

If one considers discrete Fourier transform (DFT) on the one dimensional signal \( u_o[m] \), \( m = -M, \cdots, M \),

\[
u_o[m] = \frac{1}{2M + 1} \sum_{k=-M}^{+M} \hat{u}_o[k] \exp \left( \frac{i2\pi km}{2M + 1} \right), \quad \text{where} \quad \hat{u}_o[k] = \sum_{m=-M}^{+M} u_o[m] \exp \left( -\frac{i2\pi km}{2M + 1} \right). \quad (4.3)
\]

One may truncate the high frequency terms treating high oscillations as noise:

\[
\hat{u}[m] = \frac{1}{2M + 1} \sum_{k=-\tilde{M}}^{+\tilde{M}} \hat{u}_o[k] \exp \left( \frac{i2\pi km}{2M + 1} \right),
\quad (4.4)
\]

for some \( 0 \leq \tilde{M} < M \). It is a crude denoising in the frequency domain and this truncated DFT is also related to the solution of the heat equation by separation of variables which leads to Fourier series expansion, since the high frequency terms exponentially decay.

This type of truncation is also related to the lossy compression. Assuming the evenness of the image, DFT is reduced to discrete cosine transform (DCT), and a proper selection and thresholding of the DCT coefficients is the main idea of the popular image compression technique jpeg. Such frequency filtering is crucial in many places, such as wireless communication and imaging areas including MRI and CT images.

5. SHRINKAGE OR THRESHOLDING

To reach the idea of wavelet shrinkage by Donoho and Johnstone [10], we follow the logic in [16]. For such purpose, we start from the Bayesian decision theory, which is originated from Bayes’ formula (3.2). The risk of the denoiser \( D \) of \( u_o \) is the average loss with respect to the probability distribution of the noise \( n \):

\[
r(D, u) = \mathbb{E}\{\|u - Du_o\|^2\} \quad (5.1)
\]

with \( u_o[m] = u[m] + n[m] \), \( m = 0, \cdots, M - 1 \), similar to (2.1). If one assumes the prior probability distribution \( \pi \), we can define Bayes risk, which is the expected risk with respect to \( \pi \):

\[
r(D, \pi) = \mathbb{E}_\pi \{r(D, u)\}. \quad (5.2)
\]

Definitely we want to minimize Bayes risk, which yields minimum Bayes risk:

\[
r(\pi) = \inf_{D \in \mathcal{O}} r(D, \pi),
\]

where \( \mathcal{O} \) is the set of all operators. If we restrict such \( D \) on the set \( \mathcal{O}_l \) of all linear operators, the optimal operator is called the Wiener filter and it can be derived by using covariance matrices:
**Theorem 5.1** (Wiener filter). *If the signal \( u \) and noise \( n \) are independent with the covariance matrices \( R_u \) and \( R_n \), respectively, then the Wiener filter that minimizes \( E\{\|\tilde{u} - u_o\|^2\} \) is*

\[
\tilde{u} = R_u (R_u + R_n)^{-1} u_o.
\]  

(5.3)

Notice that if \( R_u \) and \( R_n \) are uncorrelated, and so are diagonal with the variances \( \sigma_u^2[m] \) and \( \sigma_n^2[m] \), the relation (5.3) is the following ratio:

\[
\tilde{u}[m] = \frac{\sigma_u^2[m]}{\sigma_u^2[m] + \sigma_n^2[m]} u_o[m].
\]  

(5.4)

We can clearly interpret it; if the variance of the signal is relatively larger than that of the noise, we trust the observed data \( u_o[m] \) and don’t shrink much. On the other hand, if that of the signal is relatively smaller, it is very possible to be noise so we shrink.

Since the covariance operator is symmetric and positive semi-definite, it can be diagonalized using eigenvectors with decreasing order of eigenvalues, called the *Karhunen-Loève basis* or *Principal Component Analysis (PCA)*. If the covariance operators \( R_u \) and \( R_n \) are diagonalizable under the same Karhunen-Loève basis, the equation (5.4) is accomplished. We also remark that the standard heat operator is also symmetric and positive semi-definite, it can be diagonalizable by sinusoidal waves. Its discrete version is given in (4.3), and then one may speculate the resemblance between (4.4) and (5.4).

It is generally not possible to compute the optimal Bayes estimator. To avoid such complexity, classical strategies choose a linear operator, although the minimum risk among linear estimators may be far beyond the minimum risk from all estimators. Thus we consider a particular class of nonlinear estimators that are diagonal in a basis \( B \).

In the basis \( B = \{v_m\}_{0 \leq m < M} \), \( u_o \) has the following basis expansion:

\[
u_o = \sum_{m=0}^{M-1} u_o^B[m]v_m \text{ where } u_o^B[m] = \langle u_o, v_m \rangle.
\]

A *diagonal operator* estimates each \( u^B[m] \) by multiplying \( u_o^B[m] \) by a factor \( a_m(u_o^B[m]) \) independently:

\[
\tilde{u} = Du_o = \sum_{m=0}^{M-1} u_o^B[m]a_m(u_o^B[m])v_m.
\]  

(5.5)

For such \( a_m(\cdot) \), one may choose the *hard thresholding HT* \( \lambda \) with the parameter \( \lambda \):

\[
HT_\lambda(x) = \begin{cases} 
  x & \text{if } x \geq \lambda, \\
  0 & \text{if } |x| < \lambda, \\
  x & \text{if } x \leq -\lambda.
\end{cases}
\]  

(5.6)

One may compare it with the truncation in (4.4) which is linear with fixed \( \tilde{M} \) and assume that low order or smooth terms are signal and high order or highly oscillatory terms are noise. The idea of hard thresholding is that if the amplitude \( u_o^B[m] \) is large enough, it is very possible to
be a signal and we keep. Otherwise, it is likely to be noise, we throw away. But if the value is often just little above or below the thresholding level \( \lambda \), is the decision by \( HT_\lambda \) proper? Furthermore, it is discontinuous.

With such concerns, one may consider the following alternative, which is called the soft thresholding \( ST_\lambda \) with the parameter \( \lambda \) [9]:

\[
ST_\lambda(x) = \begin{cases} 
  x - \lambda & \text{if } x \geq \lambda, \\
  0 & \text{if } |x| \leq \lambda, \\
  x + \lambda & \text{if } x \leq -\lambda.
\end{cases}
\]

The operator is now continuous and still shrink even if the signal strength is more than thresholding level \( \lambda \).

Now one big question is arising: How does one choose the thresholding level \( \lambda \)? What is optimal? Donoho and Johnstone [10] showed the fundamental result that the risk of thresholding estimators is close to that of the oracle projector, which is the diagonal estimator by some strong prior knowledge. For more details, we refer to [16].

**Theorem 5.2** (Donoho, Johnstone). Let \( \lambda = \sigma \sqrt{2 \ln M} \). The risk \( r_{th}(u) \) of a hard- or soft-thresholding estimator satisfies for all \( M \geq 4 \),

\[
r_{th}(u) \leq (2 \ln M + 1)(\sigma^2 + r_{pr}(u)).
\]

The factor \( 2 \ln M \) is optimal among diagonal estimators in \( \mathcal{B} \):

\[
\lim_{M \to \infty} \inf_{D \in \mathcal{O}_d} \sup_{u \in \mathcal{C}_M} \frac{E\{\|u - Du_0\|^2\}}{\sigma^2 + r_{pr}(u)} \frac{1}{2 \ln M} = 1,
\]

where \( \mathcal{O}_d \) is the set of all diagonal operator and \( r_{pr} \) is the risk of the oracle projector.

### 6. Regularization

We recall the variational form (3.3):

\[
\min_u E[u] + \frac{\lambda}{2} E[u_0|u].
\]

Under this setting, we have to choose a regularity term \( E[u] \) and a fitting term \( E[u_0|u] \). If we assume the Gaussian white noise, naturally the least square term \( \|u - u_0\|^2_2 \) comes out for the fitting term \( E[u_0|u] \). One popular choice of the regularity term \( E[u] \) is also adding the \( L^2 \) criteria \( \|u\|^2_2 \), then it is called Tikhonov regularization, which is widely used in inverse problems. One may choose the \( L^1 \) norm \( \|u\|_1 \) which enforces sparsity of \( u \), which will be discussed in the next section. Considering the regularity or differentiability of \( u \), one may choose the \( L^2 \) norm or the \( L^1 \) norm of the gradient \( \nabla u \). If the \( L^2 \) norm of the gradient of \( u \) is chosen, we consider Sobolev space as a proper function space for the original \( u \). From the energy

\[
\min_u \|\nabla u\|^2_2 + \frac{\lambda}{2} \|u - u_0\|^2_2,
\]
one may derive the following elliptic equation as Euler-Lagrange equation:

$$-\Delta u + \lambda (u - u_0) = 0.$$  

Since the Laplacian $\Delta u$ is isotropic, blur in the solution $u$ is expected by crushing edges and jumps, similar to the heat equation case (4.2).

To overcome such drawbacks, Rudin, Osher, and Fatemi [20] introduce the following energy, called total variation (TV) denoising or Rudin-Osher-Fatemi model, assuming $u$ belong to the space of bounded variations (BV):

$$\min_u \|\nabla u\|_1 + \frac{\lambda}{2} \|u - u_0\|_2^2.$$  

Its Euler-Lagrange equation is

$$-\nabla \cdot \left[ \frac{\nabla u}{|\nabla u|} \right] + \lambda (u - u_0) = 0. \quad (6.1)$$

In (6.1), the term $1/|\nabla u|$ can be understood as a diffusion coefficient or conductivity of heat; if the region is smooth or homogeneous, $|\nabla u|$ is small. Thus the diffusion coefficient $1/|\nabla u|$ is large, local fluctuations are averaged out by diffusion. Along edges and jumps, $|\nabla u|$ is very large and so $1/|\nabla u|$ is small, hence there is a little diffusion due to the low conductivity. Thus edges and jumps are kept. Actually, BV space allows discontinuity. BV space may be considered as the space of piecewise differentiable functions. Meanwhile, Sobolev space doesn’t allow such discontinuity.

TV denoising model and the anisotropic diffusion by Perona and Malik [19] triggered the researches in image processing by mathematicians and image processing problems including denoising are popularized in applied mathematics, especially in the areas of partial differential equations and numerical analysis.

After introducing a discretization, solving a PDE such as (6.1) numerically becomes the iteration of a discrete filter, similar to (4.2). Then, how is it different from designing a discrete filter directly? Basically, PDE theory and numerical analysis answer the stability issues and the behavior of the limit by iterations. Even when we apply a discrete filter, if denoising is not satisfactory, we may try to apply the filter more than one time. To forecast what happens eventually, one may investigate PDE-related to the filter and invoke the theories for stabilities and limits.

Usually the solution of a PDE satisfies some types of smoothness. However, so-called texture such as patterns in furs, rocks and soils is not smooth at all. Due to the regularity assumptions, details and fine structures behave as noise in function aspects. Thus, the denoising scheme induced by PDE washes out them. Human beings also recognize them as meaningful or geometric objects. It is a typical defect in the variational PDE methods.

Starting from Rudin-Osher-Fatemi model, Yves Meyer considers the ‘texture + noise’ component as an ‘oscillating pattern’ which is defined by Besov norm estimates and develops so called Meyer $G$ norm [17]. In other words, if one wants to analyze ‘texture’ mathematically, various Besov-type function spaces may be necessary. Combining TV model and oscillatory
functions by Meyer, Osher, Sole and Vese [18] developed a model for image restoration and image decomposition into cartoon and texture by using the negative Sobolev space \( H^{-1} \).

7. Data Adapted Methods

Buades, Coll and Morel introduced the following nonlocal means algorithm [3]. Shortly speaking, it estimates the value of \( x \) as an average of the values of all the pixels whose Gaussian neighborhood looks like the neighborhood of \( x \):

\[
    u(x) = NL(u_o)(x) = \frac{1}{C(x)} \int_{\Omega} \exp \left( -\frac{(G_a * |u_o(x.+)-u_o(y.+)|^2)(0)}{h^2} \right) u_o(y) dy,
\]

(7.1)

where \( G_a \) is the Gaussian kernel with the standard deviation \( a \), \( h \) acts as a filtering parameter, and \( C(x) = \int_{\Omega} \exp \left( -\frac{(G_a * |u_o(x.+)-u_o(z.+)|^2)(0)}{h^2} \right) dz \) is the normalizing factor. Lastly,

\[
    G_a * |u_o(x.+)-u_o(y.+)|^2(0) = \int_{\mathbb{R}^2} G_a |u_o(x+t)-u_o(y+t)|^2 dt,
\]

which measures the distance between the neighborhoods of \( x \) and \( y \) under Gaussian weights.

They noticed that every small window in a natural image has many similar windows in the same image. In that sense it is highly redundant. Instead of acknowledging the given image as one instance, the given image can be regarded as a composition of local images, called patches. For example, one may consider the 5 × 5 neighboring patch for each pixel, then the given image consists of many small images. If there are many alike patches by selfsimilarity and we take the average among them, we obtain the ideal denoising method in (3.1). The locations of those patches may not be close together, that is why it is called nonlocal means.

The average filter and frequency filtering in Section 4 and variational PDE methods in Section 6 belong to local smoothing methods, which may reconstruct main geometrical configurations but fail to preserve the fine structure, details, and texture. But nonlocal means bypass such restriction by exploiting innate redundancy and selfsimilarity.

The truncation in (4.4), hard thresholding (5.6) and soft thresholding (5.7) try to make its coefficient sparse to achieve denoising under some orthogonal basis. If we have a redundant set consisting of clean patches or good prototypes, where the redundancy means spanning same set but possibly linearly dependent, than the chance for sparsity will increase. Such a code book is called a dictionary and those members are called atoms.

Let \( A \) be a given dictionary, where each column is an atom. Then we can consider the following optimization problem:

\[
    \min_x \|x\|_0 \quad \text{subject to} \quad \|Ax - u_o\| < \epsilon, \quad (7.2)
\]

where \( \|x\|_0 \) is the number of nonzeros in \( x \). Since the dictionary \( A \) is redundant, if \( A \) is a \( m \times p \) matrix, \( m \leq p \) and thus the equation is underdetermined. Considering \( Ax = u_o \), if one assumes at least one solution, there must be infinitely many; under the assumption that the matrix \( A \) have full rank, the dimension of the solution is \( p - m \). Among them we seek the sparsest solution by penalizing the number of nonzero in \( x \) and \( \tilde{u} = Ax \) is represented by a few
columns in $A$. If the dictionary $A$ consists of good and clean patches or prototypes, $\hat{u} = Ax$ is a denoised version of $u_o$.

Due to nonconvexity and non-differentiability, $\| \cdot \|_0$ is often replaced by $\| \cdot \|_1$ and then it is called the basis pursuit denoising problem [8]:

$$\min_x \|x\|_1 \quad \text{subject to} \quad \|Ax - u_o\| < \sigma^2,$$

(7.3)

which is closely related to compressed sensing [4, 5]. Their general form is described in (3.3) and (3.4). Note that soft thresholding (5.7) is a closed-form solution for $\min_x \lambda |y| + \frac{1}{2} |y - x|^2$, and one can quickly figure out the closed-form solution for

$$\min_x \lambda \|x\|_1 + \frac{1}{2} \|y - x\|^2.$$

With the Lagrange multiplier or thresholding level $\lambda$, soft thresholding is a basic tool to solve $L^1$ problems including (7.3).

Although pre-constructed dictionaries consisting of existing bases such as DFT, DCT, and wavelets lead to fast transforms of the complexity $O(m \log m)$, they are typically limited to sparsify the signals and images of interest. Thus, generating a dictionary to sparsify only a certain type of signals or images, called dictionary learning is developed in the machine learning point of view:

$$\min_{A, \{x_i\}_{i=1}^M} \|y_i - Ax_i\| \quad \text{subject to} \quad \|x_i\|_0 < \epsilon, \ 1 \leq i \leq M,$$

(7.4)

where $\{y\}_{i=1}^M$ is a training database consisting of typical signals/images of interest. After running a computational algorithm, The dictionary $A$ should consists of good and clean patches or prototypes by learning or experiencing through the data base $\{y\}_{i=1}^M$. K-SVD algorithm [1] is such an instance, which is closely related to k-means clustering and singular value decomposition or PCA. For more details, we refer to [12].

8. CONCLUSION

In this paper, we review noise, denoising, and its major branches in somewhat unorganized and messy fashion, which is the nature of noise. We emphasize again the fundamental principle for denoising; taking average. Actually all methods try proper averaging without obtaining several copies. The problem of denoising will long live, since most algorithms are away from a desirable level of applicability. Furthermore, more and more new problems flow into this enterprise.

We remind that we don’t restrict the dimension of the problem. Now what is that data with the dimension more than 3? They may be data from internet, social networks, medical clinics, financial transactions, to name a few; we live in a deluge of data. Now, big data is an overused buzzword, the stuff people don’t understand but want to sell. It may be described as tremendous items in a higher dimensional space and there could be some weird phenomena which can not be observed in low dimensional spaces we got used to, such as curse of dimensionality, or concentration of measure [15]. However, they may be divided into two categories; ‘information + non-information’, like ‘signal + noise’. Thus, we may denoise them. Furthermore, we also
observe that denoising is closely related to other problems such as lossy compression and representation. They are all related to the dimensionality reduction of data.

Turning back to images, denoising is a part of the image restoration problem, since noise is a part of image degradation:

\[ u_o = K u + n, \]

where \( K \) is a blur kernel explaining blurring in the image, defocusing and motion blur, for examples. Thus, image restoration consists of denoising and deblurring. For more reading, we recommend Gonzalez and Woods [13] for general and engineering aspects. For mathematical point of view, Chan and Shen [6] and Aubert and Kornprobst [2] are recommended. Also, Chan, Shen, and Vese’s review paper [7] is also an excellent introductory. For wavelets and related signal/image processing, one has to consult with Mallat [16], and for sparse and redundant representation, Elad [12] is excellent. Toward machine learning and data analysis, Duda, Hart and Stork [11] is popular, and Hastie, Tibshirani, and Friedman [14] is advanced. Actually, there must be many many excellent books in this huge world, which the author may neither know nor explore.

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REFERENCES


ALGEBRAIC CORRECTION FOR METAL ARTIFACT REDUCTION IN COMPUTED TOMOGRAPHY

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ABSTRACT. If there are metals located in the X-ray scanned object, a point outside the metals has its range of projection angle at which projections passing through the point are disturbed by the metals. Roughly speaking, this implies that attenuation information at the point is missing in the blocked projection range. So conventional projection completion MAR algorithms to use the undisturbed projection data on the boundary of the metal trace is less efficient in reconstructing the attenuation coefficient in detailed parts, in particular, near the metal region. In order to overcome this problem, we propose the algebraic correction technique (ACT) to utilize a pre-reconstructed interim image of the attenuation coefficient outside the metal region which is obtained by solving a linear system designed to reduce computational costs. The reconstructed interim image of the attenuation coefficient is used as prior information for MAR. Numerical simulations support that the proposed correction technique shows better performance than conventional inpainting techniques such as the total variation and the harmonic inpainting.

1. INTRODUCTION

Computed Tomography (CT) is an imaging modality for providing 2-D and 3-D high-resolution tomographic images of the scanned human body from measured X-ray projections. Despite the controversy about the hazards associated with exposure to X-ray radiation, CT is a powerful tool to investigate the interior of the human body, and has been widely used for diagnostic and therapeutic purposes in various medical disciplines for decades. CT can provide high-resolution anatomic images in the absence of metallic objects such as dental fillings or prosthesis in the scanning path, but on the other hand metallic objects implanted in the human body cause severe metal streak artifacts in CT images and deteriorate reconstructed CT image quality. Usually metals are strongly attenuating objects and the detectors sensing X-ray

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beam attenuated by metals undergo severe photon starvation and thus after-log projection data become inaccurate.

Metal Artifact Reduction (MAR) to improve CT image quality is a hot issue of CT applications in clinical practice and several MAR algorithms have been proposed over past three decades. Lewitt and Bates first developed a MAR image reconstruction method from incomplete projections, in which the projection measurements through metals were assumed to be missing and were recovered by polynomial interpolation [1]. Besides linear and polynomial interpolations [2–4], wavelet interpolation [5, 6], sinogram inpainting [7, 8], and normalized MAR interpolation [9] techniques have been proposed in order to fill the missing projection data. These MAR algorithms can be classified into the projection or sinogram completion methods. During the last decade, iterative methods modeling the physics behind metal artifacts have established another class of MAR algorithms in which the noise [10, 11] and the beam hardening [12, 13] were modeled. Compared to projection completion methods, model-based iterative methods are computationally intensive and have limitations in clinical applications [14]. Recently, the hybrid method combining the projection completion method with the iterative method has been proposed [15, 16].

In the projection completion method, the missing projections on the metal trace can be filled up by continuing the flow of the uncorrupted projections adjacent to the metal trace in the help of various inpainting algorithms such as interpolation and total variation. Inpainting is the image processing technique of reconstructing lost or deteriorated parts of images and videos. For more details, see [17, 18]. These traditional methods to fill the gap of the missing projections with the uncorrupted projections on the boundary of the metal trace in the sinogram may distort the true attenuation coefficient outside metal objects.

Fig. 1 (a), (b) and (c) show a phantom model containing metals located on two white regions, its sinogram and the sinogram region corresponding to a range of angle whose projections passing through the red point, as an example, in Fig. 1 (a) are blocked by the metal region, respectively. The closer a point is to the metal region, the wider the range of angle, in which projections passing through the point are disturbed by metals, is. Therefore the traditional filling methods may cause inaccurate information to projections passing through the red spot. As a result, CT images reconstructed by the filtered back projection (FBP) method from this corrected sinogram may be less efficient to recover the details of the true image of the phantom model. The efficiency of the projection completion method has high dependence of the accuracy of synthesized data.

In this paper we propose a new MAR algorithm which is called the algebraic correction technique (ACT) using an interim image of the attenuation coefficient outside the metal region. The attenuation coefficient is usually calculated from the measured projection data by FBP and algebraic reconstruction technique (AKT) basically proposed by Kaczmarz [19]. Since FBP cannot be applied to incomplete data, we adopt ART which is expressed by a linear system with relationship between the uncorrupted projections and image values on rectangular grid. Since the matrix of the linear system is quite large and severely under-determined, various iterative algorithms to solve the linear system have been developed at the cost of computational time and memory [19–23]. In ACT, we deal with those limitations and improve the imaging
quality efficiently by generating a linear system at coarse scale that are solved in relatively short time by the least squares method equipped with Tikhonov regularization and reconstructing the corresponding low resolution interim image. The temporary image of the attenuation coefficient is exploited as prior information for MAR. Assuming that the attenuation value is identically zero on the metal region, we project the solution of the linear system into the metal trace and interpolate synthesized projections on the metal trace in order to replace the corrupted projection data. The final image of the attenuation coefficient is hence obtained from this surrogate sinogram by FBP.

This paper is organized as follows. In Section 2, we give a brief introduction to ARTs and explain the proposed algebraic correction method ACT. In Section 3, numerical results are provided to demonstrate ACT and we compare its performance with those of traditional inpainting techniques including the total variation and the harmonic inpainting. Finally we finish the paper with conclusion and future works.

2. Method

2.1. Algebraic Reconstruction Technique. Algebraic Reconstruction Technique (ART) can be traced to an iterative technique introduced by S. Kaczmarz [19] and was first applied in CT [20]. In ART, it is assumed that a certain reconstruction region, for instance rectangle or disk, is known and the region completely contains the scanned object. The region is then discretized with square grids and we obtain the following relationship between the measured projection data $p = (p_1, p_2, \cdots, p_M) \in \mathbb{R}^M$ and the unknown image $f = (f_1, f_2, \cdots, f_N) \in \mathbb{R}^N$. 

Figure 1. (a) shows a phantom model containing two metals which are located on white regions. (b) is its sinogram. Two blue and red curves indicate the traces of projections passing through blue and red spots in (a) located outside the metal regions. (c) shows the magnified image of the boxed region in (b). Projections passing through the red point are blocked by the metals in a range of projection angle. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)
\[
\sum_{j=1}^{N} w_{ij} f_j = p_i, \quad i = 1, 2, \ldots, M
\]  

(2.1)

where \( M \) is the number of projections measurements, \( N \) is the number of grids, and \( w_{ij} \) is the weighting factor which is equal to the fractional length or area of the \( j \)-th image pixel intercepted by the \( i \)-th ray. Note that most of the \( w_{ij} \)'s are zero since only a small number of image pixels contribute to any projection and the matrix \( W = (w_{ij}) \) is sparse. An iterative algorithm to solve the linear system (2.1) was basically proposed by S. Kaczmarz [19] which can be expressed as

\[
f^{(i)} = f^{(i-1)} - \frac{\langle f^{(i-1)}, w_i \rangle - p_i}{\langle w_i, w_i \rangle} w_i, \quad i = 1, 2, \ldots, M
\]

(2.2)

where \( \langle \cdot, \cdot \rangle \) is the dot product in the space \( \mathbb{R}^N \), \( w_i = (w_{i1}, w_{i2}, \ldots, w_{iN}) \in \mathbb{R}^N \), and \( f^{(i)} \) is the orthogonal projection of \( f^{(i-1)} \) onto the hyperplane \( \langle f, w_i \rangle = p_i \). The method (2.2) is based on considering the solution \( f \) as a point in \( \mathbb{R}^N \) which is the intersection of \( M \) hyperplanes. The convergence of the above iterative algorithm was investigated by K. Tanabe [21].

However, since \( N \) is often much bigger than \( M \), the linear system (2.1) is severely underdetermined. More worse, the projections obtained in CT often contain noise which would cause \( p \) not to lie in the range of the operator \( W \). Even if convergence is guaranteed, the method (2.2) would then converge very slowly to the solution. So many other iterative algorithms have been developed. Simultaneous Iterative Reconstructive Technique (SIRT, [22]) and Simultaneous Algebraic Reconstructive Technique (SART, [23]) are variations of the Kaczmarz’s method.

2.2. Algebraic Correction Technique for MAR. Fig. 2 (a) shows a phantom model containing two metal regions implanted in white areas and several regions with different attenuation values. Fig. 2 (b) shows the sinogram of the phantom model. FBP generates an image with streak artifacts caused by a metal object as shown in Fig. 2 (c). Using a simple thresholding, we can determine the metal region in the reconstructed image. Then we cut off the corrupted projections from the sinogram whose rays lie on the metal region. With the help of conventional inpainting methods such as the total variation and harmonic inpainting methods, the missing projections can be filled up so that FBP provides an improved image without streak artifacts from the synthesized sinogram. Fig. 2 (d1) and (d2) are the sinogram filled by the harmonic inpainting method and its FBP image, respectively. Fig. 2 (e1) and (e2) are the sinogram filled by the total variation method and its FBP image, respectively.

Reconstructed images (d2) and (e2) are significantly improved compared with the reconstructed image (c). However, it is difficult to distinguish two small disc regions, which are originally located above the metal region, in (d2) and (e2). The reason is as follows. Let \( r_0 \) be a point in the disc regions, for instance, the red spot in Fig. 1 (a). The red curve in Fig. 1 (b) traces the projections whose rays pass through the point \( r_0 \). Fig. 1 (c), which is a magnified one of the boxed region in Fig. 1 (b), shows the blind part of attenuation information of the red point \( r_0 \). In other words, the projection rays passing through the point \( r_0 \) at projection angle
from $\theta_1$ to $\theta_2$ are blocked by the metal objects and hence there is no projection containing attenuation information at $r_0$ on the range of angle $[\theta_1, \theta_2]$. So conventional filling of the missing projections in the box using the neighboring uncorrupted data may ignore information on the attenuation value at the point $r_0$, and hence CT images reconstructed by FBP from this manipulated sinogram are less efficient to recover the details of the true image of the attenuation coefficient of the phantom model.

In this paper, we take a different approach to filling up the metal trace in the sinogram. Instead of using the boundary projections of the metal trace, we exploit an intermediate image of the attenuation coefficient reconstructed from incomplete projection data. Let $M$ be the number of measured projection data for all projection angles and let $p = (p_1, p_2, \cdots, p_M) \in \mathbb{R}^M$ be the measured projection data. We categorize the projections $p_k$ into two parts of corrupted and uncorrupted ones. Let $p_t = (p_{t_1}, \cdots, p_{t_{m_1}}) \in \mathbb{R}^{m_1}$ be the vector of uncorrupted projections whose ray does not touch metal objects, and let $p_s = (p_{s_1}, \cdots, p_{s_{m_2}}) \in \mathbb{R}^{m_2}$ be the vector of corrupted projections whose ray passes through metal objects. Note that $m_1 + m_2 = M$. In many clinical practices, the measurement number $M$ is huge. Even though the corrupted data $p_s$ is eliminated from the whole data $p$, $m_1$ is still so big that applying the method (2.2) encounters memory limitations and requires considerable computation time. So we use coarse square grids $D_i$, $i = 1, 2, \cdots, N$, with width $\delta$ bigger than the size of a detector and let $f = (f_1, f_2, \cdots, f_N) \in \mathbb{R}^N$ be the image vector. If the area of the intersection of $D_i$ and the metal region is bigger than $\delta^2 / 2$, we suppose that the image value $f_i$ is zero. Here there are several ways to generate linear systems (2.1) with various size, but we want the size of the generated linear system to be not big. So we propose to select $N$ uncorrupted projections $p_{t_k}$ from $p_t$ so that they are evenly distributed through components of $p_t$ in index order. Let $\hat{p}_t \in \mathbb{R}^N$ be a vector whose elements are the selected uncorrupted projections $p_{t_k}$. Then we have the matrix form of the linear system (2.1)

$$Wf = \hat{p}_t$$

Here, we emphasize that the weight matrix $W$ is not so big that we apply the Kaczmarz's iterative method (2.2) or its variations such as SIRT and SART. So in order to solve (2.3), we apply the well-known least squares method equipped with the Tikhonov regularization defined by

$$\arg \min_f \left( ||Wf - \hat{p}_t||_2^2 + \alpha ||f||_2^2 \right)$$

where $\alpha$ is the regularization parameter. Then (2.4) has the unique solution

$$f = (W^TW + \alpha I)^{-1} W^T \hat{p}_t$$

where $I \in \mathbb{R}^{N \times N}$ is the identity matrix.

Here note that the solution $f$ in (2.5) is a temporary image of the attenuation coefficient on coarse grids $D_i$. We synthesize sporadic projection data from the image $f$ and interpolate the synthesized projection data into the metal trace in the sinogram in order to replace the corrupted data $p_s$. Finally, we apply FBP in order to obtain the final reconstructed image from this updated sinogram.
3. Results

3.1. Summary of the proposed method. Unlike the conventional inpainting methods to use the boundary data of the metal trace in the sinogram, the proposed ACT exploits an interim image of the attenuation coefficient as prior information for MAR. Briefly, the proposed method is based on the following steps:

Step 1: Reconstruct an FBP image from the measured projection data $p$.

Step 2: Using a simple thresholding, find the image of the metal region in the reconstructed image and cut off the metal trace from the sinogram.

Step 3: Decompose the vector $p$ into two vectors $p_s$ whose elements are projections corrupted by metal objects and $p_t$ whose elements are uncorrupted projections outside the metal trace.

Step 4: Discretize the phantom with coarse square grids $D_i$, $i = 1, 2, \cdots, N$. Select evenly $N$ uncorrupted projections from $p_t$ and let $\tilde{p}_t$ be the vector of $N$ selected projections.

Step 5: Find a solution $f$ defined in (2.5). If the area of the intersection of $D_i$ and the metal region is bigger than half of the area of $D_i$, we suppose that the image value $f_i$ is zero.

Step 6: Project the solution $f$ into the metal trace.

Step 7: Interpolate projected projections of $f$ on the metal trace in order to replace $p_s$.

Step 8: Using FBP, reconstruct the final image from the updated sinogram.

3.2. Numerical Experiment. We use an attenuation coefficient distribution as depicted in Fig. 2 (a). The image size is $128 \times 128$. We set the infinitesimal rotation angle for the simulated projection to $1^\circ$. The sinogram hence has the size of $182 \times 180$ (Fig. 2 (b)). To obtain the projection image and perform the FBP in the procedure, we use an open source project, called scikit-image [24], running on the python. Also, in order to solve the minimization problem (2.4), we use the numpy [25] which is a famous linear algebra library on the python. For the acceleration of the linear solver in the numpy, we build the numpy with Intel MKL (math kernel libraries) that enables to use all core of the Intel CPU. We test all numerical experiments on 64-bit Ubuntu 12.04. LTS with Intel i7 CPU (3.4GHz, quad core) and 16GB memory.

Fig. 3 (a) shows the interim images obtained by solving the minimization problem (2.4) on the coarse grid. Clearly, we see that the reconstruction result is less affected by metal artefact. From the obtained images, we perform the forward projection to get a prior information for the
Figure 2. (a) Original image containing metal regions. (b) Sinogram of (a). (c) FBP reconstructed image from the sinogram (b). (d1) Inpainted sinogram using the harmonic inpainting. (d2) FBP reconstructed image from the sinogram (d1). (e1) Inpainted sinogram using the total variation. (e2) FBP reconstructed image from the sinogram (e1). (f1) Sinogram using proposed correction method. (f2) FBP reconstruction image from the sinogram (f1).
The sinogram correction (see Fig. 3 (b)). Fig. 3 (c) presents the correction step using information that the fine grid is twice of coarse one in scale. We put the projection data of the coarse grid on even points of the fine grid in the metal trace region. Then, we apply the harmonic inpainting to fill the remaining points in the metal trace region (see Fig. 3 (d)).

Fig. 2 (f2) is the final result by applying FBP to Fig. 2 (f1).

To verify the numerical experiment, we illustrate the 1-D profiles of reconstructed images. Fig. 4 (a) illustrates zoomed-in images of the original, the proposed algorithm and the conventional harmonic inpainting from top to bottom, respectively. The gray, red and blue lines of Fig. 4 (b) mean the 1-D profile of the left images, respectively. Apparently, we see that the proposed method can reconstruct the small inclusion regions defected by metal artefact.

4. Conclusion

We propounded a question about efficiency of existing inpainting algorithms for MAR to use the boundary projections of the metal trace in the sinogram. In order to settle the limitation,
we proposed the algebraic correction technique (ACT) to utilize an interim image of the attenuation coefficient outside the metal region which was obtained by solving a carefully designed linear system. We demonstrated numerically that ACT improves image quality in detailed parts compared with conventional inpainting methods such as the total variation and the harmonic inpainting. This suggests its usefulness in medical diagnosis and other medical applications.

In ACT, it is an essential part to find a temporary image of the attenuation coefficient from the incomplete projection data outside the metal trace. Using the temporary image, the missing projections on the metal trace are calculated. To do so, we are forced to solve a linear system with the incomplete projection data. In this paper, in the purpose of avoiding possible computational cost in dealing with a large linear system, we generated a linear system defined on coarse grids and solved it using the Tikhonov regularized least squares method. However, there is not a study on design of a linear system and its inversion method to minimize computational cost and maximize image quality. So future works should focus on optimization of ACT.

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REFERENCES


LONG TERM MONITORING OF HYDRARGYRUM POLLUTED SOIL USING
PROJECTED IMAGE RECONSTRUCTION IN ELECTRICAL IMPEDANCE
TOMOGRAPHY

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ABSTRACT. In this paper we consider a novel reconstruction method in electrical impedance
tomography (EIT) and its application for monitoring and detecting a hydrargyrum (mercury)
polluted soil near to the surface of underground. We use electrodes placed on the surface of
land to collect the data which provides the relations of voltage and current map and to produce a
projected image of interior conductivity distribution onto the surface of land. Here the projected
image reconstruction method is used to monitor the pollution in soil underneath the ground
without any destruction and any digging into a land.

1. INTRODUCTION

Mongolia is known for its abundance of natural resources. Last few years, economy of
Mongolia has risen incredibly based on these natural resources. Despite its rapid growth
in economy, due to the lack of proper environmental protection, several serious effects have
been appeared in the nature. Currently, Mongolia is facing various environmental issues like
other developing countries, who are using natural resources for the economic development.
These can be global or local problems such as desertification, pollution of air and water, un-
derground contamination, yellow smoke and so on. Recently hydrargyrum pollution became a
big issue since it is more dangerous than the other pollutants. For instance, in human health,
high level mercury vapor and its various forms are considered as extremely harmful to hu-
mank lung since one time inhaling can lead to sudden death. Even more, the mercury toxin
can cause damages on human genetics and brain. In ecology, contaminated soil can pro-
duce mercury vapor in air, the rain take it to river and lakes. Therefore hydrargyrum pollu-
tion is not only a threat to human but to the entire natural system [14]. The hydrargyrum is

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known as the only metal that is liquid at room temperature, it quickly merges to the particles of metal and transforms to an amalgam (foil). Therefore mining industries, especially, artisanal minings are using this property to separate metal from ore. There are lots of artisanal mining in Mongolia and they uses mercury in mining operation, even, sometimes people wash ore by hands and use mercury to catch gold. Hence, in the last few years Mongolia became a high risk country by mercury pollution. Figure 1 is taken from the article mentioning about soil and river contamination with heavy metals like mercury in Mongolia (http://www.enp-beratung.de/EN/html/world_garden_mongolia.html).

Mercury pollution is not only a problem of Mongolia, in fact, it has been considered as a cause of one of fatal environmental damages. Figure 2 shows mercury pollution risks of different countries [2]. In Africa, from 15 century, mercury was started to use for gold mining. Later in 1956, after Japanese Minanata disease the whole world is aware of how disastrous the mercury pollution can be [14]. To do the laboratory testing, the random sampling which requires great amount of digging and keeping track of the samples are necessary. Since the time to refresh polluted soil takes very long and costs a lot, economical and effective way to monitor the mercury pollution of soil is needed.

Electrical impedance tomography (EIT) is non-destructive and not expensive method to visualize the conductivity distribution in electrically conducting objects. Since EIT is cheap, noninvasive and imaging tools are not massive compared to the other imaging techniques, it has been long time studied for human health proposing to apply in the clinics. Early works are starting from reconstruction of static imaging method, which is based on back-projection and sensitivity approaches [3, 4, 21]. Besides, the characteristics of conductivity of biological tissue changes depending on the frequencies of injected current as well as depending on time.
This motivates to develop many different modalities of imaging methods and feasibility studies for human tissue property images. A frequency or time difference imaging methods reconstructed from two types of set measurements give a benefit to eliminate some measurement and geometry errors [1], [22] and [10]. EIT also has been studied in geological environmental research, for instance, detect the mineral ore [17], land pollution detection [7], leaks detection in underground [20], detection of flow fluid into a land [19] and bedrock detection [5]. Nevertheless, all of the methods are based on sensitivity approaches, which is very sensitive to any small error.

Recently, we developed a new projective image reconstruction algorithm [11, 12, 15]. The corresponding new EIT system also has been developed [13]. In this paper, we modify the projective image reconstruction algorithm to use for mercury detection underneath the ground. First, we introduce the mathematical framework and then show several numerical simulation results.

2. Mercury pollution monitoring methods

2.1. Mathematical modeling. To monitor the conductivity distribution in observing domain, we propose to reconstruct the projection image of the conductivity distribution \( \sigma(t, r) \), which depending of time \( t \) and the position \( r \). Let \( \Omega \), the underground, be the half space \( \mathbb{R}^3_+ \) with boundary \( \partial \Omega \), \( xy \)-plane. Let \( E_1^\pm \), \( E_2^\pm \) be two pairs of current driving electrodes and \( \Gamma \) be the sensing surface on the ground filled with point electrodes as shown in figure 3. When a sinusoidal current \( I \sin(\omega t) \) with fixed frequency \( \omega \) is injected through the electrodes \( E_j^\pm \) the
resulting induced electrical potentials $u^j(t, r)$ satisfies following boundary value problem
\[
\begin{cases}
\nabla \cdot (\sigma(t, r) \nabla u^j(t, r)) = 0 & \text{in } \mathbb{R}^+ \times \Omega \\
-\sigma(t, r) \frac{\partial u^j(t, r)}{\partial n} = g^j(r) & \text{on } \mathbb{R}^+ \times \partial \Omega \text{ for } j = 1, 2
\end{cases}
\] (2.1)

with initial conditions $u^j(0, r) = h_1(r)$ and $\partial u^j(0, r)/\partial t = h_2(r)$, where $n$ is the unit outward normal vector to the boundary $\partial \Omega$ and $g^j$ satisfies
\[
\int_{\mathcal{E}^j} g^j(t, r) \, dS_r = \pm I, \quad \int_{\mathcal{E}^k} g^j(t, r) \, dS_r = 0 \quad (k \neq j) \quad \text{and} \quad g^j(r) = 0 \text{ on } \partial \Omega \setminus (\mathcal{E}_1^+ \cup \mathcal{E}_2^+).
\]

Define $f^j(t, r)$ as the Dirichlet data such that, for $j = 1, 2$,
\[
f^j(t, r) := u^j(t, r) \quad \text{on } \mathbb{R}^+ \times \partial \Omega.
\]

Using the measured voltage data on $\Gamma$, we are aiming to reconstruct the projected image of conductivity distribution of underneath the ground, while time is changing. From the reconstructed image, we can determine if there is any on-going mercury pollution and how fast it moves if there is.

2.2. Reconstruction algorithm. Since mercury polluted area moves and spreads out as time goes, we use time difference voltage data to analyze the relationship between the measurements and conductivity distribution of the interior. Let $D$ be the region that is filled with mercury which is compactly embedded in $\Omega$. Since $D$ changes in time, $D$ is a function of time $t$, $D(t)$, and the equation (2.1) can be rewritten as
\[
\begin{cases}
\nabla \cdot ((1 + \sigma_m \chi_{D(t)}(r)) \nabla u^j(t, r)) = 0 & \text{in } \mathbb{R}^+ \times \Omega \\
-\frac{\partial u^j(t, r)}{\partial n} = g^j(r) & \text{on } \mathbb{R}^+ \times \partial \Omega \text{ for } j = 1, 2
\end{cases}
\] (2.2)

where $1 + \sigma_m$ is the conductivity of mercury and $\chi_{D(t)}$ is a characteristic function defined as
\[
\chi_{D(t)}(r) = \begin{cases} 
1 & \text{if } r \in D(t) \\
0 & \text{otherwise}
\end{cases}
\]
Here $u^j$ satisfies the transmission condition

$$(1 + \sigma_m) \frac{\partial u^j}{\partial n} \bigg|_- - \frac{\partial u^j}{\partial n} \bigg|_+ = 0 \quad \text{on } \mathbb{R}^+ \times \partial D(t),$$

where $\pm$ denotes the limit from outside and inside of $D(t)$, respectively.

Let $\Phi(r, r')$ be a fundamental solution of Laplace equation. We define the double and single layer potentials of a function $\varphi \in L^2(\partial \Omega)$ by

$$D \varphi(r) = \int_{\partial \Omega} \frac{\partial \Phi(r, r')}{\partial n_{r'}} \varphi(r') dS_{r'} \quad \text{and} \quad S \varphi(r) = \int_{\partial \Omega} \Phi(r, r') \varphi(r') dS_{r'},$$

respectively. From trace formula for the double layer and single layer potentials, we have

$$\lim_{s \to 0^+} D \varphi(r \pm s n(r)) = \left( \mp \frac{1}{2} I + \mathcal{K} \right) \varphi(r), \quad \text{when } r \in \partial \Omega \quad (2.3)$$

$$\lim_{s \to 0^+} \langle n(r), S \varphi(r \pm s n(r)) \rangle = \left( \pm \frac{1}{2} I + \mathcal{K}^* \right) \varphi(r), \quad \text{when } r \in \partial \Omega,$$

where $I$ is the identity operator on $L^2(\partial \Omega)$ and $\mathcal{K}^*$ is the dual operator of $\mathcal{K}$ defined as

$$\mathcal{K} \varphi(r) = \int_{\partial \Omega} \frac{\partial \Phi(r, r')}{\partial n_{r'}} \varphi(r') dS_{r'}, \quad \text{for } r \in \partial \Omega.$$

**Theorem 2.1.** Let $u^j(t, r)$ be the solution to (2.2) and $f^j = u^j|_{\partial \Omega}$ for $j = 1, 2$. Then we have

$$f^j(t_1, r) - f^j(t_2, r) = \frac{2(2 + \sigma_m)}{(1 + \sigma_m)^2} \int_\Omega \left( (\chi_{D(t_1)} \mathbf{J}_1)(r') - (\chi_{D(t_2)} \mathbf{J}_2)(r') \right) \cdot \nabla_{r'} \Phi(r, r') dr'$$

in which $\mathbf{J}_1(r') = -(1 + \sigma_m) \nabla_{r'} u^j(t_1, r')$ and $\mathbf{J}_2(r') = -(1 + \sigma_m) \nabla_{r'} u^j(t_2, r')$.  

**Proof.** Let $u^j(t, r)$ be the solution to (2.2), then it can be represented as

$$u^j(t, r) = D f^j(t, r) - S g^j(r) - \int_{\partial D(t)} \frac{\partial u^j(t, r')}{\partial n_{r'}} \Phi(r, r') dS_{r'} - \int_{\partial D(t)} \frac{\partial u^j(t, r')}{\partial n_{r'}} \Phi(r, r') dS_{r'},$$

where $\frac{\partial}{\partial n_{r'}}$ and $\frac{\partial}{\partial n_{r'}}$ are interior and exterior normal directional derivatives on $\partial D$. The transmission condition on $\partial D$ yields

$$u^j(t, r) = D f^j(t, r) - S g^j(r) - \left( \frac{2 + \sigma_m}{1 + \sigma_m} \right) \int_{\partial D(t)} \frac{\partial u^j(t, r')}{\partial n_{r'}} \Phi(r, r') dS_{r'}.$$  

In the above we use (2.3) to have, for $r \in \Gamma$ with $\Gamma$ the sensing surface,

$$\left( \frac{1}{2} I - \mathcal{K} \right) f^j(t, r) = -S g^j(r) - \left( \frac{2 + \sigma_m}{1 + \sigma_m} \right) \int_{D(t)} \nabla_{r'} u^j(t, r') \cdot \nabla_{r'} \Phi(r, r') dr'. \quad (2.5)$$
For fixed time \( t = t_1 \) and \( t = t_2 \), we subtract the measurable quantities on the left hand side of (2.5) from time \( t_1 \) to \( t_2 \) to obtain

\[
\left( \frac{1}{2} I - K \right) \left( f_j(t_1, r) - f_j(t_2, r) \right) = -\left( 2 + \sigma_m \right) \int_\Omega (\chi_{D(t_1)}(r') \nabla_r u_j(t_1, r') - \chi_{D(t_2)}(r') \nabla_r u_j(t_2, r')) \cdot \nabla_r \Phi(r, r') \, dr'.
\]

Due to the fact that \( K = 0 \) for the half-space, the result holds.

The above theorem shows that the boundary voltage time difference data is related to the changes of polluted area \( D \) in time in the interior of the domain. Based on the relation shown in (2.4), we apply the projected image reconstruction algorithm introduced in [15] to reconstruct the diffused image of movements of mercury underneath the ground. Recall (2.4):

\[
f_j(t_1, r) - f_j(t_2, r) = c_m \int_\Omega \left( \chi_{D(t_1)} J_j^1(r') - \chi_{D(t_2)} J_j^2(r') \right) \cdot \nabla_r \Phi(r, r') \, dr',
\]

where \( c_m = 2(2 + \sigma_m)/(1 + \sigma_m)^2 \), \( r = (x, y, 0) \in \Gamma \) and \( r' = (x', y', z') \). The super index \( j \) denotes the case when a current is injected through \( \mathcal{E}_j^\pm \). Here we assume that \( D(t_1) \subset D(t_2) \) and \( \| J_j^1 - J_j^2 \|_{L^2(D(t_1))} \ll 1 \). Then by the following

\[
\chi_{D(t_1)} J_j^1 - \chi_{D(t_2)} J_j^2 = \chi_{D(t_1)} (J_j^1 - J_j^2) + (\chi_{D(t_1)} - \chi_{D(t_2)}) J_j^2,
\]

we have an estimation

\[
f_j(t_1, r) - f_j(t_2, r) = -c_m \int_\Omega (\chi_{D(t_2)} - \chi_{D(t_1)}) j_j^2(r') \cdot \nabla_r \Phi(r, r') \, dr' + O \left( \| J_j^1 - J_j^2 \|_{L^2(D(t_1))} \right).
\]

For notational simplicity, we let \( \mathcal{F}_j^2(r) := -c_m \int_\Omega (\chi_{D(t_2)} - \chi_{D(t_1)}) j_j^2(r') \cdot \nabla_r \Phi(r, r') \, dr' \).

The area \( D(t_2) - D(t_1) \) can be either one of the cases in Figure 4. Now we apply the integration by parts to the above to have

\[
\mathcal{F}_j^2(r) = c_m \int_\Omega j_j^2(r') \left( \nabla_r \chi_{D(t_2)} - \nabla_r \chi_{D(t_1)} \right) \Phi(r, r') \, dr'.
\]
The projected image reconstruction algorithm collects a pair of voltage data when the current is injected through \( E_j^\pm \) for \( j = 1, 2 \) for each time \( t_1 \) and \( t_2 \). Then the surface Laplacian \( \Delta_r = \partial^2_x + \partial^2_y \) is applied to \( F^j(r) \) on \( \Gamma \) such that

\[
\Delta_r F^j(r) = c_m \int_\Omega J^j_2(r') \left( \nabla r' \chi(D(t_2) - D(t_1))(r') \right) \Delta_r \Phi(r, r') \, dr'.
\]

(2.6)

The above formulations can be summarized as a system as follows. Let

\[
F = \begin{bmatrix} F^1 \\ F^2 \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} J^1_2 \\ J^2_2 \end{bmatrix} = \begin{bmatrix} J^1_{2,1} & J^1_{2,2} & J^1_{2,3} \\ J^2_{2,1} & J^2_{2,2} & J^2_{2,3} \end{bmatrix}
\]

then (2.6) can be written as

\[
\Delta_r F(r) = \left( \frac{4 + 2\sigma_m}{(1 + \sigma_m)^2} \right) \int_\Omega \Delta_r \Phi(r, r') J_2(r') \left( \nabla r' \chi(D(t_2) - D(t_1))(r') \right) \, dr'.
\]

(2.7)

**Observation 2.2.** We multiply \( J^1_2 \), a pseudo-inverse of \( J_2 \), and take surface divergence, \( \nabla r \cdot = (\partial_x, \partial_y, 0)^T \), to both sides of (2.7) to get

\[
\nabla r \cdot \left( J^\dagger_2(r) \Delta_r F(r) \right) = \left( \frac{4 + 2\sigma_m}{(1 + \sigma_m)^2} \right) \nabla r \cdot \int_\Omega \Delta_r \Phi(r, r') J^\dagger_2(r) J_2(r') \left( \nabla r' \chi(D(t_2) - D(t_1))(r') \right) \, dr'.
\]

(2.8)

The right hand side of the above has an information of the area \( D(t_2) - D(t_1) \).

In the left hand side of (2.8), \( F \) is an approximation of the measurable quantity

\[
\begin{bmatrix} f^1(t_1, r) - f^1(t_2, r) \\ f^2(t_1, r) - f^2(t_2, r) \end{bmatrix}
\]

which is only available from the sensing surface \( \Gamma \). From the analysis given in [15], the current \( J^j_2(r') \) can be replaced by the following

\[
J^j_2(r') \approx J^j_2(r) = \begin{bmatrix} J^j_{2,1}(r) & J^j_{2,2}(r) & 0 \\ -[\partial_x f^j(t_2, r) & \partial_y f^j(t_2, r) & 0] \end{bmatrix} \quad \text{for } j = 1, 2.
\]

Thus the projected image reconstruction algorithm which provides the projected image of the changes of interior conductivity distribution in time can be constructed as follows.

**Observation 2.3.** Let \( \Psi \) be the solution of

\[
-\Delta_r \Psi(r) = \left( \frac{(1 + \sigma_m)^2}{4 + 2\sigma_m} \right) \nabla r \cdot \left( \begin{bmatrix} \partial_x f^j_1(r) & \partial_y f^j_1(r) \\ \partial_x f^j_2(r) & \partial_y f^j_2(r) \end{bmatrix} \right)^{-1} \Delta_r \left( \begin{bmatrix} f^j_1(r) - f^j_1(r) \\ f^j_2(r) - f^j_2(r) \end{bmatrix} \right),
\]

(2.9)

where \( f^j_1(r) = f^j(t_1, r) \) and \( f^j_2(r) = f^j(t_2, r) \). Then \( \Psi \) provides the diffused (blurred) image of the area \( D(t_2) - D(t_1) \) which is projected on to the sensing surface \( \Gamma \).

In the following section, we perform several numerical tests to validate our analysis.
In this section, we run numerical tests based on the reconstruction algorithm described in the subsection 2.2. The electrical conductivity of soil is depending on the type and an underground levels. In general, in dry and clay area, it ranges from 0.0001 S/m to 0.005 S/m. The conductivity of mercury is known as $1.02 \times 10^6$ S/m. For more details about corresponding conductivity information, we refer to see [6] and [8].

In all numerical simulations, we set the domain as a rectangular cuboid with size $6 \times 6 \times 4$ m$^3$. The finite element structure of domain including information about mesh statistic shown in Figure 5. We use Comsol multiphysic to generate the meshes. The four driving electrodes of size $0.1 \times 0.1$ m$^2$ for current injection are placed at four corners on rectangle surface of observing domain. The distance from center of driving electrode to the two sizes of rectangular surface is 1.4m and 0.8m, respectively. The $27 \times 17 = 459$ number of voltage measuring electrodes are located between the driving electrodes in rectangular region occupying $2.6 \times 1.6$ m$^2$ area. The diameter of sensing electrode is 0.05 m.

Throughout the numerical simulations, total current of 50 A/m$^2$ flows in through $E_+$ and $E_-$ is set as the ground.

![Figure 5](image.jpg)

**Figure 5.** The computational domain with mercury stream and electrode configuration: mesh structure with 8353 nodes and 39459 elements

The mercury stream is of 3-dimensional tube shape along the sinusoidal type curve. The radius of the tube is 0.05 m and it runs through only in $(x, y)$-direction. At the beginning which is indicated as moment $t_1$, the mercury stream is in the form of a half circle. After $\Delta t$-time later, which is at moment $t_2$, it has moved a little bit as shown in Figure 6. At moment $t_3$, $\Delta t$-time later from moment $t_2$, the mercury stream moves all the way through the upper half circle as in Figure 6. In this way, the numerical simulation mimics the mercury stream propagation in long time monitoring. At each time, the voltage data is collected from the sensing surface and then time difference of voltage data is used to detect the movements of mercury stream. In this section, we observe the reconstructed images obtained from the differences of (moment $t_1$, moment $t_2$) and (moment $t_1$, moment $t_3$)-pairs.
In all the numerical tests, we examine the algorithm for two cases: when the level, \( z \)-directional position, of mercury stream is fixed as 1 m-below and 2 m-below from sensing surface as shown in Figure 7.

![Figure 6](image6.png)

**Figure 6.** Mercury stream at moments \( t_1 \), \( t_2 \) and \( t_3 \)

![Figure 7](image7.png)

**Figure 7.** The \( z \)-directional locations of mercury stream: 1 m and 2 m

3.1. **Example 1.** This example represents the situation when mercury stream passes right below the measuring area as in Figure 8.

![Figure 8](image8.png)

**Figure 8.** Top view of mercury stream of example 1

First, we set mercury stream flows 1 m below from sensing surface. Figure 9 presents the reconstructed images obtained from using the proposed algorithm. In Figure 10, all reconstructed images are re-scaled so that we can observe the behavior more clearly. For instance,
when there is a little change (from moment $t_1$ to $t_2$), reconstructed image shows relatively weak evidence of presence of mercury than other case.

Now we set the mercury stream 2 m below from the sensing surface. Figures 11 and 12 also show the reconstructed images without and with scaling, respectively.

The reconstructed images in Figures 9-12 clearly show the presence of mercury stream underneath the ground. The magnitude of reconstructed image is related the depth of mercury stream from sensing surface (compared with Figures 9, 10 and Figures 11, 12). Once can tell that if the mercury stream is located further away from sensing surface, the magnitude of sought numbers get smaller.
Figure 12. Scaled reconstructed image when mercury is located 2 m below from sensing surface: (a) using \((t_1, t_2)\)-data pair (b) using \((t_1, t_3)\)-data pair

3.2. Example 2. In this example, the mercury flow stems from outside of sensing area as in Figure 13. The same numerical tests with example 1 also have been run in this section.

Figure 13. Top view of mercury stream of example 2

Figures 14-17 show the reconstructed images when mercury flows into right below of the sensing surface.

Figure 14. Reconstructed image when mercury is located 1 m below from sensing surface: (a) using \((t_1, t_2)\)-data pair (b) using \((t_1, t_3)\)-data pair

Here, the computational domain is the size of \(6 \times 6 \times 4 \text{ m}^3\) and the polluted area is far away from the sensing surface, 1 and 2 m below from sensing surface. Even though the polluted area is located in the deep ground and the relative size of polluted area is small, the radius of polluted area in the pipe shape is 0.05 m, Figures 14-17 show the presence of mercury
FIGURE 15. Scaled reconstructed image when mercury is located 1 m below from sensing surface: (a) using \((t_1, t_2)\)-data pair (b) using \((t_1, t_3)\)-data pair

FIGURE 16. Reconstructed image when mercury is located 2 m below from sensing surface: (a) using \((t_1, t_2)\)-data pair (b) using \((t_1, t_3)\)-data pair

FIGURE 17. Scaled reconstructed image when mercury is located 2 m below from sensing surface: (a) using \((t_1, t_2)\)-data pair (b) using \((t_1, t_3)\)-data pair

underneath the ground. Since the location (the depth) of mercury area is far away from sensing surface, one get blurred image of abnormal material in the ground. The goal of this paper is to detect and monitor the long term behavior of hydrargyrum polluted soil and the numerical simulation results of proposed projected image reconstruction algorithm promise the usage of the proposed method.

CONCLUSION

Nowadays the ecological problems are increasing every where in world. In short period time using most effective and economical way to investigate those problems are becoming
more important. This novel method can be apply the other geological problems, for instance, heavy metal pollution in river area, land contamination from other source, toilette pipe leakage and so on. The drawback of this method is weak when the investigating object is away from the surface of observing domain. To overcome this difficulty next we will use this method with combining some other methods such as vibration and heat source.

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A LOCALIZED GLOBAL DEFORMATION MODEL TO TRACK MYOCARDIAL MOTION USING ECHOCARDIOGRAPHY

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ABSTRACT. In this paper, we propose a robust real-time myocardial border tracking algorithm for echocardiography. Commonly, after an initial contour of LV border is traced at one or two frame from the entire cardiac cycle, LV contour tracking is performed over the remaining frames. Among a variety of tracking techniques, optical flow method is the most widely used for motion estimation of moving objects. However, when echocardiography data is heavily corrupted in some local regions, the errors bring the tracking point out of the endocardial border, resulting in distorted LV contours. This shape distortion often occurs in practice since the data acquisition is affected by ultrasound artifacts, dropout or shadowing phenomena of cardiac walls. The proposed method deals with this shape distortion problem and reflects the motion realistic LV shape by applying global deformation modeled as affine transform partitive to the contour. We partition the tracking points on the contour into a few groups and determine each affine transform governing the motion of the partitioned contour points. To compute the coefficients of each affine transform, we use the least squares method with equality constraints that are given by the relationship between the coefficients and a few contour points showing good tracking results. Many real experiments show that the proposed method supports better performance than existing methods.

1. INTRODUCTION

The automated analysis of left ventricle (LV) using echocardiography is steadily demanding for medical assessment including motion analysis and strain analysis. Most of analysis methods are based on LV measurements in two-dimensional (2D) slices, because they are available in clinical practice [1, 2]. The LV measurements are commonly performed by manually drawing endocardial border in some 2D slices (such as end-systole (ES) or end-diastole (ED) frames) selected from the entire cardiac cycle and automatically tracking the traced LV contour over the remaining frames [3, 4]. This tracking process is accomplished by observing the speckle

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pattern, which is inherent appearance associated with soft tissues in ultrasound imaging and reflects the local echogeneity of the underlying scatterers. However, it is a difficult task to automatically track the motion of endocardial border in echocardiography data due to artifacts, shadowing phenomena, low-contrast and so on. User intervention is hence required for stable and successful tracking of endocardial border.

In the last two decades, there have been a variety of studies to trace LV wall motion using the tracking methods such as deformable models [5–8], active shape models [9–11], optical flow methods [2, 12–15], and so on. Nevertheless, they still have some limitations according to practical application to endocardial border motion in ultrasound images. The tracking methods based on deformable models or active shape models are somewhat inadequate for strain analysis related to local motion and deformation of heart, because they are not speckle tracking-based methods providing motion information of local region on the myocardium but shape-based tracking methods. They thus provide only the information of moving LV border and enable user to measure the volume inside LV. On the other hand, optical flows provide the local motion information of myocardium and are capable of measuring the LV volume as well as the myocardial wall motion analysis or strain analysis to detect LV abnormalities. But, some problems arise in cases of ultrasound images with unclear speckle pattern or weak signals. In practical environment, the contour points tracked by optical flows often present some inaccurate results due to above-mentioned ultrasound artifacts, dropouts or shadowing phenomena of cardiac wall [16]. For example, in the presence of edge dropout or indistinguishable speckle pattern in a local neighborhood of a tracking point, optical flows bring the tracking point out of the endocardial border, resulting in distorted LV contours or irregular distances between the tracked points throughout the entire cardiac cycle. These distorted results adversely affect LV volume measurement or strain analysis.

Among several attempts to overcome those difficulties, Süßling et al. [13] proposed a tracking method based on the weighted window Lucas-Kanade(LK) method [17], which estimated the displacements by minimizing the weighted least square criterion in the neighborhood of each tracking point. In their method, a linear model of multiple frames centering around the given time $t$ for the velocity along the time direction was used and more robust results were showed than that of the weighted LK method using the single frame at $t$. Compared with the approaches based on the LK method, Duan et al. [15] used the region-based tracking method (also known as the block matching or pattern matching method) with the cross-correlation coefficients as a similarity measure that is less sensitive to noise, fast motion and potential occlusions and discontinuities. However, these tracking methods still have drawback in dealing with the problem of the contour shape distortion in the presence of locally weak signal corrupted by rib shadowing and other factors.

In order to alleviate the LV shape distortion, we recently proposed a new optical flow method equipped with a global motion constraint [18]. It was designed to prevent each tracking point from getting out of the endocardial border by incorporating the LK optical flow method and a global motion constraint being approximately an affine transformation of the initial tracking points into a variational framework. Nevertheless, it still needs to be improved to deal with flexible and practical cardiac wall motion.
This paper proposes a new tracking algorithm that satisfies the requirements to alleviate the LV shape distortion and reflect realistic motion of endocardial contour. We consider both of global and local deformations of non-rigid heart motion by applying global deformation modeled as affine transform to the contour partitively. It is carried out by partitioning the tracking points on the contour into a few small groups and determining each affine transform governing the motion of the partitioned contour points. To compute the coefficients of each affine transform, we solve a least squares problem with equality constraints that are given by the relationship between the coefficients and a few contour points showing good tracking results. Numerical experiments show that the proposed method achieves better tracking performance compared to our previous algorithm [18] that outperforms the block matching tracking methods and the LK optical flows.

2. METHODS

2.1. Proposed method. Typically, heart motion is regarded as the non-rigid motion by rotation, contraction/expansion and shear [19, 20]. This non-rigid motion consists of global deformation modeled by an affine transformation and local deformation described by a free-form deformation. In order to reflect the global and local deformations of the endocardial motion simultaneously, we divide the whole tracking points into a few groups. Especially, each group is selected to include the tracking points located on a shadowing endocardial region disconnected by edge dropout and to be controlled by individual affine transform. Through the separated affine transforms, we compensate their poorly tracking results and perform the non-rigid motion of LV contour efficiently.

Let the endocardial border traced at initially selected frame (for example, end-systole or end-diastole frame) denoted by a parametric contour $C^* = \{r^*(s) = (x^*(s), y^*(s)) \mid 0 \leq s \leq 1\}$ that can be identified as its $n$ tracking points $r_1^*, \ldots, r_n^*$. Here, $0 = s_1 < s_2 < \cdots < s_n = 1$. Let $C(t) = \{r(s,t) = (x(s,t), y(s,t)) \mid 0 \leq s \leq 1\}$ be the contour deformed from $C(0) = C^*$ at time $t$. The motion of the contour $C(t)$ will be determined by an appropriately chosen velocity $U(t)$ indicating a time change of tracking points $(r_1(t), \ldots, r_n(t))$:

$$
U(t) := \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} r_1(0) \\ \vdots \\ r_n(0) \end{bmatrix} = \begin{bmatrix} r_1^* \\ \vdots \\ r_n^* \end{bmatrix}
$$

Here, we identify the contour $C(t)$ with tracking points $\{r_1(t), \ldots, r_n(t)\}$ and regard each point $r_i$ as column vector.
Figure 1. An example partitioning 13 control points into two groups. Some tracking points in the second group are located on the contour with weak edges.

For the tracking method mentioned above, we let the set of tracking points \( \{ r_1(t), \ldots, r_n(t) \} \) be partitioned into \( M \) groups as follows:

\[
G_1 = \{ r_{k_0}(t), r_{k_0+1}(t), \ldots, r_{k_1-1}(t), r_{k_1}(t) \}, \\
G_2 = \{ r_{k_1}(t), r_{k_1+1}(t), \ldots, r_{k_2-1}(t), r_{k_2}(t) \}, \\
\ldots \\
G_M = \{ r_{k_{M-1}}(t), r_{k_{M-1}+1}(t), \ldots, r_{k_M-1}(t), r_{k_M}(t) \},
\]

where \( 1 = k_0 < k_1 < \cdots < k_{M-1} < k_M = n \) and \( r_{k_0}(t), r_{k_1}(t), \ldots, r_{k_M}(t) \) mean the tracking points for obvious speckle patterns by strong signals so that they become one of both ends of each group of tracking points. That is, we assume that the displacements \( u_{k_0}(t), u_{k_1}(t), \ldots, u_{k_M}(t) \) are reliable results computed by a commonly used speckle tracking method. Since the apex and two annulus regions of endocardial borders are usually represented as good edges by strong signals, we set the number of groups to 2. Fig. 1 shows an example of 13 control points partitioned into two groups \( (n = 13 \text{ and } M = 2) \). Using these partitions of tracking points, the proposed tracking method is performed as follows.

We apply the optical flows tracking method [17] to compute the displacements \( u_1(t), \ldots, u_n(t) \). By solving the least squares method with equality constraints that are given by the relationship between the initial points \( r^*_{k_0}, r^*_{k_1}, \ldots, r^*_{k_M} \) and the displacements \( u_{k_0}(t), u_{k_1}(t), \ldots, u_{k_M}(t) \), we determine the coefficients \( a_{1,j}, \ldots, a_{6,j} \) of affine transform corresponding to each group \( G_j (j = 1, \ldots, M) \) and compute the displacements of the remaining tracking points except for both ends in each group \( G_j \) again using the determined affine transform and all the initial points belonging to \( G_j \). The proposed method is capable of compensating the poorly tracking points near endocardium with unclear speckle patterns or edge dropout by utilizing the tracking points reflecting well-tracking results by optical flows tracking method as well as applying local deformation effects.
To sum up, the proposed tracking method is performed as follows.

1. Select the first frame from the whole image frames for a heartbeat cycle.
2. If current image is the last one, go to step 3. or perform the following process:
   (1) Partition the tracking points \( \{ r_1(t), \ldots, r_n(t) \} \) into \( M \) groups. (Set \( M = 2 \) in our application.)
   (2) Compute the displacements \( u_1(t), \ldots, u_n(t) \) using the optical flows tracking method [17] as a commonly used speckle tracking method.
   (3) Solve the least squares problem with equality constraints to determine the affine transform corresponding to each group of tracking points.
   (4) Compute the displacements of the remaining tracking points except for both ends in each group \( G_j \) again using the determined affine transform and all the initial points belonging to \( G_j \).
   (5) Move to next image frame and go to step 2.
3. Terminate the process.

Additionally, we describe how to compute the affine transform corresponding to each group of tracking points in the next subsection.

2.2. Regional affine transform. Let the matrix

\[
\begin{bmatrix}
  \begin{bmatrix}
  r_{k_{j-1}}^T \\
  \vdots \\
  r_{k_j}^T 
  \end{bmatrix}
\end{bmatrix}
\]

be denoted by \( \Phi_j(C^*) \). If the \( i \)th point \( r_i(t) \) on LV contour belongs to the \( j \)th group \( G_j \) of the tracking points, the relationship between the initial points and displacements associated to \( G_j \) are represented as the following equations:

\[
\Phi_j(C^*) \begin{bmatrix}
  a_{1,j}(U_j(t)) \\
  a_{2,j}(U_j(t)) \\
  a_{3,j}(U_j(t)) \\
  a_{4,j}(U_j(t)) \\
  a_{5,j}(U_j(t)) \\
  a_{6,j}(U_j(t)) 
\end{bmatrix} = \begin{bmatrix}
  (r_{k_{j-1}}(t) + u_{k_{j-1}}(t))^T \\
  \vdots \\
  (r_{k_j}(t) + u_{k_j}(t))^T 
\end{bmatrix} \quad \text{with } U_j(t) := \begin{bmatrix}
  u_{k_{j-1}}(t) \\
  \vdots \\
  u_{k_j}(t) 
\end{bmatrix}.
\]

We solve the linear equation (2.1) using the least squares method with equality constraints that are given by the equations associated with the both ends \( r_{k_{j-1}} \) and \( r_{k_j} \) of the \( j \)th group \( G_j \):

\[
\begin{bmatrix}
  \begin{bmatrix}
  r_{k_{j-1}}^T \\
  r_{k_j}^T 
  \end{bmatrix}
\end{bmatrix} \begin{bmatrix}
  a_{1,j}(U) \\
  a_{2,j}(U) \\
  a_{3,j}(U) \\
  a_{4,j}(U) \\
  a_{5,j}(U) \\
  a_{6,j}(U) 
\end{bmatrix} = \begin{bmatrix}
  (r_{k_{j-1}}(t) + u_{k_{j-1}})^T \\
  (r_{k_j}(t) + u_{k_j})^T 
\end{bmatrix}.
\]

The description of the least squares method with equality constraints is given in Appendix.

We denote the matrix

\[
\begin{bmatrix}
  \begin{bmatrix}
  r_{k_{j-1}}^T \\
  r_{k_j}^T 
  \end{bmatrix}
\end{bmatrix}
\]

by \( B \) and compute the \( QR \)-factorization of its transpose matrix \( B^T \) as follows:

\[
B^T = Q \begin{bmatrix}
  R \\
  0 \\
\end{bmatrix},
\]
where $Q$ is a $3 \times 3$ matrix of the form $Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ and $R$ is a $2 \times 2$ matrix defined by $\begin{bmatrix} 1 & r_{12} \\ 0 & r_{22} \end{bmatrix}$.

Let $A_1$ and $A_2$ be the matrices defined by

$$A_1 := \Phi_j(C^*) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \text{ and } A_2 := \Phi_j(C^*)q_3,$$

respectively.

Then the coefficients $a_{1,j}(U), \cdots, a_{6,j}(U)$ are determined by the equations

$$\begin{bmatrix} a_{1,j}(U) & a_{3,j}(U) \\ a_{2,j}(U) & a_{4,j}(U) \\ a_{5,j}(U) & a_{6,j}(U) \end{bmatrix} = Q \begin{bmatrix} \begin{bmatrix} R^{-T} \begin{bmatrix} r_{kj-1} \\ r_{kj} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} r_{kj-1} \\ r_{kj} \end{bmatrix} \end{bmatrix} \\ (A_2^T A_2)^{-1} A_2^T \begin{bmatrix} \begin{bmatrix} r_{kj-1} \\ r_{kj} \end{bmatrix} \end{bmatrix} - A_1 R^{-T} \begin{bmatrix} r_{kj-1} \\ r_{kj} \end{bmatrix} \end{bmatrix}, \quad (2.3)$$

where $R^{-T}$ means the inverse matrix of $R^T$.

3. EXPERIMENTAL RESULTS

We tested the performance of the proposed algorithm in clinical setting using many real data. The proposed algorithm was compared with our previous algorithm [18] that outperforms the block matching tracking methods and the LK optical flows. For experiments, we used the 10 cases of $420 \times 512$ size 2D echocardiography data acquired using a Samsung Medison V10 ultrasound system (Seoul, Korea) and a phased array transducer P2-4BA (2-4 MHz). Each method was performed to track both of the endocardial and epicardial borders simultaneously using 26 tracking points. After that, LV contour was made by connecting the points using the natural cubic spline. We used the standard finite difference method to compute $u \cdot \nabla I + \frac{\partial I}{\partial t}$ of optical flows method. All the experiments were conducted using MATLAB 8.0 and computer (Intel Core i7-3770 CPU at 3.40 GHz and 16GB RAM) and the computational time was about 9 milliseconds at each frame.

In order to describe that the proposed method shows better performance than the previous method, we selected a representative case among the 10 cases of 2D echocardiography data. The LV tracking results on the representative case were depicted in Fig. 2. The LV border shown in the chosen case includes outer wall part of weak edges. Two contours were initially traced along the inside and outside of LV myocardial wall for endocardial and epicardial borders. Each contour was identified with 26 points. The first row shows manually traced LV contours by a clinical expert for images at ED, ES and next ED frames within the entire cycle. The next two rows exhibit the tracking results obtained by two tracking methods of our previous method and the proposed method. Overall, both of two tracking methods have tracked well dealing with the edge dropouts by weak signals. However, the performance difference was shown in the third column of Fig. 2. The contour obtained by performing the previous method did not reflect the endocardial border sufficiently, while the proposed method has tracked well
the endocardial border very approximately compared to the manual tracing contour. The proposed method kept regular distribution of tracking points and successfully have tracked local speckle patterns.

3.1. Assessment of LV border tracing. For quantitative evaluation of the performance of the proposed tracking algorithm, we used the Hausdorff distance $\varepsilon_H$ [27, 28] to compare the automated LV contours produced by algorithms with manually traced contours by a clinical expert. Here, the Hausdorff distance between the contour $C_1$ and $C_2$ is given by

$$
\varepsilon_H(C_1, C_2) = \max \left\{ \sup_{r_1 \in C_1} \left( \min_{r_2 \in C_2} \| r_1 - r_2 \| \right), \sup_{r_2 \in C_2} \left( \min_{r_1 \in C_1} \| r_1 - r_2 \| \right) \right\}.
$$

Fig. 3 shows the Hausdorff distances $\varepsilon_H$ between contours drawn manually and contours generated automatically for the entire cycle from an ED frame to the next ED frame. For the case used in Fig. 2, the Hausdorff distances $\varepsilon_H$ were computed for the whole heartbeat cycle. Blue dotted and red solid lines present the Hausdorff distances $\varepsilon_H$ computed for the tracking results of the previous and proposed tracking methods, respectively. We observed that the proposed method provides the smaller errors compared to the previous tracking.

![Figure 3](image)

**Figure 3.** Comparison of tracking performance using the Hausdorff distance $\varepsilon_H$. For the case used in Fig. 2, the Hausdorff distance $\varepsilon_H$ between the LV contours produced by two tracking methods and reference contours by manually tracing in the whole of images are computed.

3.2. Assessment of tracking errors for individual point. For more detailed performance evaluation of the proposed algorithm, we used an additional assessment method regarding on repeatability of local point along the forward and backward entire cardiac cycle. It was proposed in our previous work [18]. Let $t_R$ be a time interval of one heartbeat cycle from ED frame to the next ED frame. Using one cycle image $I(r, t)$, $0 \leq t \leq t_R$, we generate a forward-backward image defined by

$$
\bar{I}(r, t) = \begin{cases} 
I(r, t) & \text{if } 0 \leq t \leq t_R \\
I(r, 2t_R - t) & \text{if } t_R \leq t \leq 2t_R.
\end{cases}
$$

Let $\{r_1^{**}, \ldots, r_n^{**}\}$ be the set of tracking points returned at time $t = 2t_R$ by performing an automated tracking algorithm on the forward-backward image $\bar{I}(r, t)$.

Using the forward-backward image $\bar{I}(r, t)$, $0 \leq t \leq 2t_R$ and the returning tracking position $r_j^{**}$ at time $t = 2t_R$, we computed local tracking errors defined by the distance between the initial position $r_j^{*}$ and the corresponding returning position $r_j^{**}$:

$$
\text{Returning Tracking Error (RTE)} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} |r_i^{*} - r_i^{**}|^2}.
$$

(3.1)
For the representative case shown in Fig. 2, we obtained the RTE values of 1.3123 and 0.8134 pixels corresponding to the tracking results by previous and proposed methods, respectively. That is, both of overall and regional assessments of tracking results by using $\varepsilon_H$ and RTE support that the proposed method outperforms the previous method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Mean of errors</th>
<th>standard deviation of errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Previous method</td>
<td>2.2432</td>
<td>0.7037</td>
</tr>
<tr>
<td>Proposed method</td>
<td>1.0364</td>
<td>0.5648</td>
</tr>
</tbody>
</table>

**Table 1.** Comparison of tracking performance using the RTE. For the total experimental data set of 10 cases, the RTE values of the contour points produced by the two tracking algorithms are measured. (in pixels)

We computed the RTE values of the tracking points produced by the two tracking algorithms for the experimental data set of 10 cases. In Table 1, the mean and standard deviation of the RTEs obtained by the previous method and the proposed method were described. Table 1 presented that the proposed method provides improved performance compared with the conventional tracking methods.

**4. DISCUSSION AND CONCLUSION**

In this paper, we proposed a new tracking algorithm that satisfies simultaneously the requirements to alleviate the LV shape distortion and reflect realistic motion of endocardial contour. In order to improve the performance of our previous method related to detailed LV motion, we considered both of global and local deformations of non-rigid heart motion by applying global deformation modeled as affine transform to the contour partitively. It is carried out by partitioning the tracking points on the contour into a few small groups and determining each affine transform governing the motion of the partitioned contour points. To compute the coefficients of each affine transform, we modeled a least squares problem with equality constraints that are given by the relationship between the coefficients and a few contour points showing good tracking results. By doing this, the proposed method keeps regular distribution of tracking points and tracks local speckle patterns successfully by applying local deformation effects as well as compensating the poorly tracking points near endocardium with unclear speckle patterns or edge dropout by utilizing the tracking points reflecting well-tracking results by optical flows tracking method.

We have experimentally demonstrated that the proposed method is capable of performing robust real-time LV border tracking even in the presence of indistinguishable portions of the LV walls in echocardiography data. Especially, the numerical experiments showed that the proposed method achieves better tracking performance compared to our previous algorithm [18] that outperforms commonly used tracking methods including the block matching tracking methods and the LK optical flows.

We used the performance evaluation method for LV tracking that is based on the forward-backward tracking error estimation RTE as shown in section 3.2. The conventional evaluation
of global tracking performance using the delineated LV contours have some limitations in estimating errors of individual tracking points; in the case when tracking points erroneously move along LV border, the LV contour connecting the tracking points can not reveal those individual tracking errors. The forward-backward point tracking error estimation RTE provides a better local tracking performance assessment in the whole cycle.

APPENDIX

Least Squares with Equality Constraints. For \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^m, d \in \mathbb{R}^p, \) \( \text{rank}(B) = p, \) a constrained least square problem is expressed as a least squares problem with a solution \( x^* \):

\[
x^* = \min_x \| Ax - b \|_2
\]

with equality constraints:

\[
Bx = d.
\]

To solve this constrained least square problem, we compute the \( QR \)-factorization of \( B^T \):

\[
B^T = Q \begin{bmatrix} R \\ 0 \end{bmatrix}
\]

with

\[
Q \in \mathbb{R}^{n \times n}, R \in \mathbb{R}^{p \times p} \text{ and } 0 \in \mathbb{R}^{(n-p) \times p}.
\]

And then we consider the following identities:

\[
AQ = [A_1 A_2], Q^T x = \begin{bmatrix} y \\ z \end{bmatrix} \text{ with } A_1 \in \mathbb{R}^{n \times p} \text{ and } A_2 \in \mathbb{R}^{n \times (n-p)}. \tag{4.4}
\]

Using these identities, we compute \( Bx \) and \( Ax \):

\[
Bx = (Q \begin{bmatrix} R \\ 0 \end{bmatrix})^T x = [R^T 0] Q^T x = [R^T 0] \begin{bmatrix} y \\ z \end{bmatrix} = R^T y \tag{4.5}
\]

and

\[
Ax = (AQ) (Q^T x) = [A_1 A_2] \begin{bmatrix} y \\ z \end{bmatrix} = A_1 y + A_2 z. \tag{4.6}
\]

The problem given in the form of (4.1) hence becomes

\[
\min_{y, z} \| A_1 y + A_2 z - b \|_2 = \min_{y, z} \| A_2 z - (b - A_1 y) \|_2
\]

subject to

\[
R^T y = d. \tag{4.8}
\]

Since \( y \) is determined directly by (4.8), we can find the solution \( z \) of the least square problem (4.7) and finally obtain

\[
x = Q \begin{bmatrix} y \\ z \end{bmatrix}.
\]
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REFERENCES


COMPACTNESS AND DIRICHLET’S PRINCIPLE

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ABSTRACT. In this paper we explore the emergence of the notion of compactness within its historical beginning through rigor versus intuition modes in the treatment of Dirichlet’s principle. We emphasize on the intuition in Riemann’s statement on the principle criticized by Weierstrass’ requirement of rigor followed by Hilbert’s restatement again criticized by Hadamard, which pushed the ascension of the notion of compactness in the analysis of PDEs. A brief overview of some techniques and problems involving compactness is presented illustrating the importance of this notion.

Compactness is discussed here to raise educational issues regarding rigor vs intuition in mathematical studies. The concept of compactness advanced rapidly after Weierstrass’s famous criticism of Riemann’s use of the Dirichlet principle. The rigor of Weierstrass contributed to establishment of the concept of compactness, but such a focus on rigor blinded mathematicians to big pictures. Fortunately, Poincaré and Hilbert defended Riemann’s use of the Dirichlet principle and found a balance between rigor and intuition. There is no theorem without rigor, but we should not be a slave of rigor. Rigor (highly detailed examination with toy models) and intuition (broader view with real models) are essentially complementary to each other.

1. INTRODUCTION

The purpose of this article is to provide motivations, intuition, and applications for the notion of compactness. Rigor, intuition, and applications in mathematics are each important to the advancement of scientific knowledge. Intuition helps to invent theories, and methodological rigor ascertains their reliability. Emphasis on rigor should not overshadow a more conceptual overview or a work’s physical motivation. Simmons remarked [27] “Mathematical rigor is like clothing: in its style it ought to suit the occasion, and it diminishes comfort and restricts freedom of movement if it is either too loose or too tight.” Indeed, students majoring in mathematics learn the concept of compactness without knowing its physical or historical

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background. This type of educational approach prevents students from expanding their ability to conduct very complex or advanced science.

The concept of compactness and its introduction was highlighted by the famous debate between Riemann (1826-1866) and Weierstrass (1815-1897) regarding the convergence issue of the minimization problem in Dirichlet’s principle. Riemann used the Dirichlet principle as follows: If \( u \) is the solution of Dirichlet’s problem \( \Delta u = 0 \) in a bounded smooth domain \( \Omega \subset \mathbb{R}^3 \) with boundary data \( u|_{\partial \Omega} = \phi \in C(\partial \Omega) \), then \( u \) can be obtained by the limit of the minimizing sequence \( \{v_n\} \) of the energy functional

\[
f(v) := \int_{\Omega} |\nabla v|^2 \, dx \tag{1.1}
\]

within an admissible set such as \( A := \{ v \in C^2(\Omega) \cap C(\bar{\Omega}) : v|_{\partial \Omega} = \phi \} \). In short, Riemann’s observation was that \( f \) attains its minimum at a function \( u \) in the admissible class.

Riemann’s use of Dirichlet’s principle had been attacked by many mathematicians until the work of Hilbert [18, 24, 25], which was also insufficiently rigorous. In particular, Weierstrass strongly criticized Riemann’s use of the Dirichlet principle, because this method had no rigorous evidence of the existence of the limit of the sequence \( \{v_n\} \) within the admissible set. Moreover, in 1870, he presented an example of a 1D minimization problem whose minimizers do not exist: He considered the following energy functional over the set \( A = \{ v \in C^1([-1,1]) : v(-1) = 0 \text{ & } v(1) = 1 \} \):

\[
f(v) = \int_{-1}^{1} \left| x \frac{\partial}{\partial x} v(x) \right|^2 \, dx
\]

To be precise, the sequence given by

\[
v_n(x) = (\sin \frac{n \pi x}{2})^2 \chi_{[0,1/n]}(x) + \chi_{(1/n,1]}(x)
\]

is contained in the set \( A \). Given that \( \lim_{n \to \infty} f(v_n) \to 0 \), it is a minimizing sequence. However, the sequence \( v_n \) is bounded in \( W^{1,1}([-1,1]) \) because \( \|\frac{\partial v}{\partial x}\|_{L^1} = 1 \); thus, it will converge inside the larger space \( BV([-1,1]) \) of functions with bounded variation to the Heaviside function, which does not belong to \( A \).

This famous riposte is not a proper counterexample attacking Riemann’s use of the Dirichlet principle, because it corresponds to the degenerate partial differential equation (PDE)

\[
\frac{d}{dx} \left( x^2 \frac{d}{dx} u \right) = 0
\]

whose coefficient is not away from zero. (See Lax-Milgram theorem.) This degenerate PDE is very different from the Laplace equation having coercivity.

The problem lays with Riemann’s incorrect use of the Dirichlet principle. He had not understood a suitable hypothesis, and was therefore considered to be wrong. Weierstrass chose to attack and attempt to discredit Riemann on that principle, rather than correct the principle by considering the right hypothesis. Had he done so, he might have noticed that it is necessary for the space on which the minimizing problem is set to have some specific properties that guarantee the limit of the minimizing sequence to stay inside it. Weierstrass demonstrated
his prowess through his counter-example disproving Riemann’s result on the Dirichlet principle, but by failing to attempt to re-formulate the hypothesis in order to correct it, he also demonstrated his own limitations.

Weierstrass’s focused criticism of Riemann’s work - which sought a fatal blow rather than constructive refinement - prevented him from finding the right correction. His studies on analytic functions would lead to his achievements regarding the compactness of the minimizing sequence. His works on analytic functions and compactness were some of his main achievements in that period, along with the Bolzano–Weierstrass theorem.

Therefore, Riemann and Weierstrass’s confrontation over the tradeoff between rigor and intuition during their work on the Dirichlet principle was an early demonstration of the importance of the notion of compactness. Weierstrass’s meticulous approach might be connected to his prior teaching work, while Riemann’s earlier theological studies suggest an intuitive character able to forgo rigor for a more conceptual outlook.

Weierstrass may also have had a personal interest in discrediting Riemann, because Riemann had published a paper on the Abelian functions, taking the same title as an earlier work on that subject by Weierstrass and writing a substantially better contribution. Thus aggrieved, Weierstrass would later focus on undermining Riemann. Weierstrass criticized [6] “Riemann’s disciples are making the mistake of attributing everything to their master, while many had already been made by and are due to Cauchy, etc. Riemann did nothing more than to dress them in his manner for his convenience”. Weierstrass also pointed out the incorrectness of Cauchy’s belief that a continuous function is differentiable except at some isolated points. Many of Weierstrass’ results, including his example of a continuous non-differentiable function as well as his counterexample to Dirichlet’s principle, were motivated by his criticism of Riemann’s methods, and his distrust in Riemann’s results [6]. Weierstrass provided the following continuous function that is nowhere differentiable (except a set of measure zero): \( \phi(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{\pi n^2} \).

In the early 20th century, mathematicians again looked at the Dirichlet principle after long neglecting it. Hilbert improved it, correcting the statement by considering the right hypothesis on the domain and the boundary values.

Weierstrass’s rigor certainly contributed to establishing the concept of compactness. However, such focusing on rigor can blind mathematicians to the broader context of their work. Rigor should not prevent researchers looking at the big picture to find new approaches. In 1900, some leading mathematicians warned of narcissists practicing ”malignant self-love in narrow-scope mathematics”. Poincaré mentioned [15]: “For half a century there has been springing up a host of weird functions, which seem to strive to have as little resemblance as possible to honest functions that are of some use. No more continuity, or else continuity but no derivatives, etc. ... Formerly, when a new function was invented, it was in view of some practical end. Today they are invented on purpose to show our ancestors reasonings at fault, and we shall never get anything more out of them.”

There is no theorem without rigor, but we should not be a slave of rigor. Mathematicians’ work should not lend credence to the following well-known joke: Physicists think that the real world approximates equations; engineers think that equations approximate the real world;
mathematicians simply don’t see the relation between them. We should find a balance between rigor (detailed analysis using a toy model) and intuition (broader overview using a real model).

The following sections summarize some theories related to compactness, illustrating the importance of this notion in analysis. The examples are presented with limited details to maintain ease of reading.

2. DIRICHLET’S PRINCIPLE IN THE ABSENCE OF RIGOR

In the mid-nineteenth century, the Dirichlet principle was presented by Gauss and Dirichlet for the study of potential theory in electrostatics and Newtonian gravity. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) (or \( \mathbb{R}^2 \)) with its connected Lipschitz boundary. For a given \( \phi \in C(\partial \Omega) \), they considered a function in \( A_\phi := \{ v \in C^2(\Omega) \cap C(\overline{\Omega}) : v|_{\partial \Omega} = \phi \} \) which minimizes the integral

\[
 f(v) = \int_{\Omega} |\nabla v|^2. \tag{2.1}
\]

The following Dirichlet principle was stated without rigorous proof.

- Any function \( u \) minimizing \( f(v) \) satisfies \( \Delta u = 0 \) in \( \Omega \).
- Any function \( u \) which satisfies \( \Delta u = 0 \) in \( \Omega \) with \( u|_{\partial \Omega} = \phi \in C(\partial \Omega) \) is a minimizer.

However, the first statement is not completely accurate in terms of the existence of a minimizer. Indeed, the space \( C^2(\Omega) \) is not the right space in which to set the admissible set for the existence of a minimizer due to the lack of completeness with respect to the distance \( f(u_n - u_m) \) in terms of energy between two functions \( u_n \) and \( u_m \). In practice, the existence of a minimizer is obvious provided that the given system is physically existing or concerns observable quantities. Hence, it requires to find the correct space of physically meaningful functions with a proper distance concept. The correct space for solutions of the Laplace equation is the Sobolev space \( \{ v \in H^1(\Omega) : v|_{\partial \Omega} = \phi \} \) equipped with the norm \( \|v\| = \sqrt{\int_{\Omega} |v|^2 + |\nabla v|^2} \), which measures both the size and regularity of a function. Before the twentieth century, the Hilbert space \( H^1(\Omega) \) and measure theory had not been introduced; thus, there was insufficient knowledge to validate Dirichlet’s principle in a rigorous way. This issue of ascertaining the existence of a minimizer might have been a possible motivation for the development of the notion of compactness.

The second statement is also not correct. (Non-physical solutions should be excluded for the uniqueness.) Hadamard [23] gave the following counter example of a solution of Dirichlet’s problem in the unit disk \( \Omega \) whose energy blows up:

\[
u(x) = \sum_{n=1}^{\infty} 2^{-n} |x|^{2n} \cos(2^{2n} \theta) \quad (\tan \theta = x_2/x_1). \tag{2.2}\]

In this example, \( u \in A_\phi \) has the boundary data \( \phi = \sum_{n=1}^{\infty} 2^{-n} \cos(2^{2n} \theta) \) which is continuous but not differentiable almost everywhere. For a matter of fact, the following is an example of an harmonic function which does not have the standard maximum principle. Consider the two
dimensional function [32]:
\[ u(x) = (|x|^{-3/2} - |x|^{3/2})\sin\left(\frac{3}{2}\theta\right), \quad (\tan \theta = \frac{x_2}{x_1}) \]
defined in the domain
\[ \Omega := \{(r \cos \theta, r \sin \theta) : 0 < r < 1, 0 < \theta < \frac{3\pi}{2}\}. \]
Clearly, \( u \) is not a minimizer \((f(u) = \infty)\) although it satisfies
\[ \nabla^2 u = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0. \]
In this example, \( u \in C^\infty(\overline{\Omega} \setminus \{0\}) \) but \( u \notin C(\overline{\Omega}). \)

Let us deal with this issue of the existence of a minimizer from a mid-nineteenth century perspective by considering a minimizing sequence \( \{u_n\} \) in \( A_\phi \) such that
\[ \lim_{n \to \infty} f(u_n) = \inf_{v \in A_\phi} f(v). \]
Clearly, the existence of a minimizer essentially depends on whether a limit function of the set \( \{u_n : n = 1, 2, \ldots\} \) exists in \( A_\phi \). The \( L^2 \)-norm or the \( H^1 \)-norm were not known in the nineteenth century; thus, the distance between \( u_n \) and \( u_m \) was considered to be \( \|u_n - u_m\| = \sup_{x \in \Omega} |u_n(x) - u_m(x)| \). (According to the Pythagorean formula, the \( L^2 \)-distance between two functions resembles the Euclidean distance between two points in \( \mathbb{R}^n \). The sup-norm is an inappropriate distance to deal with the convergence issue.) Hence, \( \{u_n\} \) must be contained in the closed and bounded set
\[ K = \{v \in A : \|v\| \leq a\}, \quad (2.3) \]
where \( a = \sup_{x \in \partial \Omega} |\phi(x)| \). This leads us to ask the following questions about the compactness properties of \( K \).

- Does every sequence in the set \( K \) have a limit in \( K \)?
- Does \( f \) defined in \( K \) attain its minimum in \( K \)?

Before the development of Ascoli-Arzelà theorem in the late nineteenth century, the concepts of a sequence of functions being uniformly equicontinuous or uniformly convergent were unknown. (The sequence \( \{u_n\} \) has a limit in \( K \) provided the gradient \( \nabla u_n \) is uniformly bounded.) Without knowledge of \( H^1 \)-norm, it is very difficult to extract a suitable compactness for the function space.

The difficulties facing nineteenth century mathematicians trying to handle rigorously the space of functions led them to study oversimplified models (called the toy models) of the admissible set. They studied the closed interval \([0, a]\) instead of the admissible set \( K \) in (2.3). The following basic compactness properties were obtained using the toy models.

- Every continuous function \( f \) on the closed bounded interval \([0, a]\) attains its minimum because of its compactness property (Bolzano-Weierstrass [37, 31]).
  - The \([0, a]\) has no hole, that is, every open cover has a finite subcover (Heine-Borel [31]).
  - Every sequence in \([0, a]\) has at least one limit point in \( K \) (Fréchet’s approach involving nested intersection).
• This compactness concept can be extended to any closed and bounded set in n-dimensional Euclidean space. Basically, any sequence in a closed and bounded domain contains a convergent (Cauchy) subsequence because any compact domain can be divided into a finite number of subdomains whose diameters are less than the half of its diameter.

However, this compactness property cannot be applied to the closed and bounded set $K$ in (2.3) which lies in infinite dimensional space. Indeed, $K$ is not compact, that is, there exist a sequence $\{v_n\}$ in $K$ that does not have a limit in $K$. Moreover, there exist many examples of bounded sequence $\{v_n\} \subset K$ such that $f(v_n) \to \infty$. (For example, $v_n := \cos n|x|$ is bounded by 1, but $f(v_n) = \int_\Omega |
abla \cos n|x||^2\,dx \to \infty$.)

Hilbert restated Dirichlet’s principle in 1900, and by the following year he published two papers that effectively revived it. His statement was based on the hypothesis that both the boundary data and the boundary $\partial\Omega$ are analytic. Nevertheless, Hadamard criticized the instability of his statement since his approach cannot be extended to the case of only continuous boundary data $\phi$. To be precise, let $\Omega$ be the unit disc and $\phi$ the boundary data in Hadamard’s example (2.2). There exist a sequence of analytic functions $\{\phi_n\}$ such that $\phi_n \to \phi$ in $L^\infty$ sense. Clearly, $\phi_n$ has an extension $\tilde{\phi}_n \in C^2(\overline{\Omega})$ to the domain $\Omega$. Writing $g_n = \Delta \tilde{\phi}_n$, the Dirichlet problem is changed into the following Poisson’s equation:

$$-\Delta(u_n - \tilde{\phi}_n) = g_n \quad \text{in } \Omega \quad \text{with} \quad (u_n - \tilde{\phi}_n)|_{\partial\Omega} = 0 \quad (2.4)$$

The corresponding energy functional is

$$f(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \int_\Omega g_nv$$

within the set $\{v \in C^2(\overline{\Omega}) : v|_{\partial\Omega} = 0\}$. According to Hilbert’s statement, there exists a solution $u_n - \tilde{\phi}_n$ to Poisson’s equation which is a minimizer of the corresponding energy functional with the source term $g_n$. This solution can be written as

$$u_n - \tilde{\phi}_n = \Delta^{-1}g_n.$$ 

Since $\Delta^{-1}$ is a compact operator in the Sobolev space $H^{-1}(\Omega)$, the sequence $u_n$ seems to have a limit as far as $g_n$ is bounded in $H^{-1}(\Omega)$. Due to Ascoli-Arzelà theorem, boundedness of $g_n$ leads to the uniform equicontinuity of the sequence $\{u_n - \tilde{\phi}_n\}$. However, in the example of Hadamard, $\phi$ does not have an extension to $H^1(\Omega)$, and therefore $g_n$ cannot be bounded in $H^{-1}(\Omega)$. A more complete proof requires the condition $\phi \in H^{1/2}(\partial\Omega)$ in order to have $\|\tilde{\phi}_n - \tilde{\phi}\|_{H^1(\Omega)} \to 0$, whereas the Sobolev spaces were not introduced yet until that period.

Before ending this section, let us mention the mathematical term well-posedness stemmed from Hadamard [22]. Mathematical models of physical phenomena require to meet the following three properties: (1) Existence: at least one solution exists. (2) Uniqueness: only one solution exists. (3) Continuity or stability: a solution depends continuously on the data.
3. SOBOLEV COMPACT EMBEDDINGS AND COMPENSATED COMPACTNESS

Sobolev spaces, named for the Russian mathematician Sergei Lvovich Sobolev [34], constitute one of the main function spaces used in the study of PDEs. The space $W^{k,p}(\Omega)$ when $k \in \mathbb{N}$ and $p \geq 1$ is the space of $L^p(\Omega)$ functions for which any distributional derivative of the order $\alpha$ is in $L^p(\Omega)$ for any multi-index $\alpha$ such that $|\alpha| \leq k$. When endowed with the norm
\[
||u||_{W^{k,p}} = \left( \sum_{|\alpha| \leq k} ||\partial^{\alpha} u||_{L^p}^p \right)^{\frac{1}{p}},
\]
they are Banach spaces, and in the special case $p = 2$ where $W^{k,2}(\Omega) = H^k(\Omega)$, it is an Hilbert space endowed with the scalar product
\[
\langle u, v \rangle_{H^k} = \sum_{|\alpha| \leq k} \langle \partial^{\alpha} u, \partial^{\alpha} v \rangle_{L^2}.
\]

When considering minimizing sequences in the resolution of a PDE, belonging to a Sobolev space, some compactness properties are required for the convergence of these sequences. Using the space $W^{1,p}(\Omega)$, with $1 < p < +\infty$ being reflexive, we can thus extract from any bounded sequence a weak convergent subsequence.

By definition, weak convergence is the convergence of a sequence to some limit that holds through the dual space when multiplying in an appropriate way by any element of the dual space. For Sobolev spaces, the dual space may be difficult to identify with another function space. Therefore, we consider the following characterization of weak convergence: a sequence $u_n$ converges weakly to $u$ in $W^{1,p}(\Omega)$, $1 < p < +\infty$, if and only if $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$ and all the partial derivatives $\frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i}$ weakly in $L^p(\Omega)$. Note that the weak convergence in $L^p(\Omega)$ is clearly defined via Riesz theorem, because we have that the dual of $L^p(\Omega)$ is $L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We also notice analogous weak convergence in the case $p = +\infty$, thus adding a star, due to $L^1(\Omega)$ being a strict subspace of the dual of $L^\infty(\Omega)$, which gives the weak$^*$ convergence of a sequence $u_n$ to $u$ in $L^\infty(\Omega)$, denoted as $u_n \rightharpoonup^* u$, if and only if
\[
\int_{\Omega} u_n \varphi \to \int_{\Omega} u \varphi
\]
for every $\varphi \in L^1(\Omega)$. The case of bounded sequences in $W^{1,1}(\Omega)$ will be evoked in the following sections when discussing the behavior of energies with linear growth.

Moreover, the following Sobolev compact embeddings are given by the Rellich-Kondrachov theorem [30, 28], which says that when $\Omega$ is a bounded regular domain in $\mathbb{R}^N$ then the embeddings of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ with $1 \leq q < \frac{NP}{N-p}$ for $1 \leq p < N$ or $1 \leq q < +\infty$ for $p = N$, are compact. When $p > N$, then $W^{1,p}(\Omega)$ is compactly embedded in $C^{0,\gamma}(\overline{\Omega})$, the space of Hölderian functions of order $0 \leq \gamma < 1 - \frac{N}{p}$. The compact embedding meaning that any bounded sequence in $W^{1,p}(\Omega)$ will contain a convergent subsequence in the larger space.

As a consequence of the Rellich-Kondrachov theorem in the case $p = 2$, if a sequence $u_n$ is bounded in $H^1(\Omega)$ converging weakly to some $u$ in $L^2(\Omega)$ and if $v_n$ is bounded in $L^2(\Omega)$
converging weakly to some \( v \) in \( L^2(\Omega) \), then the product \( u_nv_n \) converges to \( uv \) weakly in \( \mathcal{D}'(\Omega) \), the dual space of \( C_c^\infty(\Omega) \), i.e.

\[
\int_\Omega u_nv_n \varphi \to \int_\Omega uv \varphi
\]

for every \( \varphi \in C_c^\infty(\Omega) \). Murat-Tartar theory [29, 36] of compensated compactness permits to obtain the same result when we do not control all the partial derivatives of \( u_n \) but controlling some other partial derivatives of \( v_n \) in a complementary way. For example, if \( u_n, v_n : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \) satisfy

\[
\|u_n\|_{L^2(\Omega; \mathbb{R}^2)} + \|\text{div} u_n\|_{L^2(\Omega)} \leq \alpha \quad \text{and} \quad \|v_n\|_{L^2(\Omega; \mathbb{R}^2)} + \|\text{rot} v_n\|_{L^2(\Omega; \mathbb{R}^2)} \leq \beta
\]

for constants \( \alpha, \beta > 0 \), and if \( u_n \rightharpoonup u, v_n \rightharpoonup v \) weakly in \( L^2(\Omega; \mathbb{R}^2) \), then the product \( u_nv_n \) converges to \( uv \) weakly in \( \mathcal{D}'(\Omega) \).

The compensated compactness was used also in the study of the convergence of Dirichlet problems with strongly oscillating coefficients, namely the convergence of the solutions of

\[
\begin{align*}
\text{div}(A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x)) & = f \quad \text{for} \quad x \in \Omega \\
u_\varepsilon & = 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

when \( \varepsilon \) goes to zero. For \( f \in H^{-1}(\Omega) \) and \( A \) verifying some growth and coercivity conditions, we have \( \sup_{\varepsilon > 0} \|u_\varepsilon\|_{H^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \) and \( u_\varepsilon \) has a weak-limit. However, \( A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) \) may not have any weak-limit. The compensated compactness theory can be used to get a weak-limit.

4. STONE-WEIERSTRASS, ASCOLI-ARZELÀ AND SHANNON SAMPLING THEOREMS

Let us look into the main result behind the Sobolev compact embeddings. Two of the fundamental theorems in analysis are the Stone-Weierstrass Theorem and the Ascoli-Arzelà theorems. These results are very important in the study of sets of continuous functions. The first theorem permits the assertion that the polynomials are dense in the space of the continuous functions and thus enables the possibility to approximate a continuous function with a sequence of polynomial functions that are very regular and easy to handle.

The Stone-Weierstrass [42, 35] theorem states that if \( \mathcal{A} \) is an algebra containing the constant function 1 in the space \( C(X) \) of continuous functions defined on a compact metric space \( X \), then \( \mathcal{A} \) is dense in \( C(X) \) if and only if for every \( x, y \in X \) there exists \( f \in \mathcal{A} \) such that \( f(x) \neq f(y) \). It is then easy to verify that the family of polynomials has the right assumptions to conclude the result mentioned above.

The Ascoli-Arzelà [3, 2] theorem states that a subset of \( C(X) \) is relatively compact if and only if it is bounded and equicontinuous. This theorem could be useful in resolving some kind of problems with some compactness properties, which can be approximated by a sequence of problems for which we have more information. The extraction of a convergent subsequence would then give the solution for the initial problem.

It is the case in Péano theorem for the resolution of the Cauchy problem that consists in considering two open domains \( I \subset \mathbb{R} \) and \( \Omega \subset \mathbb{R}^N \), \( (t_0, y_0) \in I \times \Omega \). Then, using Ascoli-Arzelà
compactness theorem, we can find a neighborhood $J$ of $t_0$ in $I$, and a function $y : J \rightarrow \Omega$ solution of the equation

$$\begin{cases}
y'(t) = F(t, y(t)), & t \in J \\
y(t_0) = y_0,
\end{cases}$$

where $F : I \times \Omega \rightarrow \mathbb{R}^N$ is continuous.

Ascoli-Arzelà theorem is also the basis of the Sobolev compact embeddings. In fact, the case $p > N$ is a direct consequence, while the case $p \leq N$ uses the $L^p$-version of it which is the Riesz-Fréchet-Kolmogorov theorem. This theorem asserts that if a bounded family $F$ in $L^p(\Omega)$ is such that for an $\omega \subset \subset \Omega$ we have $\forall \varepsilon > 0$, $\exists 0 < \delta < \text{dist}(\omega, \Omega)$ such that $||\tau_h f - f||_{L^p(\omega)} < \varepsilon$ $\forall |h| < \delta$, $f \in F$, with $\tau_h f(x) = f(x + h)$, then $F|_\omega$ is relatively compact in $L^p(\omega)$. Its proof being in its turn based on the Ascoli-Arzelà theorem.

The proof in Ascoli-Arzelà theorem relies on the fact that every compact metric set contains a countable dense subset. It is on this set that the uniformly continuous limit application and the extracted sequence are constructed. There is, therefore, some discrete-to-continuous procedure behind this compactness theory. It appears also in completely non-related problems where we expect similar ideas of compactness to be hidden.

Sampling theory in signal processing was first derived for communication signal processing in the first half of the twentieth century. The Shannon sampling theorem [33] states that if a continuous $L^1$ function defined on $\mathbb{R}$ has bounded width $B$, that is the support of the Fourier transform $\text{supp} \hat{u} \subset [-B, B]$ (no frequencies higher than B), $u$ is completely determined by giving its ordinates at a series of points, called a sample, whose points are separated by a distance smaller than $\frac{1}{2B}$ (2B being the sample rate called Nyquist rate). In fact, when the frequencies are less than $B$, the ordinates of $u$ in points spaced by $\frac{1}{2B}$ gives the Fourier coefficients of $\hat{u}$ which gives $\hat{u}$ itself and thus $u$.

The set $A = \{v \in C([0, 1]) : \text{supp} \hat{v} \subset [-B, B], \sup |v| \leq 1\}$ is compact according to Bolzano-Weierstrass and Shannon sampling theorem. Since any function in the set $A$ can be viewed as a vector in a finite dimensional space with the dimension higher than $\frac{1}{2B}$, $A$ can be considered as a closed bounded set in the finite dimensional space. The assumption on the bandwidth enable us to control the oscillations of the function preventing them from being too fast (or too high) within a distance smaller than $\frac{1}{2B}$. Exactly the same principle applies to compactness. The equicontinuity assumption in the Ascoli-Arzelà theorem also prevents such behavior of the oscillations. This means that the notion of compactness is also behind the sampling process by identifying infinite-dimensional function spaces as finite dimensional vector spaces when controlling the oscillations or the variations.

5. COMPACT OPERATOR AND LAYER POTENTIALS FOR LAPLACE EQUATION

In the 1830s, Gauss and Weber [20] began collaborating on the theory of terrestrial magnetism. Weber was a physics professor in Göttingen who wrote important contributions on electricity. They used the harmonic property of magnetic potential to compute the horizontal intensity of the magnetic field; the magnetic potential is dictated by the Laplace equation from
Gauss’s law ($\nabla \cdot \mathbf{B} = 0$ for magnetic field $\mathbf{B}$) and divergence-free magnetic induction. They mentioned Dirichlet’s principle without proof. Gauss used a layer potential approach to handle the Dirichlet problem. An electrical potential $u$ in electrostatics can be expressed as a single layer potential

$$ S\rho(r) = \int_{\partial\Omega} \frac{1}{4\pi|r - r'|} g(r') \, ds_{r'} \quad (r \in \mathbb{R}^3 \setminus \partial\Omega) \quad (5.1) $$

where $\rho$ represents the distribution of electric charge. Given that $S\rho$ satisfies $\Delta u = 0$ in $\mathbb{R}^3 \setminus \partial\Omega$, we have

$$ \int_{\partial\Omega} \rho \, S\rho \, ds = \int_{\mathbb{R}^3 \setminus \partial\Omega} |\nabla S\rho|^2. $$

Indeed, the Dirichlet problem with the boundary data $\phi$ can be reduced to finding $\rho$ such that $S\rho = \phi$ on $\partial\Omega$. However, solving this integral equation computationally is not simple at all due to compactness of the operator $S$ on $L^2(\partial\Omega)$. Let us consider the computational issue of solving the Dirichlet problem. According to the Riemann mapping theorem, for any simply connected open set $\Omega$ in complex domain $\mathbb{C}$, there exists a conformal mapping $\Phi$ from $\Omega$ to the unit disk. Noting that complex analytic functions have their real and imaginary parts obeying the Cauchy-Riemann equations, the two dimensional Dirichlet problem in any simply connected domain ($\nabla^2 u = 0$ in $\Omega$ with $u|_{\partial\Omega} = \phi$) can be reduced (through the conformal mapping) to solve the much simpler Dirichlet problem in the unit disk; due to the rectangular geometry of the unit disc in polar coordinates, Fourier analysis and separable variable technique enable us to solve the corresponding Dirichlet problem constructively. At a first glance, it appears that conformal mappings can be effectively used for computing solutions to the two dimensional Dirichlet problem in a simply connected domain. Unfortunately, this is not true when considering the fact that the Poisson kernel (the normal derivative of Green’s function or Randon-Nikodym derivative of the harmonic measure with respect to the surface measure over $\partial\Omega$) is very sensitive (and highly nonlinear) to any perturbation of the geometry of $\partial\Omega$. Except in some special cases, it is impossible to find a reliable method of identifying conformal mapping. In short, difficulty in identifying conformal mapping is equivalent in difficulty to finding the Green function.

In the contrast to Gauss’s use of single layer potential in solving Dirichlet problem, Neumann used double layer potential:

$$ D\rho(r) = \int_{\partial\Omega} \frac{\langle r' - r, \mathbf{n}(r') \rangle}{4\pi|r - r'|^3} \rho(r') \, dS_{r'} \quad (r \in \mathbb{R}^3 \setminus \partial\Omega) \quad (5.2) $$

The reason is that $D_{\Omega}\phi$ satisfies the Laplace equation $\nabla^2 u = 0$ in $\Omega$ and solving the Dirichlet problem is reduced to find $\phi$ such that

$$ \rho(r) = \lim_{t \to 0^+} D_{\Omega\phi}(r - t\mathbf{n}(r)) = \left(\frac{1}{2} I + K\right)\rho(r), \quad (r \in \partial\Omega) \quad (5.3) $$
where $I$ is the identity operator and $\mathcal{K}$ is the trace operator given by

$$
\mathcal{K}\rho(r) = \int_{\partial\Omega} \frac{\langle r' - r, n(r') \rangle}{4\pi|r - r'|^3} \rho(r') \, dS_{r'} \quad \text{for } r \in \partial\Omega.
$$

(5.4)

In the case where $\Omega$ is smooth and convex, the solution $u$ can be expressed as Neumann series

$$
u(r) = 2 \sum_{n=0}^{\infty} D((-2\mathcal{K})^n \phi)(r) \quad (r \in \Omega).
$$

(5.5)

The Calderón-Zygmund school of harmonic analysis [9] made a significant contribution in developing layer potential techniques for solving Dirchlet problem. For the ease of explanation, $\Omega$ is assumed to be three dimensional domain with its connected Lipschitz boundary $\partial\Omega$. The boundary value problem of the Laplace equation can be solved by double layer potential.

If the operator $\frac{1}{2}I + \mathcal{K}$ is invertible in a ”proper” Banach space, the solution of the Dirichlet problem

$$\nabla^2 u = 0 \quad \text{in } \Omega \quad \text{with } u|_{\partial\Omega} = \phi
$$

can be expressed as

$$u(x) = D(\frac{1}{2}I + \mathcal{K})^{-1} \phi(x) \quad \text{in } \Omega.
$$

If $\partial\Omega$ is $C^1$, $\mathcal{K}$ is a compact operator on $L^2(\partial\Omega)$ so that we can apply Fredholm theory[14].

- Since the operator $\mathcal{K} : L^2(\partial\Omega) \rightarrow L^2(\partial\Omega)$ is compact, for every bounded sequence $\{\rho_n\}$ in $L^2(\partial\Omega)$, the sequence $\{\mathcal{K}\rho_n\}$ has a limit in $L^2(\partial\Omega)$.

- There exist eigenvalues $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ and the corresponding orthonormal eigenfunctions $\{\rho_n : n = 1, 2, \cdots\}$ of $\mathcal{K}^* \mathcal{K}$ such that

$$\mathcal{K}^* \mathcal{K}\rho = \sum_n \lambda_n^2 \langle \rho, \rho_n \rangle \rho_n.
$$

- If the range of $\mathcal{K}$ is infinite dimensional, then $\lim_{n \rightarrow \infty} \sigma_n = 0$.

- Hence, $\frac{1}{2}I + \mathcal{K} \approx \frac{1}{2}I$ except a finite dimensional space $X_N$ spanned by eigenfunctions $\{\rho_n : n = 1, 2, \cdots, N\}$ for large $N$. The invertibility of $\frac{1}{2}I + \mathcal{K}$ on $X_N$ is like invertibility of $N \times N$ matrix.

When $\partial\Omega$ is only in $C^{0,1}$, $\mathcal{K}$ is no more compact so that Fredholm theory can not be applied.

Until 1980, the Calderón-Zygmund school worked toward the proof of the boundedness of the trace operator $\mathcal{K}\Omega$ on $L^2(\partial\Omega)$ which was solved by a deep knowledge on the harmonic analysis [11]. It turns out that the boundedness of $\mathcal{K}$ on $L^2(\Omega)$ is equivalent to $\mathcal{K}1 \in BMO$ [12]. In 1984, the invertibility of $\frac{1}{2}I + \mathcal{K}$ on $L^2(\partial\Omega)$ was proven using Rellich type identity that substitutes the compactness [41].

6. $\Gamma$-CONVERGENCE AND COMPACTNESS

Another example where compactness is involved in variational problem is illustrated with the $\Gamma$-convergence theory. In the 1970s, Ennio de Giorgi [13] introduced the notion of $\Gamma$-convergence, a kind of convergence that was mainly used in problems related to the calculus of
variations. This convergence gives the limit of a sequence of minimizing problems instead of computing only the limit of a sequence of functional. It was very useful in treating several variety of problems like relaxation, homogenization, dimension reduction etc.. The limit problem having the advantage of being relaxed which guarantees the existence of solution even when there is no solution for the initial minimizing problems.

We recall that a sequence of functions $F_n$ from a metric space $X$ into $\mathbb{R}$ is said to $\Gamma$-converge into $F$ for the topology of $X$ if the following two conditions are satisfied for every $x \in X$:

$$\begin{align*}
\forall x_n \to x, \liminf F_n(x_n) &\geq F(x) \\
\exists y_n \to x, \lim F_n(y_n) &= F(x).
\end{align*}$$

The $\Gamma$-convergence has mainly two compactness properties. The first concerns the set of functions from $X$ into $\mathbb{R}$ which have a sequential compactness property with respect to $\Gamma$-convergence, in the sense that any sequence $F_n : X \to \mathbb{R}$ admits a $\Gamma$-convergent subsequence. The $\Gamma$-convergence verifies also the Urysohn property of convergence structures, in the sense that a sequence of functions $F_n : X \to \mathbb{R}$ $\Gamma$-converges to a function $F$ if and only if, every subsequence of $F_n$ contains a further subsequence which $\Gamma$-converges to $F$. In the practice, once we obtain the $\Gamma$-limit for the subsequences and verify that it does not depend on the chosen subsequence, we can conclude to the $\Gamma$-convergence of the initial sequence.

The second property concerns the minimizers of a sequence $F_n : X \to \mathbb{R}$. It states that if the minimizers stay in a compact set of $X$ for all $n$, then, their limit points are minimizer of the $\Gamma$-limit $F$. The proof of this result is very simple and uses only the definition (6.1).

The computation of the $\Gamma$-limit of a sequence of functions consists in two fundamental steps. The first dealing with the lower bound proving the first statement in the definition (6.1) for any sequence $x_n \to x$. The second consists in constructing the recovery sequence $y_n$ appearing in the second statement of the definition (6.1) which permits to have the upper bound for the $\Gamma$-limit and thus reaching the result combining the two steps.

In the first step, we consider a sequence $x_n \to x$ in the topology of $X$ and we suppose that $F_n(x_n)$ is bounded otherwise the result is obvious. This boundedness combined with eventual coercivity properties of the functions $F_n$ implies the compactness of the sequence $x_n$ in some function space, thus the convergence of the sequence in some sense that permits to obtain the desired inequality and then the lower bound. Thus, the importance of having some coercivity properties realized by the sequence of functions $F_n$.

It is worthwhile to mention that in some problems, as in dimension reduction, the $\Gamma$-convergence technique replaced the usual method of formal asymptotic expansion which lies on an Ansatz supposing that the deformations has special form. The $\Gamma$-convergence procedure does not suppose such an Ansatz and thus gives more rigorous results.

An example underlying the importance of compactness in problems involving $\Gamma$-convergence techniques, is about relaxation or dimension reduction for energies of the type

$$E(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) dx$$
with \( u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m \), \( \nabla u \) its gradient, when \( f \) has linear growth, i.e. there exist \( \alpha, \beta > 0 \) such that
\[
\alpha \|A\| \leq f(x, u, A) \leq \beta (1 + \|A\|),
\]
for every \( (x, u, A) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{m \times N} \). Minimizing sequences with finite energy will be bounded in the Sobolev space \( W^{1,1}(\Omega; \mathbb{R}^m) \). However, because of the lack of reflexivity of \( W^{1,1}(\Omega; \mathbb{R}^m) \), such minimizing sequences will only be relatively compact in the larger space \( BV(\Omega; \mathbb{R}^m) \) of functions with bounded variation firstly invoked by Jordan [26] in the scalar case, then by Tonelli [40] in the vectorial case. It is the space of \( L^1 \) functions whose first distributional derivatives are Radon measures. Thus, the limit energy, in both cases of relaxation or dimension reduction, will be decomposed for every \( u \in BV(\Omega; \mathbb{R}^m) \) into three parts, according to the Besicovitch decomposition Theorem, written with respect to the three mutually singular measures; \( \mathcal{L}^N, \mathcal{H}^{N-1}_{|J_u} \) and \( D^c u \), where \( \mathcal{L}^N \) is the Lebesgue measure in \( \mathbb{R}^N \), \( \mathcal{H}^{N-1}_{|J_u} \) is the \((N - 1)\)-dimensional Hausdorff measure restricted to the jump set of \( u \) and \( D^c u \) the Cantor part of \( Du \). Thus, underlying how much the compactness can change the issue of the result.

7. Conclusion

A broad conclusion of this work is that rigor and intuition are both fundamental to proper mathematical computing, both in teaching and in research. They should never be separated, but they must be used in the right way. As we have seen, the intuition in the work of Riemann led to several important discoveries and results, while the rigorous approach taken by Weierstrass in criticizing Riemann’s result led to other interesting studies and results. Weierstrass’s meticulous criticism not only identified a wrong result, eventually prompting its correction, but its implications also aided other works, some in unrelated areas. Compactness is one of the most important notions that emerged from the study of the Dirichlet principle and its correction. It has since found applicability in several mathematical fields related to many important problems.

In treating mathematical problems, especially new ones, and when trying to advance on them, the first step should be intuitive. Intuition should provide either the correct result or something closely resembling it. If the result is not perfect, rigorous study should point out its flaws and identify the necessary corrections. This should also be the way in which mathematics is studied. There is often a key point in a long proof of a result. This key point is usually not reached by systematic rigor, but often through intuition. Good intuition is honed by practice, but needs curiosity and imagination. Therefore, it is important for scientists to take an interest in fields not necessarily related to their work. This interest can lend a perspective on the broader context of their work, thus stimulating the innovative ideas required to solve their own difficult problems.

This article, however, should not give a wrong message that rigor is relatively less important than intuition. Mathematical rigor is important tool to destroy wrong intuition where mathematicians have advantage over physicists and engineers. Intuition can be obtained by everlasting effort to try to find connection between mathematical output and relevant physical problems based on rigorous mathematics.
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REFERENCES


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