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HOW TO PREPARE FOR RETIREMENT?
OPTIMAL SAVING, LABOR SUPPLY, AND INVESTMENT STRATEGY

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ABSTRACT. In this paper we study consumption-labor supply decision of an agent who prepares for retirement at a known time in the future. The agent is assumed to have a preference which is represented by the von Neumann-Morgenstern utility function in which the felicity function has constant relative risk aversion over the composite of consumption and leisure. The composite is obtained by the Cobb-Douglas function. A general problem has been studied by Bodie et al. (2004). We contribute to the literature by deriving the Slutsky equations and conducting comparative statics. In particular, we show that wealth effect can exhibit an interesting property depending upon the time until retirement, as the interest rate increases.

1. INTRODUCTION

A mathematical method to derive the optimal consumption and investment in one-period model has been studied by Markowitz(1952) [1]. His mean-variance portfolio selection model was simple, and hence had strong impact on the financial market although it ignores many things in the real market. Merton(1969) [2] and Samuelson(1969) [3] have improved Markowitz’s model to the many period life cycle model and shown the explicit solution of the optimal intertemporal consumption and portfolio. Merton and Samuelson used the stochastic dynamic programming to solve their problem.

On the above quantitative models, the lifetime consumption-leisure choice problem has been developed by Bodie et al. (1992) [5]. And Bodie et al. (2004) [7] have improved the consumption-leisure/labor decision problem with habit formation, stochastic opportunity set, stochastic wages and labor supply flexibility.

The model that implicitly selects the time to retire has been studied by Choi et al. (2008) [8]. They used the constant elasticity of substitution (CES) function combined with a CRRA utility.

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We study the optimal consumption and leisure choice of an individual who has a CRRA utility function with retirement in continuous time model. Our model is one-case of Bodie et al. (2004) [7] with no habit formation and constant wages.

We assume that an individual knows his/her retirement time. And we contribute to derive the Slutsky equation which helps explain the causes of economic factors. The substitution effect and income effect of the optimal consumption and labor is calculated in the market where the risky asset is not tradable. The method to derive the Slutsky equation has been shown by Grandville(1989) [4].

Section 2 presents a model for continuous-time optimal consumption, leisure/labor, and portfolio selection problem where the tradable asset is the only risk-free asset. Our agent is assumed to have a composition of CRRA utility and Cobb-Douglas type utility showing substitutability of consumption and leisure. We use the Lagrangian method to calculate the optimal choice values.

In Section 3, by using the component of shadow price, we describe an explicit form of the solution and the Slutsky equation to decompose the effects of the interest rate change into the income effect and substitution effect. We use the property of the dual value function to calculate the Slutsky equation.

Section 4 introduces a model with stock investment. Section 5 gives a closed form solution of the model in Section 4.

2. THE MODEL

We consider a consumption and portfolio selection problem when an agent prepares for retirement. We assume that the agent has a time separable von Neumann-Morgenstern utility, so that he/she will try to maximize his/her utility given initial wealth amount, $X(0)$, as follows:

$$ V(X(0)) = \max_{\{c(t), \ell(t)\}} \left[ \int_0^T e^{-\rho t} u(c(t), \ell(t)) dt + e^{-\rho T} U(X(T)) \right] $$

(2.1)

where $c(t) \triangleq \{c(s) | t \leq s \leq T\}$, $\ell(t) \triangleq \{\ell(s) | t \leq s \leq T\}$ denote the agent’s consumption stream and leisure choice, respectively.

We assume that there is a tradable asset in the form of bond. Then, the wealth evolution equation is given by

$$ dX(t) = \left( rX(t) - c(t) + w(t)(\bar{L} - \ell(t)) \right) dt $$

(2.2)

where $\bar{L} - \ell_t$ is interpreted as the labor that the agent supplies, and $w(t)$ is the wage rate of the labor supply.
3. Main Results

Lemma 3.1. For any given initial wealth level, \( X_0 \), at time 0, the wealth process (2.2) satisfies the following equation:

\[
X_0 = e^{-rT}X(T) + \int_0^T e^{-rt}\left(c(t) - w(t)(\mathcal{L} - \ell(t))\right)dt. \tag{3.1}
\]

Proof. We can change equation (3.1) into the following form:

\[
e^{-rt}dX(t) - re^{-rt}X(t)dt = e^{-rt}\left(-c(t) + w(t)(\mathcal{L} - \ell(t))\right)dt
\]

where LHS equals to the derivative of \( e^{-rt}X(t) \). Thus, if we take an integral between \( t \) and \( T \), we get

\[
e^{-rt}X(t) = e^{-rT}X(T) + \int_t^T e^{-rs}\left(c(s) - w(s)(\mathcal{L} - \ell(s))\right)ds.
\]

Now, we assume that the form of utility function and the bequest function are all of the constant relative risk aversion (CRRA) type:

\[
u(c, \ell) = v\left(c^{\alpha}(\ell^{1-\alpha})\right), \quad U(X(T)) = Kv(X(T)),
\]

where \( v(x) = \frac{x^{1-\gamma}}{1-\gamma} \). Using Lemma 3.1, we get the following result.

Proposition 3.2. We obtain the optimal consumption, \( c^*_t \), and the optimal labor choices, \( \ell^*_t \), the optimal level of terminal wealth, \( X^*(T) \), to the optimization problem (2.1) as follows:

\[
c^*(t) = A\left(\lambda e^{(\rho-r)T}\right)^{-1/\gamma}w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}, \tag{3.2}
\]

\[
\ell^*(t) = \frac{1 - \alpha}{\alpha w(t)}c^*(t), \tag{3.3}
\]

\[
X^*(T) = \left(\frac{\lambda e^{(\rho-r)T}}{K}\right)^{-1/\gamma}, \tag{3.4}
\]

where

\[
A = \left(\alpha \left(\frac{1 - \alpha}{\alpha}\right)^{(1-\alpha)(1-\gamma)}\right)^{\frac{1}{\gamma}}. \tag{3.5}
\]

The Lagrange multiplier to the budget constraint, \( \lambda \), can be obtained by plugging \( c^*(t), \ell^*(t), \) and \( X^*(T) \) into (3.1).

If \( w(t) \) is given with \( w(t) = w_0 e^{gt} \), then the Lagrange multiplier, \( \lambda^* \), to the budget constraint is given by the following equation:

\[
(\lambda^*)^{\frac{1}{\gamma}} = \left(\alpha \left(\frac{1 - \alpha}{w_0}\right)^{1-\alpha}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\mathcal{R} + \mathcal{G}} \left(1 - e^{-\mathcal{R} + \mathcal{G}T}\right) + \frac{1}{\gamma} \frac{K}{\mathcal{R} - \mathcal{G}} e^{-\frac{\mathcal{R}T}{\gamma}} X_0 + \frac{w_0 L}{\mathcal{R} - \mathcal{G}} \left(1 - e^{-\frac{(\mathcal{R} - \mathcal{G})T}{\gamma}}\right), \tag{3.6}
\]
where
\[ R \triangleq r + \frac{\rho - r}{\gamma}, \quad \text{and} \quad G \triangleq \frac{(1 - \gamma)(1 - \alpha)}{\gamma} g. \] (3.7)

**Proof.** The first order conditions (FOCs) in the Lagrangian are as follows:
\[
\begin{align*}
&u_c(c(t), \ell(t)) = \lambda e^{(\rho - r)t}, \\
&u_\ell(c(t), \ell(t)) = \lambda w(t)e^{(\rho - r)t}, \\
&V'(X(T)) = \lambda e^{(\rho - r)T}.
\end{align*}
\]

By the assumption, \( u_c(c, \ell) = v'(c^a\ell^{1-a})\alpha c^{\alpha-1}\ell^{1-a} \) and \( u_\ell(c, \ell) = v'(c^a\ell^{1-a})(1-\alpha)c^\alpha\ell^{-\alpha} \).
Since the first two conditions in FOCs imply \( u_c(c(t), \ell(t)) = w(t)u_\ell(c(t), \ell(t)) \), we get
\[ \ell(t) = 1 \]
which confirms (3.3). Substituting this into \( u_c(\cdot) \) gives us
\[
\begin{align*}
u_c(c(t), \ell(t))) &= \alpha \left( \frac{1 - \alpha}{\alpha w(t)} \right)^{1-a} c(t)^{-\gamma} \left( \frac{1 - \alpha}{\alpha w(t)} \right)^{(1-a)} \\
&= \alpha \left( \frac{1 - \alpha}{\alpha w(t)} \right)^{(1-a)(1-\gamma)} c(t)^{-\gamma} \\
&= w(t)^{-1}(1-\alpha)(1-\gamma) \left( \frac{c(t)}{A} \right)^{-\gamma} \quad \text{by (3.5)},
\end{align*}
\]
and hence, we get the optimal level of consumption: equation (3.2). Similarly, from the third equation in the FOCs, we get the optimal level of terminal wealth: equation (3.4).

**Remark.**
(i) \( \lambda \) can be interpreted as the lifetime income since \( \lambda \) equals to \( X_0 + \int_0^T e^{-rt} \)
\[ w(t) (L - \ell^*(t)) dt + \int_0^T e^{-rt} w(t) \ell^*(t) dt. \]
(ii) \( \Lambda_2 \) is associated with the cost of consumption and leisure before retirement. \( \Lambda_2 = \int_0^T e^{-rt} \cdot (\lambda^*)^{1/\gamma} \cdot (c^*(t) + w(t)\ell^*(t)) dt \)
(iii) \( \Lambda_3 \) is associated with the cost of wealth at retirement. \( \Lambda_3 = e^{-RT} \cdot (\lambda^*)^{1/\gamma} \cdot W^*(T) \)

**Lemma 3.3.** The value function can be represented as the lifetime income times shadow price divided by 1 - \( \gamma \):
\[ V(X_0) = \frac{\Lambda_1}{1 - \gamma}. \]
Proof. The proof is derived by substituting (3.2), (3.3), (3.4), and (3.8) into (2.1).

Now, consider an expenditure problem which is dual to the our primal problem:

\[
E = \min_{c(t), \ell(t)} \left[ e^{-rT} X(T) + \int_0^T e^{-r t} \left( c(t) - \omega(t)(L - \ell(t)) \right) dt \right]
\]

with constraint

\[
\int_0^T e^{-\rho t} u(c(t), \ell(t)) dt + e^{-rT} V(W(T)) \geq U
\]

where \( U \) is a given utility. By using lemma 3.3 and the property of duality, we can derive the expenditure function as

\[
E = X_0 = V^{-1}(U) = \left( (1 - \gamma) \cdot U \right)^{-\frac{1}{\gamma}} \cdot (\Lambda_2 + \Lambda_3)^{-\frac{\gamma}{1 - \gamma}} - \frac{\omega_0 T}{r - g} \left( 1 - e^{-(r-g)T} \right). \tag{3.9}
\]

Then, the Hicksian version of consumption should equal to

\[
c_H(t) = A \left( \lambda_H e^{(\rho-r)t} \right)^{-1/\gamma} w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}} \tag{3.10}
\]

for \( \lambda_H = \left( \frac{\Lambda_2 \Lambda_3}{\Lambda_H} \right)^{-\gamma} = \left( \frac{E + w_0 L_0 T e^{-(r-g)T}}{\Lambda_2 + \Lambda_3} \right)^{-\gamma}. \) By using equations (3.2) and (3.10), we get both the substitution and income effect w.r.t. (with respect to) the interest rate denoted by \( r. \)

Proposition 3.4. (i) The substitution effect (SE) to the consumption:

\[-c^s(0) \cdot \frac{1}{1 - \gamma} \cdot \frac{\Lambda_2' + \Lambda_3'}{\Lambda_2 + \Lambda_3}.\]

(ii) The income effect (IE) to the consumption:

\[c^i(0) \cdot \left( \frac{\Lambda_1'}{\Lambda_1} + \frac{\gamma}{1 - \gamma} \cdot \frac{\Lambda_2' + \Lambda_3'}{\Lambda_2 + \Lambda_3} \right).\]

Proof. Derivatives of optimal consumption and shadow price w.r.t interest rate are as follows:

\[
\frac{\partial c^s(0)}{\partial r} = -\frac{1}{\gamma} \cdot \frac{c^s(0)}{\lambda} \cdot \frac{\partial \lambda}{\partial r} \quad \text{and} \quad \frac{\partial \lambda}{\partial r} = -\gamma \lambda^{\frac{1 + \gamma}{\gamma}} \cdot \frac{(\lambda_1' \Lambda_2 - \lambda_1' \Lambda_3') + (\lambda_1' \Lambda_3 - \lambda_1' \Lambda_2')}{(\Lambda_2 + \Lambda_3)^2}.
\]

Substituting \( \frac{\partial \lambda}{\partial r} \) into \( \frac{\partial c^s(0)}{\partial r} \), using the definition of \( \lambda, \) (3.8), we calculate \( \frac{\partial c^s(0)}{\partial r} \):

\[c^s(0) \cdot \left( \frac{\Lambda_1'}{\Lambda_1} - \frac{\Lambda_2'}{\Lambda_2} + \frac{\Lambda_3'}{\Lambda_3} \right).\]

In the similar way to deriving \( \frac{\partial c^r(0)}{\partial r} \), the substitution effect of \( c^s(0) \) is as follows:

\[
\frac{\partial c_H}{\partial r} = c^s(0) \cdot \left( \frac{\Lambda_1'}{\Lambda_H} - \frac{\Lambda_2'}{\Lambda_2} + \frac{\Lambda_3'}{\Lambda_3} \right) \quad (\because c^s(0) = c_H(0)).
\]
By using definition of $\Lambda_H$, $\Lambda^*_H$ can be written as the following
\[
((1-\gamma) \cdot U)^{\frac{1}{1-\gamma}} \cdot (\Lambda_2 + \Lambda_3) - \frac{\gamma}{1-\gamma}.
\]
Since $\Lambda_2$ and $\Lambda_3$ are function of $r$, we get the proposition (3.4)-(i).
IE can be derived by using the Slutsky equation:
\[
-\frac{\partial c^*}{\partial X} \cdot \frac{\partial E}{\partial r} = \frac{\partial c^*(0)}{\partial r} - \frac{\partial c_H}{\partial r}.
\]

We introduce the elasticity of substitution between the cost of total consumption, which is the cost of consumption and leisure for lifetime, and the lifetime income w.r.t. the interest rate as
\[
\frac{(\Lambda_2 + \Lambda_3)'/(\Lambda_2 + \Lambda_3)}{\Lambda'_1/\Lambda_1} : ESCIR.
\]
And note that IE can be rewritten as $-SE - \frac{(\Lambda^*)'c^*(0)}{\gamma \lambda^*}$.

Remark. The following statements are the effects of the interest rate to the consumption for $\gamma > 1$; it is obvious that the SE is less than 0 because both $\Lambda'_2$ and $\Lambda'_3$ are less than 0.
(i) If the ESCIR is in $(0, \frac{\gamma-1}{\gamma})$, then the IE and the total effect(IE+SE) are less than 0.
(ii) If the ESCIR is in $[\frac{\gamma-1}{\gamma}, 1]$, then the IE is larger than 0 but the total effect is less than 0.
(iii) If the ESCIR is in $(1, \infty)$, then the IE and the total effect are larger than 0.

In a similar manner, we can also derive the substitution and income effects of labor and terminal wealth w.r.t. the interest rate:

<table>
<thead>
<tr>
<th>Substitution Effect</th>
<th>Income Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Labor $(\ell - \ell^*(0))$</td>
<td>$\ell^*(0) \cdot \frac{1}{1-\gamma} \cdot \frac{\Lambda'_2 + \Lambda'_3}{\Lambda_2 + \Lambda_3} &gt; 0$</td>
</tr>
<tr>
<td>Terminal Wealth $(X^*(T))$</td>
<td>$X^*(T) \cdot \frac{1}{1-\gamma} \cdot \frac{\Lambda'_2 + \Lambda'_3}{\Lambda_2 + \Lambda_3}$</td>
</tr>
</tbody>
</table>

The SE and IE to the labor have the opposite sign to consumption. Additionally, the sign of SE to the terminal wealth (or income) depends on how much time remains from now. When an agent faces retirement, the SE to the terminal wealth has a negative sign. However, if he/she has plenty of time until his retirement, then his/her terminal wealth will increase.

The substitution and wealth effects w.r.t. the growth rate of wage are the following:

<table>
<thead>
<tr>
<th>Substitution Effect</th>
<th>Income Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption $(c^*(0))$</td>
<td>$-c^*(0) \cdot \frac{1}{1-\gamma} \cdot \frac{\Lambda'_2 + \Lambda'_3}{\Lambda_2 + \Lambda_3}$</td>
</tr>
<tr>
<td>Labor $(\ell - \ell^*(0))$</td>
<td>$\ell^*(0) \cdot \frac{1}{1-\gamma} \cdot \frac{\Lambda'_2 + \Lambda'_3}{\Lambda_2 + \Lambda_3}$</td>
</tr>
<tr>
<td>Terminal Wealth $(X^*(T))$</td>
<td>$-X^*(T) \cdot \frac{1}{1-\gamma} \cdot \frac{\Lambda'_2 + \Lambda'_3}{\Lambda_2 + \Lambda_3}$</td>
</tr>
</tbody>
</table>
By Itô’s theorem, we know that the following equation:

\[ X_t = X_0 + \int_0^t r X_s \, ds + \int_0^t \sigma X_s \, dB_s, \]

where \( X_t \) denotes the agent’s consumption stream, leisure choice and investment decision, respectively.

Then, we have the following result.

\[ \text{Lemma 5.1.} \quad \text{For any given initial wealth level, } X_0, \text{ at time } 0, \text{ the wealth process (4.2) satisfies the following equation:} \]

\[ X_T = X_0 + \int_0^T (r X_s + \mu - r) \Pi(s) + w(s)(\ell(s) - c(s)) \, ds + \sigma \Pi(s) dB(s). \]

### 4. The Model With Stock Investment

Now, we assume that there are two tradable assets: bond and stock. So, an agent has the following expected utility:

\[ V(X(0)) = \max_{c(t), \ell(t), \Pi(t)} E_0 \left[ \int_0^T e^{-\rho t} u(c(t), \ell(t)) \, dt + e^{-\rho T} U(X(T)) \right], \quad (4.1) \]

where \( c(t) \triangleq \{ c(s) \mid t \leq s \leq T \} \), \( \ell(t) \triangleq \{ \ell(s) \mid t \leq s \leq T \} \), \( \Pi(t) \triangleq \{ \Pi(s) \mid t \leq s \leq T \} \) denote the agent’s consumption stream, leisure choice and investment decision, respectively.

The price of bond, \( S_0(t) \), is given by \( dS_0(t) = rS_0(t) \, dt \). We assume the price of stock, \( S_1(t) \), follows a geometric Brownian motion, with drift, \( \mu \), and volatility, \( \sigma \):

\[ dS_1(t) = \mu S_1(t) \, dt + \sigma S_1(t) \, dB(t), \]

where \( B(t) \) denotes the standard Brownian motion. Then, the wealth amount of the agent, \( X(t) \), at time \( t \) is given by

\[ dX(t) = (rX(t) + (\mu - r)\Pi(t) + w(t)(\ell(t) - c(t)) - c(t)) \, dt + \sigma \Pi(t) dB(t). \quad (4.2) \]

### 5. The Martingale Method And Results

Let us denote \( \theta \triangleq \frac{\mu - r}{\sigma} \) and define \( H \) by

\[ H(t) \triangleq e^{-\left[(r + \frac{1}{2}\sigma^2)t + \theta B(t)\right]} \]

Then, we have the following result.

**Lemma 5.1.** For any given initial wealth level, \( X_0 \), at time \( 0 \), the wealth process (4.2) satisfies the following equation:

\[ X_T = X_0 + \int_0^T H(t)(c(t) - w(t)(\ell(t) - c(t))) \, dt + H(T)X(T). \]

**Proof.** By Itô’s theorem, we know that \( dH = -H(r dt + \theta dB) \). We apply Itô’s theorem to \( H(t)X(t) \) again, and substitute the above \( dH \) and \( dX \) in (4.2), so that we have

\[ d(HX) = X \, dH + H \, dX + dH \cdot dX \]

\[ = -H(c - w(\ell - c)) \, dt - H(\sigma \Pi - \sigma \theta) dB. \]
Changing this into the integral form between \( t \) and \( T \), we get

\[
H(t)X(t) = H(T)X(T) + \int_t^T H(s)(c(s) - w(s)(\ell(s) - \ell^*(s)))\,ds + \int_t^T H(s)(\theta X(s) - \sigma \Pi(s))\,dB(s).
\]

Since the last term in the above equation is martingale and \( H(0) = 1 \), we get the desired result.

Now, we assume that the form of utility function and the bequest function are all of constant relative risk aversion (CRRA) type:

\[
u(c, \ell) = v(c^{\alpha \ell^{1-\alpha}}), \quad U(X(T)) = K v(X(T)),
\]

where \( v(x) = x^{1-\gamma} / \gamma \). By using Lemma 5.1, we get the following result.

**Proposition 5.2.** We obtain the optimal consumption, \( c^*_t \), and the optimal labor choices, \( \ell^*_t \), the optimal level of terminal wealth, \( X^*(T) \), to the optimization problem (4.1) as follows:

\[
c^*(t) = A(\lambda e^{\rho t} H(t))^{-1/\gamma} w(t)^{-\frac{(1-\alpha)(1-\gamma)}{\gamma}}, \tag{5.2}
\]
\[
\ell^*(t) = \frac{1 - \alpha}{\alpha w(t)} c^*(t), \tag{5.3}
\]
\[
X^*(T) = \left(\frac{\lambda e^{\rho T} H(T)}{K}\right)^{-1/\gamma}. \tag{5.4}
\]

The Lagrange multiplier to the budget constraint, \( \lambda \), can be obtained by plugging \( c^*(t) \), \( \ell^*(t) \), and \( X^*(T) \) into (5.1).

**Proof.** The proof is similar to proposition (3.2).

**Remark.** Now, consider the optimized wealth process \( X^*(t, B(t)) \) for all \( 0 \leq t \leq T 

\[
X^*(t, B(t)) \triangleq E_t\left[ \int_t^T \frac{H(s)}{H(t)} \left( c^*(s) - w(s)(\ell^*(s) - \ell^*(s)) \right)\,ds + \frac{H(T)}{H(t)} X^*(T) \right].
\]

By (5.3), we have

\[
X^*(t) = E_t\left[ \int_t^T \frac{H(s)}{H(t)} \left( c^*(s) - w(s)\ell^*(s) \right)\,ds + \frac{H(T)}{H(t)} X^*(T) \right]. \tag{5.5}
\]

Comparing this to equation (4.2), we can derive the optimal cash amount invested in the risky asset, \( \Pi^*(t) \), as follows:

\[
\Pi^*(t) = \frac{1}{\sigma} \frac{\partial X^*(t, B(t))}{\partial B(t)}.
\]

To get an explicit form of the solution, we assume that the wage rate grows with rate \( g \) from the initial value \( w(0) = w_0 \); \( w(t) = w_0 e^{gt} \). Then, we get the following result.
Proposition 5.3. If $w(t)$ is given with $w(t) = w_0 e^{st}$, then the Lagrange multiplier, $\hat{\lambda}^*$, to the budget constraint is given by the following equation:

$$\hat{\lambda}^* = \left( \frac{\Lambda_1}{\hat{\lambda}_2 + \Lambda_3} \right)^{-\gamma}, \quad (5.7)$$

where

$$\hat{R} \triangleq r + \frac{\rho - r}{\gamma} - \frac{1 - \gamma}{2\gamma^2}\theta^2,$$

$$\hat{\lambda}_2 \triangleq \left( \alpha \left( \frac{1-\alpha}{w_0} \right) (1-\alpha) \right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\hat{R} + G} \left( 1 - e^{-(\hat{R} + G)T} \right),$$

$$\hat{\lambda}_3 \triangleq K^\gamma e^{-\hat{R}T}.$$  

And the optimal cash amount invested in the risky asset is given with

$$\Pi^*(t) = \frac{1}{\sigma} \cdot \frac{\theta}{\gamma} \left( X^*(t) + E_t \left[ \mathbb{L} \int_t^T H(s) \frac{H(t)}{H(t)} w(s) ds \right] \right). \quad (5.8)$$

Proof. The proof is in the appendix. □

Since Lemma 3.3 also satisfies for the model of stock investment

$$V(x) = \frac{\Lambda_1}{1 - \gamma} \hat{\lambda} \quad \text{for} \quad X_0 = x,$$

we derive the similar result to the previous model. By using proposition 5.2, we derive the substitution effect and income effect which is the same as proposition 3.4 in the market with the risky asset.

6. Conclusion

In this paper, we have studied the optimal consumption and portfolio selection problem when an agent chooses his/her labor supply. We have classified the IE, SE, and their sum depending on the size of ESCIR:

(i) the signs of IE and SE for the consumption and labor have been shown by the interval of size of ESCIR.
(ii) the signs of IE and SE for the terminal wealth depends on 2 factors, which are the size of ESCIR and the length of remaining time to the retirement.

It is interesting to conduct the comparative static analysis of variables in the future as Grandville(1989) [4] did.

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$^1\hat{R}$ in equation (5.7) is corresponding to the constant, $\nu$, in Merton(1969) [2].
APPENDIX

Proof of Proposition 5.3. Equation (5.5) at time 0 gives an alternative form of the budget constraint:

\[ X_0 = \frac{1}{\alpha} E_0 \left[ \int_0^T H(t) e^s(t) dt \right] - \mathbb{L} E_0 \left[ \int_0^T H(t) w(t) dt \right] + E_0[H(T)X(T)]. \]

Note that \( B(T) - B(t) \) is normally distributed (\( \sim N(0, T-t) \)) and independent of \( B(t) \) which implies

\[ E_t[e^{\mu T + \theta B(T)}] = e^{\mu T + \theta B(t) + \frac{\theta^2}{2}(T-t)}, \quad \text{for all } 0 \leq t \leq T. \] (A.1)

Applying (A.1) to the last term in the above equation for budget constraint gives us

\[ E_0[H(T)X^*(T)] = \left( \frac{\lambda e^{\theta T}}{K} \right)^{-\frac{1}{2}} E_0[H^{1-\frac{1}{2}}] 
= \left( \frac{\lambda e^{\theta T}}{K} \right)^{-\frac{1}{2}} E_0 \left[ e^{\frac{1}{2} \gamma} \left( (r + \frac{1}{2} \theta^2)T + \theta B(T) \right) \right] 
= \left( \frac{\lambda}{K} \right)^{-\frac{1}{2}} e^{-RT}. \]

We now use the assumption that \( w(t) = w_0 e^{gt} \) to calculate the first and the second term. Applying (A.1) and Fubini’s theorem to the second term, we have

\[ E_0 \left[ \int_0^T H(t) w(t) dt \right] = w_0 \int_0^T E_0[H(t)e^{gt}] dt 
= w_0 \int_0^T e^{-(r-g)t} dt 
= \frac{w_0}{r-g} \left( 1 - e^{-(r-g)T} \right). \]

Similarly,

\[ E_0 \left[ \int_0^T H(t) c^s(t) dt \right] = A \lambda^{-1/\gamma} \int_0^T E_0 \left[ e^{-\frac{2}{\gamma} t} H(t) \frac{\gamma}{\gamma} w(t) \frac{(1-\alpha)(1-\gamma)}{\gamma} \right] dt 
= A \lambda^{-1/\gamma} w_0 \frac{(1-\alpha)(1-\gamma)}{\gamma} \int_0^T e^{-(R+G)t} dt 
= A \lambda^{-1/\gamma} w_0 \frac{(1-\alpha)(1-\gamma)}{\gamma} \frac{1}{R+G} \left( 1 - e^{-(R+G)T} \right). \]

Substituting the above results in the budget constraints, we have

\[ X_0 = \frac{1}{\alpha} A \lambda^{-1/\gamma} w_0 \frac{(1-\alpha)(1-\gamma)}{\gamma} \frac{1}{R+G} \left( 1 - e^{-(R+G)T} \right) - \mathbb{L} \frac{w_0}{r-g} \left( 1 - e^{-(r-g)T} \right) + \left( \frac{\lambda}{K} \right)^{-\frac{1}{2}} e^{-RT}, \]

or

\[ X_0 + \frac{w_0 \mathbb{L}}{r-g} \left( 1 - e^{-(r-g)T} \right) = \lambda^{-1/\gamma} \left[ \left( \alpha^\gamma \left( 1 - \frac{1}{w_0} \right)^{1-\gamma} \right)^{\frac{1}{2}} \frac{1}{R+G} \left( 1 - e^{-(R+G)T} \right) + \frac{1}{K} \right] e^{-RT}. \]
This is equivalent to equation (5.7). The calculations conditioned at time \( t \) is nearly the same as those at time 0, so that we have

\[
E_t[H(T)X^*(T)] = \left( \frac{\lambda e^{\theta T}}{K} \right)^{-\frac{1}{2}} E_t[H^{1-\frac{1}{2}}]
\]

\[
= \left( \frac{\lambda}{K} \right)^{-\frac{1}{2}} e^{-RT-\frac{1}{2}\left(\frac{1-\gamma}{1-\rho}\right)^2\theta^2 t + \frac{1-\gamma}{1-\rho} \theta B(t)}.
\]

We also have

\[
E_t\left[ \int_t^T H(s)w(s)ds \right] = w_0 \int_t^T E_t[H(s)e^{\theta s}] ds
\]

\[
= w_0 e^{-\theta B(t)-\frac{\theta^2 t}{2}} \int_t^T e^{-(r-g)s} ds
\]

\[
= \frac{w_0}{r-g} e^{-\theta B(t)-\frac{\theta^2 t}{2}} (e^{-(r-g)t} - e^{-(r-g)T}),
\]

and

\[
E_t\left[ \int_t^T H(s)c^*(s)ds \right] = A\lambda^{-1/\gamma} \int_t^T E_t\left[ e^{-\frac{\theta s}{\gamma}} H(s) \frac{1-\alpha(1-\gamma)}{\gamma} w(s) \right] ds
\]

\[
= A\lambda^{-1/\gamma} w_0 \left( \frac{1-\alpha(1-\gamma)}{\gamma} e^{-\frac{1}{2}\left(1-\frac{1-\gamma}{1-\rho}\right)^2\theta^2 t + \frac{1-\gamma}{1-\rho} \theta B(t)} \int_t^T e^{-\left(R+G\right)s} ds \right)
\]

\[
= A\lambda^{-1/\gamma} w_0 \left( \frac{1-\alpha(1-\gamma)}{\gamma} e^{-\frac{1}{2}\left(1-\frac{1-\gamma}{1-\rho}\right)^2\theta^2 t + \frac{1-\gamma}{1-\rho} \theta B(t)} e^{-\left(R+G\right)t} - e^{-\left(R+G\right)T} \right).
\]

By differentiating the terms in (5.5) partially in \( B(t) \), we have

\[
\frac{\partial E_t[H(T)X^*(T)]}{\partial B(t)} = \frac{1-\gamma}{\gamma} \theta E_t[H(T)X^*(T)]
\]

\[
\frac{\partial E_t\left[ \int_t^T H(s)w(s)ds \right]}{\partial B(t)} = -\theta E_t\left[ \int_t^T H(s)w(s)ds \right]
\]

\[
\frac{\partial E_t\left[ \int_t^T H(s)c^*(s)ds \right]}{\partial B(t)} = \frac{1-\gamma}{\gamma} \theta E_t\left[ \int_t^T H(s)c^*(s)ds \right],
\]

which give us

\[
\frac{\partial (HX^*)}{\partial B} = \frac{1-\gamma}{\gamma} \theta E_t[H(T)X^*(T)]
\]

\[
= \frac{\theta}{\gamma} \left( LE_t\left[ \int_t^T H(s)w(s)ds \right] + (1-\gamma)HX^* \right).
\]
Since $\frac{\partial H}{\partial B} = -\theta H$ and $\Pi^*(t) = \frac{1}{\sigma} \frac{\partial X^*}{\partial B}(t)$, we have

$$\Pi^*(t) = \frac{1}{\sigma} \left( \theta X^*(t) + \frac{1}{H(t)} \frac{\partial (HX^*)}{\partial B}(t) \right).$$

By canceling $\theta X^*(t)$ in the right-hand side of the above equation, we finally have

$$\Pi^*(t) = \frac{1}{\sigma} \frac{1}{H(t)} \left( \mathbb{E}_t \left[ \int_t^T H(s) w(s) ds \right] + H(t) X^*(t) \right),$$

which is similar to equation (5.8).

\[ \square \]

\textbf{References}


ANALYSIS OF THE MMPP/G/1/K QUEUE WITH A MODIFIED
STATE-DEPENDENT SERVICE RATE

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ABSTRACT. We analyze the MMPP/G/1/K queue with a modified state-dependent service rate. The service time of customers upon service initiation is changed if the number of customers in the system reaches a threshold. Then, the changed service time is continued until the system becomes empty completely, and this process is repeated. We analyze this system using an embedded Markov chain and a supplementary variable method, and present the queue length distributions at a customer’s departure epochs and then at an arbitrary time.

1. INTRODUCTION

In this paper, a finite capacity queueing system with a modified state-dependent service rate is analyzed. Queueing systems with finite buffers exist in a wide variety of applications such as computer systems, telecommunication networks, and production lines, among others. While operating systems with a queue, in which the arrivals at the systems and the service of customers (packets or lots) occur randomly, some customers may suffer long delays or be blocked. This can finally lead to a situation in which the delay requirements of users are not satisfied. Possible solutions to this problem is to control the arrival or the service rate. For the queueing model with queue length dependent arrival rate, refer to Choi et al. [3]. A variable service rate depending on the queue length considered in this paper operates as follows: when the number of customers in the system exceeds the threshold, the service rate is increased to a certain value to serve customers more quickly. The increased service rate continues until the

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system becomes empty for the first time, and then the service rate is reduced to its original value.

The scheme-variable service characteristics based on the state of the system have been extensively applied in real-world applications. For example, systems adapting service speed based on the queue length can be found in telecommunication systems such as that proposed by Choi et al. [4]. By applying a cell-discarding scheme for voice packets in ATM (Asynchronous Transfer Mode) networks, Choi et al. [4] analyzed the $M/G_1, G_2/1$ queue. Also, the applications in call centers can be found in Bekker et al. [1].

For analytical approaches, Choi et al. [4] obtained analytical expressions by means of integral representations and devised an asymptotic approximation for the system size distribution. Choi et al. [5] analyzed the $M^X/G/1$ model with queue-length dependent service times using Markov renewal theory and presented the queue length distributions including the transient distribution at time $t$ and its limiting distributions. Also, the virtual waiting time distribution was presented. Choi et al. [4] and Choi et al. [5] aptly summarized previous work on queueing systems with queue-length dependent service times. These results dealt with one threshold policy, and the two thresholds policy can be found in Dudin [6], Nishimura and Jiang [7], Nobel and Tijms [8], Zhernovyi and Zhernovyi [9, 10]. Specifically, Zhernovyi and Zhernovyi [9, 10] analyzed the finite queueing model with the two thresholds policy using the Korolyuk potential method. They gave the Laplace transform for the distribution of the number of customers during a the busy period, the distribution function for the busy period, the mean duration of the busy period, and the formula for the stationary distribution of the number of customers and other measures. Our model differs from previous works in that it considers a Markov-modulated Poisson process (MMPP) as the arrival process of customers. In many realistic situations, particularly in telecommunication systems, there is a correlation between the inter-arrival times of customers (or packets) and the degree of burstiness. It requires the use of correlated arrival models rather than models assuming Markovian arrival streams [2]. The MMPP is used to model traffic streams with bursty characteristics and time correlations between inter-arrivals. For example, traffic such as voice and video in telecommunication networks has these properties. We claim that our model extends previous ones on queues with a state-dependent service rate with one threshold by considering the MMPP. The MMPP is assumed as the arrival process of customers that have not been considered; this assumption can be more suitably utilized in real-world problems such as telecommunication systems.

The remainder of this paper is organized as follows. Section 2 describes the mathematical model. In Section 3, queue length distributions upon departure and arbitrary epochs are presented. We first derive the queue length distribution at a customer’s departure epochs using an embedded Markov chain. Next, the queue length distribution at an arbitrary time is obtained using a supplementary variable method. Other performance indices, such as the loss probability and the mean queue length, are also presented. Section 4 gives numerical examples and Section 5 concludes the paper.
2. Model description

There are a single server and a buffer with finite capacity $K$ including a customer in service. Customers arrived when the buffer is full are blocked and lost. The customers are served by first-come, first-served (FCFS) approach based on their arrival order. The service times of customers are different depending on queue length. Specifically, if the number of customers in the system is less than the threshold value $L$ at the service initiation, the customers have the service time $S_1$ with distribution function $G_1$, a mean of $\mu_1$ and the Laplace transform $G_1^*(s)$. If the number of customers in the system is equal to or greater than the threshold value $L$ at the service initiation, the customers have the service time $S_2$ with distribution function $G_2$, a mean of $\mu_2$, and the Laplace transform $G_2^*(s)$. This service time $(S_2)$ of customers continues until the system becomes empty. Then, the customers are served by the service time $S_1$ again until the number of customers in the system reaches the threshold $L$. We assume $\mu_2 \leq \mu_1$ because the faster services are required when there are relatively more customers in the buffer.

The arrival process of customers is assumed to follow an MMPP with representation $(Q, \Lambda)$. Here, the matrix $Q$ is the infinitesimal generator matrix of an underlying Markov process $J(t)$ with state space $\{1, 2, \cdots, N\}$. And the matrix $\Lambda = \text{diag}(\lambda_i)$ is the arrival rate matrix. The stationary probability vector $\pi$ of the underlying Markov process $J(t)$ is given by solving the equation

$$\pi Q = 0, \quad \pi e = 1,$$

where $e$ and $0$ are vectors of size $N$ consist of all ones and zeroes, respectively. Let $M(t)$ be the number of arrivals by $\Lambda$ during the interval $(0, t]$. At this stage, we define the conditional probabilities:

$$p_{i,j}(n, t) = Pr\{M(t) = n, J(t) = j \mid M(0) = 0, J(0) = i\}, \quad n \geq 0.$$

Then, the matrix $P(n, t)$ is defined as $P(n, t) \triangleq (p_{i,j}(n, t))_{1 \leq i, j \leq N}$.

3. Analysis

3.1. Queue length distribution at departure epochs. First, the queue length distribution at the departure epochs of customers is considered. We term the period in which the service time of customers is generated by the service time $S_1$ as the underload period and the period in which the service time of customers is generated by the service time $S_2$ as the overload period. As soon as the system becomes empty, the underload period is started. The overload period starts from when the number of customers in the system at service initiation reaches the threshold $L$ to the instant when the system becomes empty.

Now we introduce the notations:

$$\tau_n = \text{the } n\text{th customer’s departure epoch, } n \geq 1, \tau_0 = 0,$$

$$N_n = \text{the number of customers in the system at time } \tau_n+,$$
Also, the vectors are defined.

\[ \xi_n = \begin{cases} 1, & \text{if the system is in the underload period at time } \tau_n + \\ 2, & \text{if the system is in the overload period at time } \tau_n + \end{cases} \]

and

\[ J_n = \text{the state of the underlying Markov process at time } \tau_n +. \]

Then, the process \( \{(N_n, \xi_n, J_n), n \geq 0\} \) form a Markov chain with finite state space in lexicographic order:

\[
\{(0, 1, 1), \ldots, (0, 1, N), (0, 2, 1), \ldots, (0, 2, N), \ldots,
(L - 1, 1, 1), \ldots, (L - 1, 1, N), \ldots, (L, 1, 1), \ldots, (L, 1, N), \ldots,
(L + 1, 2, 1), \ldots, (L + 1, 2, N), \ldots, (K - 3, 1, 1), \ldots, (K - 3, 2, N), \ldots,
(K - 2, 1, 1), \ldots, (K - 2, 2, N), (K - 1, 1, 1), \ldots, (K - 1, 2, N)\}.
\]

Note that if \( N_n = 0 \), then \( \xi_n = 1 \), and if \( N_n \geq L \), then \( \xi_n = 2 \). We define the steady-state probability of the Markov chain \( \{(N_n, \xi_n, J_n), n \geq 0\} \) as follows:

\[ x_{k, r, j} = \lim_{n \to \infty} \Pr\{N_n = k, \xi_n = r, J_n = j\}, \quad 0 \leq k < K, r = 1, 2, j = 1, 2, \ldots, N. \]

Also, the vectors are defined.

\[ x_{k, r} = (x_{k, r, 1}, \ldots, x_{k, r, N}) \]

\[ x_k = (x_{k, 1}, x_{k, 2}) \]

\[ x = (x_0, x_1, \ldots, x_{K - 1}). \]

Note that \( x_{0, 2} = 0, x_{k, 1} = 0 \) for \( L \leq k < K \). We introduce the following probability matrices:

\[ A_n^r = \int_{0}^{\infty} P(n, x) dG_r(x), r = 1, 2 \]

\[ A'_n = \int_{0}^{\infty} P(0, t) dt \Lambda A_n^1 = (\Lambda - Q)^{-1} \Lambda A_n^1, \]

\[ \overline{A}_n = \sum_{k=n}^{\infty} A_k^r, \quad \overline{A}'_n = \sum_{k=n}^{\infty} A'_k. \]

Also, the following matrices are introduced:

\[ B_k = \begin{pmatrix} A^r_k & 0 \\ 0 & A^r_k \end{pmatrix}, \quad 0 \leq k \leq L - 1, \quad B'_k = \begin{pmatrix} 0 & A^r_k \\ 0 & 0 \end{pmatrix}, \quad k \geq L, \]

\[ C_k = \begin{pmatrix} 0 & A^r_k \\ 0 & 0 \end{pmatrix}, \quad C'_0 = \begin{pmatrix} 0 & 0 \\ A^r_0 & 0 \end{pmatrix}, \quad C'_k = \begin{pmatrix} 0 & A^r_k \end{pmatrix}, \]

\[ D_k = \begin{pmatrix} 0 & 0 \\ 0 & A^r_k \end{pmatrix}, \quad k \geq 0, \]

and

\[ \overline{B}_n = \begin{pmatrix} 0 & \overline{A}_n^1 \\ 0 & 0 \end{pmatrix}, \quad \overline{C}_n = \begin{pmatrix} 0 & \overline{A}_n^1 \\ 0 & \overline{A}_n^2 \end{pmatrix}, \quad \overline{D}_n = \begin{pmatrix} 0 & 0 \\ 0 & \overline{A}_n^2 \end{pmatrix}. \]
Then, the transition probability matrix $\mathbf{Q}$ of the Markov chain $\{(N_n, \xi_n, J_n), n \geq 0\}$ is given by

$$
\mathbf{Q} = \begin{pmatrix}
    b_0 & b_1 & b_2 & \cdots & b_{L-1} & b_{L} & b'_{L+1} & \cdots & b'_{K-3} & b'_{K-2} & b'_{K-1} \\
    c_0 & c_1 & c_2 & \cdots & c_{L-1} & c_{L} & c'_{L+1} & \cdots & c'_{K-3} & c'_{K-2} & c'_{K-1} \\
    0 & c_0 & c_1 & \cdots & c_{L-2} & c'_{L-1} & c'_L & \cdots & c'_K & c'_{K-2} & c'_{K-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & c_1 & c'_2 & c'_3 & \cdots & c'_{K-L-1} & c'_{K-L} & c'_{K-L+1} \\
    0 & 0 & 0 & \cdots & D_0 & D_1 & D_2 & \cdots & D_{K-L-2} & D_{K-L-1} & D_{K-L} \\
    0 & 0 & 0 & \cdots & 0 & D_0 & D_1 & \cdots & D_{K-L-3} & D_{K-L-2} & D_{K-L-1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & D_1 & D_2 & D_3 \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & D_0 & D_1 \\
    0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & D_0 \\
\end{pmatrix}
$$

The steady-state probability vector $\mathbf{x}$ of the Markov chain $\{(N_n, \xi_n, J_n), n \geq 0\}$ is given by solving the equations:

$$
\mathbf{x} \mathbf{Q} = \mathbf{x}, \quad \mathbf{x} \mathbf{e} = 1,
$$

where $\mathbf{e} = (1, 1, \cdots, 1)^T$.

3.2. Queue length distribution at an arbitrary time. In this subsection, we derive the probability distribution of the queue length at an arbitrary time. Let $N(t)$ and $J(t)$ be the number of customers in the system and the state of the underlying Markov process at time $t$, respectively. In addition,

$$
\xi(t) = \begin{cases} 
    1, & \text{if the system is in the underload period at time } t, \\
    2, & \text{if the system is in the overload period at time } t.
\end{cases}
$$

We define the stationary probabilities:

$$
y_n(j) = \lim_{t \to \infty} \Pr\{N(t) = n, J(t) = j\}, \quad 0 \leq n \leq K.
$$

$$
y_n = (y_n(1), y_n(2), \cdots, y_n(N)).
$$

First, by the key renewal theorem, we have

$$
y_0(j) = \text{jth component of } \left[ \frac{1}{E} x_{0,1} (\Lambda - Q)^{-1} \right],
$$

where $E = x_{0,1} \left[(\Lambda - Q)^{-1} e + \mu_1\right] e + \sum_{n=1}^{L-1} x_{n,1} e \mu_1 + \sum_{n=1}^{K-1} x_{n,2} e \mu_2$ is the mean interdeparture time of customers.

Next, we derive the probabilities $y_n(n \geq 1)$ by using a supplementary variable method. Let $\tilde{T}$ and $T$ be the elapsed and remaining service time for the customer in service, respectively.
Furthermore, we define the stationary joint probability distribution of the number of customers in the system and the remaining service time for the customer in service:

\[ \alpha_r(n, j, x)dx = \lim_{t \to \infty} \Pr\{N(t) = n, \xi(t) = r, J(t) = j, x < T \leq x + dx\}, \]

\[ n \geq 1, \quad r = 1, 2, \]

and the Laplace transform of \( \alpha_r(n, j, x) \)

\[ \alpha_r^*(n, j, s) = \int_0^\infty e^{-sx} \alpha_r(n, j, x)dx. \]

\[ \alpha_r^*(n, s) = (\alpha_r^*(n, 1, s), \cdots, \alpha_r^*(n, N, s)). \]

In order to derive the queue length distribution at an arbitrary time, the number of arrivals of customers during the elapsed service time should be obtained. Thus, we also define the following conditional probability \( \beta_r(n, j_1, j_2, x)dx \) as follows:

\[ \beta_r(n, j_1, j_2, x)dx = \lim_{t \to \infty} \Pr\{n \text{ arrivals of customers during } T, \xi(t) = r, J(t) = j_2, x < T \leq x + dx|J(\bar{T}) = j_1\}, \quad n \geq 0, \ r = 1, 2, \]

where \( \bar{T} \) is the service starting time of the customer serving at time \( t \). We also define the Laplace transform \( \beta_r^*(n, j_1, j_2, s) \) of \( \beta_r(n, j_1, j_2, x) \) and matrix \( \beta_r^*(n, s) \) with \( \beta_r^*(n, j_1, j_2, s) \) as \((j_1, j_2)\) - elements:

\[ \beta_r^*(n, j_1, j_2, s) = \int_0^\infty e^{-sx} \beta_r(n, j_1, j_2, x)dx \]

\[ \beta_r^*(n, s) = (\beta_r^*(n, j_1, j_2, s))_{1 \leq j_1, j_2 \leq N}, \quad r = 1, 2. \]

By conditioning the queue length at last service completion epoch before time \( t \), \( \alpha_r^*(n, s) \) satisfy the following equations:

For \( 1 \leq n < K \),

\[ \alpha_1^*(n, s) = \frac{\mu_1}{E} \left[ x_{0,1} \beta_1^*(n - 1, s) + \sum_{k=1}^{\min\{n, L-1\}} x_{k,1} \beta_1^*(n - k, s) \right], \]

\[ \alpha_2^*(n, s) = \frac{\mu_2}{E} \left[ \sum_{k=1}^n x_{k,2} \beta_2^*(n - k, s) \right]. \]

Using the method introduced by Choi et al. [4], \( \beta_r^*(n, s) \) is given as follows:

\[ \beta_r^*(n, s) = \frac{1}{\mu_r} \left[ \sum_{k=0}^n A_k^r R_{n-k}(s) - G_r^*(s) R_n(s) \right], \quad r = 1, 2, \]

where \( R_n(s) = (sI - \Lambda + Q)^{-1}\{\Lambda (\Lambda - sI - Q)^{-1}\}^n \). Substituting \( \beta_r^*(n, s)(r = 1, 2) \) into above equations, and putting \( s = 0 \), we obtain the following stationary queue length probabilities at
an arbitrary time. For $1 \leq n < K$,
\[
y_n = \alpha_1^*(n, 0) + \alpha_2^*(n, 0)
\]
\[
= \frac{1}{E} \left[ x_{0,1} \sum_{k=0}^{n-1} A_k^1 (Q - \Lambda)^{-1} \{\Lambda (\Lambda - Q)^{-1}\}^{n-1-k} - (Q - \Lambda)^{-1} \{\Lambda (\Lambda - Q)^{-1}\}^{n-1} \right]
\]
\[
+ \sum_{k=1}^{\min\{n, L-1\}} \sum_{m=0}^{n-k} A_{k,1}^m (Q - \Lambda)^{-1} \{\Lambda (\Lambda - Q)^{-1}\}^{n-k-m} - (Q - \Lambda)^{-1} \{\Lambda (\Lambda - Q)^{-1}\}^{n-k} \]
\[
+ \sum_{k=1}^{n} \sum_{m=0}^{n-k} A_{k,2}^m (Q - \Lambda)^{-1} \{\Lambda (\Lambda - Q)^{-1}\}^{n-k-m} - (Q - \Lambda)^{-1} \{\Lambda (\Lambda - Q)^{-1}\}^{n-k} \]
\]
and
\[
y_K = \pi - \sum_{n=0}^{K-1} y_n.
\]

Finally, we obtain the following performance measures using the stationary queue length distribution $\{y_n, n \geq 0\}$:
(a) The loss probability ($P_{\text{loss}}$):
\[
P_{\text{loss}} = \frac{y_K \Lambda e}{\sum_{k=0}^{K} y_k \Lambda e}.
\]
(b) The mean queue length:
\[
M = \sum_{i=1}^{K} iy_i e.
\]
(c) By Little’s law, we obtain the mean waiting time in the system:
\[
W = \frac{M}{\lambda^*(1 - P_{\text{loss}})},
\]
where $\lambda^* = \pi \Lambda e$.

4. Numerical results

In this section, we present numerical results on the effects of the modified state-dependent service rate on the mean waiting time and loss probability. We set the capacity of the buffer to $K = 10$ and the threshold value to $L = 5$. We assume that the arrivals of customers follow an MMPP with
\[
Q = \begin{pmatrix}
-q_{12} & q_{12} \\
q_{21} & -q_{21}
\end{pmatrix}, \quad \Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
\text{other} & \lambda_2
\end{pmatrix},
\]
and $q_{12} = q_{21} = 0.1$ and $\lambda_2/\lambda_1 = 10$. The effective arrival rate $\lambda^*$ for this MMPP is given by $\lambda^* = \pi \Lambda e$. 

To investigate the effect of the modified state dependent service rate, we consider three cases. For the ‘High’ and ‘Low’ cases below, queueing systems have two service distributions \( S_1 \) and \( S_2 \).

- **‘without Threshold’ case**: The ordinary system without threshold values is assumed. Its service time distribution having mean 2 is assumed to follow hyper-exponential distribution with the probability density function \( p \theta_1 e^{-\theta_1 t} + (1 - p) \theta_2 e^{-\theta_2 t} \), where \( p = \frac{1}{3}, \theta_1 = \frac{1}{4}, \) and \( \theta_2 = 1 \).

- **‘High’ case**: \( S_2 \) is a hyper-exponential variable with mean 0.5 (\( p = \frac{1}{4}, \theta_1 = 3, \theta_2 = \frac{9}{5} \)). \( S_1 \) is assumed to have the same distribution of the ‘without Threshold’ case (mean=2).

- **‘Low’ case**: \( S_2 \) is a hyper-exponential variable with mean 1 (\( p = \frac{1}{4}, \theta_1 = \frac{3}{2}, \theta_2 = \frac{9}{10} \)). \( S_1 \) is assumed to have the same distribution of the ‘without Threshold’ case (mean=2).

Fig. 1 and Fig. 2 show the mean waiting time and loss probability as a function of effective arrival rate, respectively. These figures show that the mean waiting time and the loss probability generally increase as the effective arrival rate increases. Furthermore our model outperforms the ordinary queueing system without threshold values. Also Fig.1 shows that the mean waiting times converge to certain values: (the capacity of the buffer) \( \times \) (mean service time). Figures 3 and 4 also present the mean waiting time and the loss probability in which all conditions are identical except the mean of the service time distribution of ‘without Threshold’ case is 4.

In Fig. 1 and Fig. 3, it is observed that the mean waiting time increases in the beginning but decreases later as the effective arrival rate increases. It can be interpreted as follows. As the effective arrival rate increases, the mean queue length also increases. When the mean queue length is larger than a threshold value, customers are more likely to be served by an increased service rate. As a result, the mean waiting time decreases.

**Figure 1.** Mean waiting time over the \( \lambda^* \) (effective arrival rate)
Figure 2. Loss probability over the $\lambda^*$ (effective arrival rate)

Figure 3. Mean waiting time over the $\lambda^*$ (effective arrival rate)

Figure 4. Loss probability over the $\lambda^*$ (effective arrival rate)
5. Conclusion

In this paper, we analyzed an MMPP/G/1/K queueing system with queue length-dependent service rates. Several results including the queue length distributions, loss probability and mean queue length (mean waiting time) are presented. However, determining the optimal thresholds was not discussed. Determining an optimal threshold policy, such as the number of threshold values and specified threshold values, which minimize the long-run average cost with consideration of some reasonable cost factors can be suggested for future research. Holding cost per a customer, switching-over cost and operating cost for each service mode can be considered.

References

CONVERGENCE ANALYSIS ON GIBOU-MIN METHOD FOR THE SCALAR FIELD IN HODGE-HELMHOLTZ DECOMPOSITION

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ABSTRACT. The Hodge-Helmholtz decomposition splits a vector field into the unique sum of a divergence-free vector field (solenoidal part) and a gradient field (irrotational part). In a bounded domain, a boundary condition needs to be supplied to the decomposition. The decomposition with the non-penetration boundary condition is equivalent to solving the Poisson equation with the Neumann boundary condition. The Gibou-Min method is an application of the Poisson solver by Purvis and Burkhalter to the decomposition.

Using the $L^2$-orthogonality between the error vector and the consistency, the convergence for approximating the divergence-free vector field was recently proved to be $O(h^{1.5})$ with step size $h$. In this work, we analyze the convergence of the irrotational in the decomposition. To this end, we introduce a discrete version of the Poincare inequality, which leads to a proof of the $O(h)$ convergence for the scalar variable of the gradient field in a domain with general intersection property.

1. INTRODUCTION

The Hodge-Helmholtz decomposition theorem [6] states that any smooth vector field $U^*$ can be decomposed into the sum of a gradient field $\nabla p$ and a divergence-free vector field $U$. The decomposition is unique and orthogonal in $L^2$. The Hodge projection of a vector field is defined as the divergence-free component in its Hodge-Helmholtz decomposition.

One of the main applications of the decomposition is the incompressible fluid flow, whose phenomenon is represented by the Navier-Stokes equations. Consisting of the conservation equation of momentum and the state equation of divergence-free condition, the equations can be described by a convection-diffusion equation with the Hodge projection applied at every
moment. Chorin’s seminal approximation [3] for the fluid flow first solves the convection-diffusion equation in a usual manner, and then applies the Hodge projection. Other successful fluid solvers such as Kim-Moin’s [7], Bell et al.’s [1], Gauge method [8] are in the same direction as Chorin’s.

The Hodge-Helmholtz decomposition $U^* = U + \nabla p$ in a domain $\Omega$ can be implemented through the Poisson equation $-\Delta p = -\nabla \cdot U^*$. In a bounded domain, the equation needs to be supplied with boundary condition. There are two types fluid boundary conditions. One is the non-penetration boundary condition, $U \cdot n = 0$ on $\Gamma = \partial \Omega$, and the other is the free boundary condition, $p = \sigma \kappa$ on $\Gamma$ [10]. The free boundary condition is, in other words, the Dirichlet boundary condition of the Poisson equation, and the non-penetration boundary condition corresponds to the Neumann boundary condition, $\frac{\partial p}{\partial n} = U^* \cdot n$ on $\Gamma$.

To approximate the Poisson equation, we consider the standard finite volume method. A standard finite difference/volume method for the Poisson equation with the Neumann boundary condition was introduced by Purvis and Burkhalter [11]. Though implemented in uniform grid, the method can handle arbitrarily shaped domains. It is a simple modification of the standard five-point finite difference method, and it constitutes a five-banded sparse linear system that is diagonally dominant, symmetric and positive semi-definite. Due to these nice properties, the linear system can be efficiently solved by the Conjugate Gradient method with various efficient ILU preconditioners.

The Gibou-Min method [4, 9] is an application of the Purvis-Burkhalter method on the Hodge-Helmholtz decomposition. In implementing the Hodge decomposition, the Neumann boundary condition takes the divergence form $\frac{\partial p}{\partial n} = \nabla \cdot U^*$.

Using the orthogonality condition between the error $U - U^h$ and the consistency of the method, the method was proved in [13] to provide 1.5 order of accuracy in approximating the divergence-free vector field $U$ of the Hodge projection.

In this work, we estimate the convergence of the pressure $p$ given in the Hodge-Helmholtz decomposition. Using the orthogonality, we obtain the estimate $\|Gp - Gp^h\|_{L^2} = O(h^{1.5})$ for the gradient of the pressure error. On introducing a discrete version of the Poincare inequality, we derive $\|p - p^h\| = O(h^{-0.5} \cdot h^2)$ for $h_{min}$ the smallest distance from grid nodes inside to the boundary. Our estimate reads that for many domains with $h_{min} = O(h^2)$, for instance, domains with general intersection property introduced in [12], the pressure convergence is $O(h)$. According to our numerical tests, even though the estimate $\|U - U^h\|_{L^2} = O(h^{1.5})$ is tight, however, the estimate does not meet the observed order $\|p - p^h\| = O(h^2)$. We put it off to future research to improve the estimate.

2. Numerical method

In this section, we briefly review the Gibou-Min method [4, 9] for the Hodge decomposition with the non-penetration boundary condition. Given a vector field $U^*$ in a bounded and connected domain $\Omega$, the following Poisson equation is solved for scalar $p$ with the Neumann boundary condition.
\[
\begin{cases}
- \Delta p = - \nabla \cdot U^* & \text{in } \Omega \\
\frac{\partial p}{\partial n} = U^* \cdot n & \text{on } \Gamma
\end{cases}
\] (2.1)

Then a vector field \( U \), which is defined as \( U = U^* - \nabla p \), is the desired Hodge projection of \( U^* \) that satisfies the divergence-free condition \( \nabla \cdot U = 0 \) in \( \Omega \), and the non-penetration boundary condition \( U \cdot n = 0 \) on \( \Gamma \). The Gibou-Min method samples the vector fields and scalar field on the Marker-and-Cell (MAC) staggered grid [5]. Let \( h\mathbb{Z}^2 \) denote the uniform grid in \( \mathbb{R}^2 \) with step size \( h \). For each grid node \((x_i, y_j) \in h\mathbb{Z}^2 \), \( C_{ij} \) denotes the rectangular control volume centered at the node, and its four edges are denoted by \( E_{i\pm \frac{1}{2}j} \) and \( E_{ij\pm \frac{1}{2}} \) as follows.

\[
\begin{align*}
C_{ij} & := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \\
E_{i\pm \frac{1}{2}j} & := x_{i\pm\frac{1}{2}} \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}] \\
E_{ij\pm \frac{1}{2}} & := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times y_{j\pm\frac{1}{2}}
\end{align*}
\]

Based on the MAC configuration, we define the node set and the edge sets.

**Definition 2.1** (Node and edge sets). By \( \Omega^h := \{(x_i, y_j) \in h\mathbb{Z}^2 | C_{ij} \cap \Omega \neq \emptyset \} \) we denote the set of nodes whose control volumes intersecting the domain. In the same way, we define the edge sets by \( E_x^h := \{(x_i, y_j) | E_{i\pm \frac{1}{2}j} \cap \Omega \neq \emptyset \} \) and \( E_y^h := \{(x_i, y_{j+\frac{1}{2}}) | E_{ij\pm \frac{1}{2}} \cap \Omega \neq \emptyset \} \), and then \( E^h := E_x^h \cup E_y^h \).

By the standard central finite differences, a discrete gradient operator is defined.

**Definition 2.2** (Discrete gradient). Given \( p : \Omega^h \rightarrow \mathbb{R} \), its gradient \( Gp : E^h \rightarrow \mathbb{R} \) is defined as

\[
(G^xp)_{i+\frac{1}{2}j} = \frac{p_{i+1,j} - p_{ij}}{h} \\
(G^yp)_{i,j+\frac{1}{2}} = \frac{p_{ij+1} - p_{ij}}{h}
\]

Whenever \( E_{i+\frac{1}{2}j} \cap \Omega \neq \emptyset, C_{ij} \cap \Omega \neq \emptyset \) and \( C_{i+1,j} \cap \Omega \neq \emptyset \), since \( E_{i+\frac{1}{2}j} \subset C_{ij}, C_{i+1,j} \). Hence the above definition is well posed for \( G^xp \), and so is for \( G^yp \). Discrete gradient was simply defined by the finite differences, however discrete gradient can not be defined so. For each \((x_i, y_j) \in \Omega^h \), its four neighboring edges may not be in \( E^h \), since \( C_{ij} \cap \Omega \neq \emptyset \) neither imply \( E_{i+\frac{1}{2}j} \cap \Omega \neq \emptyset \) nor \( E_{ij+\frac{1}{2}} \cap \Omega \neq \emptyset \). A proper definition comes from the following identity.

\[
\int_{C_{ij} \cap \Omega} \nabla \cdot U \, dx = \int_{\partial\Omega} U \cdot \vec{n} \, ds \\
0 = \int_{\partial C_{ij} \cap \Omega} U \cdot \vec{n} \, ds + \int_{C_{ij} \cap \Gamma} U \cdot \vec{n} \, ds
\] (2.2)
With the non-penetration boundary condition $U \cdot \vec{n} = 0$, the identity represents an integral value of the divergence by a line integral over the fraction of edges. To measure the fraction, the following Heaviside functions are defined on the edge set.

**Definition 2.3 (Heaviside function).** For each edge,

$$H_{i+1/2,j} = \frac{\text{length}(E_{i+1/2,j} \cap \Omega)}{\text{length}(E_{i+1/2,j})}, \quad \text{and} \quad H_{i,j+1/2} = \frac{\text{length}(E_{i,j+1/2} \cap \Omega)}{\text{length}(E_{i,j+1/2})}.$$

Note that $H_{i+1/2,j}, H_{i,j+1/2} \in [0, 1]$. Its value 1 implies that the edge is totally inside the domain, and value 0 implies completely outside. Using the Heaviside function, now we define discrete divergence operator.

**Definition 2.4 (Discrete divergence).** Given $U = (u, v) : E^h \to \mathbb{R}$, its discrete divergence $DU : \Omega^h \to \mathbb{R}$ is defined as

$$(DU)_{ij} = \left( u_{i+1/2,j} H_{i+1/2,j} - u_{i-1/2,j} H_{i-1/2,j} \right) \cdot h + \left( v_{i,j+1/2} H_{i,j+1/2} - v_{i,j-1/2} H_{i,j-1/2} \right) \cdot h.$$

Note that the calculation of the discrete divergence involves the vector field only in $E^h$. The edges not in $E^h$, whose Heaviside function values are zero, are ignored in the calculation.

Given a vector field $U^* : \Omega \cup \Gamma \to \mathbb{R}^2$, we define a discrete vector field $U^* = (U_x^*, U_y^*)$ on $E^h$ as

$$(U_x^*)_{i+1/2,j} := \frac{1}{H_{i+1/2,j}^h} \int_{E_{i+1/2,j}^h \cap \Omega} U_x^*(x, y) dy$$

and

$$(U_y^*)_{i,j+1/2} := \frac{1}{H_{i,j+1/2}^h} \int_{E_{i,j+1/2}^h \cap \Omega} U_y^*(x, y, j+1/2) dy.$$

With the vector field $U^* : E^h \to \mathbb{R}^2$, the Gibou-Min method computes a vector field $U^h : E^h \to \mathbb{R}$ and a scalar field $p^h : \Omega^h \to \mathbb{R}$ such that $DU^h = 0$ in $\Omega^h$ and $U^* = U^h + Gp^h$ in $\Omega^h$. Substituting $U^h$ with $U^* - Gp^h$ in $DU^h = 0$, we have the equation for $p^h$,

$$-DGp^h = -DU^* \quad \text{in} \quad \Omega^h. \quad (2.3)$$

After $p^h$ is obtained by solving the above linear system, the solenoidal vector field $U^h = U^* - Gp^h$ is calculated.
3. CONVERGENCE ANALYSIS FOR THE HODGE PROJECTION $U^h$

From now on, we consider the convergence for the Gibou-Min method. Let $L^h := DG$ denote its associated linear operator, then it maps a discrete function $p^h : \Omega^h \to \mathbb{R}$ to another function $L^h p^h : \Omega^h \to \mathbb{R}$ such that

$$
(L^h p^h)_{ij} = H_{i+\frac{1}{2},j} \left( p_{i+1,j}^{h} - p_{i,j}^{h} \right) - H_{i-\frac{1}{2},j} \left( p_{i,j}^{h} - p_{i-1,j}^{h} \right) + H_{ij+\frac{1}{2}} \left( p_{i,j+1}^{h} - p_{ij}^{h} \right) - H_{ij-\frac{1}{2}} \left( p_{ij}^{h} - p_{ij-1}^{h} \right),
$$

for each $(x_i, y_j) \in \Omega^h$.

Most of lemmas and theorems in this section will be just stated without proofs, which we refer to [13] for details, for our main theme of this work is to introduce the convergence of the gradient in the Hodge-Helmholtz decomposition.

In this setting, we have that $\text{Ker}(L^h) = \text{span}\{1_{\Omega^h}\}$ and for a vector field $U^* : E^h \to \mathbb{R}$, $-L^h p^h = -DU^*$ has a unique solution $p^h \in \{1_{\Omega^h}\}^\perp$.

To analyze the convergence for the scheme, we introduce two inner products defined on $E^h$ and $\Omega^h$.

**Definition 3.1.** Let $E^h$ and $\Omega^h$ be the sets of edges and grid nodes, respectively.

(i) (Inner product between vector fields) Given two vector fields $U^1, U^2 : E^h \to \mathbb{R}$, their inner product is defined as

$$
\langle U^1, U^2 \rangle_{E^h} := h^2 \sum_{i,j} H_{i+\frac{1}{2},j} u^1_{i+\frac{1}{2},j} u^2_{i+\frac{1}{2},j} + \sum_{i,j} H_{i,j+\frac{1}{2}} v^1_{i,j+\frac{1}{2}} v^2_{i,j+\frac{1}{2}}.
$$

(ii) (Inner product between scalar fields) Given two discrete functions $p^1, p^2 : \Omega^h \to \mathbb{R}$, their inner product is defined as

$$
\langle p^1, p^2 \rangle_{\Omega^h} := h^2 \sum_{i,j} p^1_{i,j} p^2_{i,j}.
$$

With the two inner product spaces, we can see in the following lemma that $G$ is the adjoint operator of $\frac{1}{h^2} D$.

**Lemma 3.2** (Integration-by-parts). Let $G$ and $D$ be the discrete gradient and divergence operators, respectively. Then for any discrete function $p$ on $\Omega^h$ and vector field $U$ on $E^h$, we have

$$
\langle Gp, U \rangle_{E^h} = -\left\langle p, \frac{1}{h^2} DU \right\rangle_{\Omega^h}.
$$

The integration-by-parts leads to a discrete version of the Helmholtz decomposition.

**Theorem 3.3.** Given vector field $U^* : E^h \to \mathbb{R}$, there exists a unique $p^h \in \{1_{\Omega^h}\}^\perp$ such that $DGp = DU^*$. Therefore, the decomposition

$$
U^* = U^h + Gp^h \quad \text{with } DGp^h = DU^*
$$

is unique. Furthermore, the decomposition is orthogonal, i.e, $\langle U^h, Gp^h \rangle_{\Omega^h} = 0$. 
Here we used the facts that $DG$ follows from (i) and (ii). From the decompositions (3.3), we have

**Proof.** From the decompositions (3.3), we have $U - U^h = (U^* - \nabla p) - (U^* - Gp^h)$. Since $G$ is the standard central finite difference operator, $\nabla p - Gp = O(h^2)$. Hence, the estimate (i) follows from (ii) and it suffices to show $\|Gp - Gp^h\| = O(h^{1.5})$. Lemma 3.2 shows

$$\left< \frac{1}{h} d^h - G(p - p^h), G(p - p^h) \right>_{E^h} = -\frac{1}{h^2} \left< \frac{1}{h} Dd^h - L^h(p - p^h), p - p^h \right>_{\Omega^h} = 0 \quad (3.4)$$

Here we used the facts that $DG = L^h$ and $\frac{1}{h} Dd^h = c^h = L^h e^h$. Equation (3.4) means that $\frac{1}{h} d^h - G(p - p^h)$ is orthogonal to $G(p - p^h)$, which implies

$$\| \frac{1}{h} d^h \| = \| \frac{1}{h} d^h - G(p - p^h) \|^2_{E^h} + \| G(p - p^h) \| \geq \| G(p - p^h) \|.$$
On the other hand, the pointwise estimate of \( d_h \) given in (3.2) gives
\[
\langle \frac{1}{h}d^h, \frac{1}{h}d^h \rangle_{E_h} = \sum_{ij} H_{i+\frac{1}{2},j} \left( d_{i+\frac{1}{2},j} \right)^2 + \sum_{ij} H_{i,j+\frac{1}{2}} \left( d_{i,j+\frac{1}{2}} \right)^2
\]
\[
= \sum_{H_{i+\frac{1}{2},j}, H_{i,j+\frac{1}{2}} = 1} O(h^6) + \sum_{0 < H_{i+\frac{1}{2},j}, H_{i,j+\frac{1}{2}} < 1} O(h^4)
\]
\[
= O(h^6)O(h^{-2}) + O(h^4)O(h^{-1}) = O(h^3).
\]
Here, we used the fact that the number of inside edges, \( H_{i+\frac{1}{2},j} = 1 \) and \( H_{i,j+\frac{1}{2}} = 1 \), grows quadratically so that it becomes \( O(h^{-2}) \), and that of edges near the boundary is \( O(h^{-1}) \). Consequently, we have \( \|GP - Gp^h\| = O(h^{1.5}) \), which completes the proof. \( \square \)

4. CONVERGENCE ANALYSIS FOR PRESSURE \( p \)

In order to estimate the convergence error using the gradient estimation, we need the Poincare-Friedrichs inequality for piecewise constant functions as follows. Let \( D \in \mathbb{R}^2 \) be a bounded and connected polygonal domain and \( T \) a simplicial triangulation of \( D \). By \( \mathcal{E}^1(T) \), we denote the set of the interior edges of \( T \). For an interior edge \( e \) shared by two triangles \( T_1 \) and \( T_2 \) in \( T \), we define a jump \( \left[ [w] \right] \) across \( e \) as
\[
\left[ [w] \right] = w_1n_1 + w_2n_2
\]
where \( n_j \) is the outer normal unit vector of \( T_j \) and \( w_j = w_{iT_j} \) for \( j = 1, 2 \). Then we have the Poincare-Friedrichs inequality for piecewise constant functions with respect to \( T \) ([2, Lemma 10.6.6]).

**Lemma 4.1.** There exists a constant \( C > 0 \) depending only on the minimum angle of \( T \) such that
\[
\|c\|_{L^2(D)} \leq C \left[ \int_D |c dx| + \left( \sum_{e \in \mathcal{E}^1(T)} |e|^{-1}\|\left[ [c] \right]\|_{L^2(e)}^2 \right)^{1/2} \right]
\]
for any piecewise constant function \( c \) with respect to \( T \).

**Theorem 4.2.** Let \( u : \Omega^h \to \mathbb{R} \) be a discrete function with
\[
\int_{\Omega^h} u(P) dP = \sum_{P \in \Omega^h} u(P) \text{vol}(\Omega_P) = 0, \quad (\Omega_P = C_P \cap \Omega).
\]
Then there exists a constant \( C \) independent of the step size \( h \) such that
\[
C \frac{h_{\text{min}}}{h} \|u\|_{L^2(\Omega^h)}^2 \leq \sum_{Q \in E_h} \left( \frac{u(Q^+) - u(Q^-)}{h} \right)^2 H_Q h^2.
\] (4.1)
Proof. For each \((x_i, y_j)\), we take a control volume \(D_{ij} \subset [x_i - \frac{h}{2}, x_i + \frac{h}{2}] \times [y_j - \frac{h}{2}, y_j + \frac{h}{2}]\) as \(D_{ij}\) is a union of triangles such that \(\text{vol}(D_{ij}) = \text{vol}(\Omega_{ij})\) and
\[
D_{ij} \cap D_{i+1,j} = L_{i+\frac{1}{2},j}, \quad D_{ij} \cap D_{i,j+1} = L_{i,j+\frac{1}{2}}
\]
and any angle \(\theta\) of the triangles is bounded as
\[
\theta_1 \leq \theta \leq \theta_2
\]
where \(\theta_1\) and \(\theta_2\) are independent of \(h\). In this setting, we can see that for every edge \(e\), either \(e \cap E_Q = e\) for some \(Q \in E^h\) or \(|e \cap E_Q| = 0\) for all \(Q \in E^h\). From this setting, the number of triangles in \(D_{ij}\) sharing the edge \(L_{i+\frac{1}{2},j}\) is \(O((hH_{i+\frac{1}{2},j})/h_{\text{min}})\) where \(L_{i+\frac{1}{2},j} = hH_{i+\frac{1}{2},j}\) and \(h_{\text{min}} = \min\{|L_{i+\frac{1}{2},j}|, |L_{i,j+\frac{1}{2}}|\}\).

Now, we define a piecewise constant function \(u_c\) as \(u_c = u_{ij}\) on \(D_{ij}\). Then we have
\[
\int_{\cup D_{ij}} |u_c(x)|^2 dx = \int_{\Omega_h^2} |u(P)|^2 dP \quad \text{and} \quad \int_{\cup D_{ij}} u_c(x) dx = \int_{\Omega_h^2} u(P) dP = 0.
\]
We note that \(||[u_c]||_{L^2(e)} = 0\) if \(e \cap L_{i+\frac{1}{2},j} = 0\) and \(||[u_c]||_{L^2(e)}^2 = |e|(u_{i+1,j} - u_{ij})^2\) if \(|e \cap L_{i+\frac{1}{2},j}| = |e|\). Applying \(u_c\) to Lemma 4.1, we verify that there exists a constant \(C\) independent of \(h\) such that
\[
\int_{\cup D_{ij}} |u_c(x)|^2 dx \leq C \sum_{Q \in E_h} (u_c(Q^+) - u_c(Q^-))^2 \frac{H_Q h}{h_{\text{min}}}
\]
\[
= C \frac{h}{h_{\text{min}}} \sum_{Q \in E_h} \left(\frac{u(Q^+) - u(Q^-)}{h}\right)^2 H_Q h^2
\]
which completes the proof.

We note that from the argument used for the proof of Theorem 4.2, we can see that we have an equality with \(h/h_{\text{min}} = 1\) in the case when there are two positive constants \(C_1\) and \(C_2\) independent of \(h\) such that
\[
C_1 \leq \frac{H_{i+\frac{1}{2},j}}{H_{i+\frac{1}{2},j} + H_{i,j+\frac{1}{2}}} \leq C_2, \quad \text{for all} \ (x_i, y_j) \in \Omega_h.
\]
Since \(p + c\) is also an analytic solution of equation (2.1) for an analytic solution \(p\) and a constant \(c\), we may assume that
\[
\sum_{P \in \Omega_h^2} (p - p^h)(P)\text{vol}(\Omega_P) = 0
\]
for the numerical solution \(p_h\).
Theorem 4.3 (Convergence of pressure). Let $p$ be an analytic solution to (2.1) and let $p^h$ be the numerical solution to (3.1) such that
\[ \sum_{P \in \Omega^h} (p - p^h) \, (P) \text{vol}(\Omega_P) = 0. \]
Then we have
\[ \|p - p^h\|_{L^2(\Omega^h)} \leq O(h^2) \frac{1}{\sqrt{h_{\min}}} \]
with $h_{\min} = \min\{|L_{i,\pm \frac{1}{2},j}|, |L_{i,j,\pm \frac{1}{2}}|\}$.

Proof. Applying the convergence error $e^h = p - p^h$ to Theorem 4.2, we have
\[ \|e^h\|^2_{L^2(\Omega^h)} \leq C \frac{h}{h_{\min}} \sum_{Q \in E_h} \left( \frac{e^h(Q^+) - e^h(Q^-)}{h} \right)^2 H_Q h^2 = C \frac{h}{h_{\min}} \|Ge^h\|^2. \]
On the other hand, we showed $\|Ge^h\|^2 \leq O(h^3)$ in Theorem 3.5 (ii). Consequently, we have the convergence accuracy as
\[ \|e^h\|^2_{L^2(\Omega^h)} \leq O(h^4) \frac{1}{h_{\min}} \]
which shows the theorem. \qed

We observed in [12] that for many domains, however, we have $h_{\min} = O(h^2)$ as $h$ tends to zero.

Definition 4.4. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We say that $\Omega$ has the general intersection property if the cumulative distribution function $p(\nu)$ defined by
\[ p(\nu) := |\{(x_i, y_j) \in \Omega_h : \text{dist} ((x_i, y_j), \Gamma_h) \leq \nu\}| \quad (4.2) \]
is almost linear, i.e., $p(\nu) = O(h^{-2}\nu)$.

Many domains with smooth boundary as well as rectangular and circular shapes have the general intersection property. Note that when the domain $\Omega$ has the property, the set $\Omega_h^{\tau^*}$ becomes empty as $h$ tends to zero so that the threshold treatment works nothing.

Theorem 4.5. Let $\Omega$ be a bounded open domain with smooth boundary. Assume that $\Omega$ has the general intersection property. Then, for sufficiently small $h$, we have
\[ \|e^h\|_{L^2(\Omega^h)} = \|p - p^h\|_{L^2(\Omega^h)} = O(h). \]

Proof. Assume that $\Omega$ has the general intersection property. Then, we have $h_{\min} = O(h^2)$ as $h$ tends to zero because $p(h^\alpha) = O(h^{\alpha-2}) < 1$ for any $\alpha > 1$. In this case, Theorem 4.3 implies
\[ \|p - p^h\|_{L^2(\Omega^h)} \leq O(h^2) \frac{1}{\sqrt{h_{\min}}} = O(h) \]
and it shows the theorem. \qed
5. Numerical test

5.1. Two dimensional example. In $\Omega = \{(x, y) | x^2 + y^2 < 1\}$, we take a vector field $U = (u, v)$ with $u(x, y) = -2xy + \frac{xy}{\sqrt{x^2 + y^2}}$ and $v(x, y) = 3x^2 + y^2 - \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}$, and choose a scalar variable $p(x, y) = e^{x-y}$. Note that $U \cdot \vec{n} = 0$ on $\partial \Omega$ and $\nabla \cdot U = 0$ in $\Omega$. The run of the Gibou-Min method on $U^* = U + \nabla p$ is reported in Table 1.

<table>
<thead>
<tr>
<th>grid</th>
<th>$|U - U^h|_{L^2}$</th>
<th>order</th>
<th>$|p - p^h|_{L^2}$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$40^2$</td>
<td>$6.67 \times 10^{-4}$</td>
<td>1.33</td>
<td>$1.33 \times 10^{-3}$</td>
<td></td>
</tr>
<tr>
<td>$80^2$</td>
<td>$2.48 \times 10^{-5}$</td>
<td>1.42</td>
<td>$2.49 \times 10^{-4}$</td>
<td>2.41</td>
</tr>
<tr>
<td>$160^2$</td>
<td>$8.14 \times 10^{-6}$</td>
<td>1.60</td>
<td>$6.59 \times 10^{-5}$</td>
<td>1.92</td>
</tr>
<tr>
<td>$320^2$</td>
<td>$3.05 \times 10^{-6}$</td>
<td>1.41</td>
<td>$1.32 \times 10^{-5}$</td>
<td>2.31</td>
</tr>
<tr>
<td>$640^2$</td>
<td>$1.01 \times 10^{-6}$</td>
<td>1.58</td>
<td>$3.73 \times 10^{-6}$</td>
<td>1.82</td>
</tr>
</tbody>
</table>

Figure 1. Convergence order

5.2. Three dimensional example. In $\Omega = \{(x, y, z) | x^2 + y^2 + z^2 < 1\}$, we take a vector field $U = (x^2z + 3y^2z, -2xyz, -x^3 - xy^2)$ and a scalar variable $p(x, y, z) = e^{x-y+z}$. Note that $U \cdot \vec{n} = 0$ on $\partial \Omega$ and $\nabla \cdot U = 0$ in $\Omega$. The run of the Gibou-Min method on $U^* = U + \nabla p$ is reported in Table 2.
Table 2. Convergence order

<table>
<thead>
<tr>
<th>grid</th>
<th>$|U - U^h|_{L^2}$ order</th>
<th>$|p - p^h|_{L^2}$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20$^3$</td>
<td>$1.11 \times 10^{-2}$</td>
<td>$3.47 \times 10^{-3}$</td>
</tr>
<tr>
<td>40$^3$</td>
<td>$3.89 \times 10^{-3}$</td>
<td>$1.51 \times 10^{-4}$</td>
</tr>
<tr>
<td>80$^3$</td>
<td>$1.31 \times 10^{-3}$</td>
<td>$1.57 \times 10^{-4}$</td>
</tr>
<tr>
<td>160$^3$</td>
<td>$4.43 \times 10^{-4}$</td>
<td>$1.56 \times 3.50 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

6. Conclusion

In this work, we performed convergence analysis for the Gibou-Min method that calculates the Hodge-Helmholtz decomposition. Using the $L^2$-orthogonality between the error vector $U - U^h$ and the consistency $d^h$, we proved the estimate $\|U - U^h\|_{L^2} = O(h^{1.5})$ and $\|Gp - Gp^h\|_{L^2} = O(h^{1.5})$ for the gradient of the pressure error, as well. We then introduced a discrete version of the Poincare inequality, which led us to the result $\|p - p^h\| = O(h^{-0.5} h^2)$ with $h_{\text{min}} = \min\{L_{i,j+\frac{1}{2}}, L_{i+\frac{1}{2},j}\}$. Our estimate reads that for many domains with $h_{\text{min}} = O(h^2)$, for instance, domains with general intersection property, the pressure convergence is $O(h)$. According to our numerical tests, even though the estimate $\|U - U^h\|_{L^2} = O(h^{1.5})$ is tight, however, the estimate does not meet the observed order $\|p - p^h\| = O(h^2)$. We put it off to future research to improve the estimate.

References

GLOBAL ANALYSIS FOR A DELAY-DISTRIBUTED VIRAL INFECTION MODEL WITH ANTIBODIES AND GENERAL NONLINEAR INCIDENCE RATE

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ABSTRACT. In this work, we investigate the global stability analysis of a viral infection model with antibody immune response. The incidence rate is given by a general function of the populations of the uninfected target cells, infected cells and free viruses. The model has been incorporated with two types of intracellular distributed time delays to describe the time required for viral contacting an uninfected cell and releasing new infectious viruses. We have established a set of conditions on the general incidence rate function and determined two threshold parameters $R_0$ (the basic infection reproduction number) and $R_1$ (the antibody immune response activation number) which are sufficient to determine the global dynamics of the model. The global asymptotic stability of the equilibria of the model has been proven by using Lyapunov theory and applying LaSalle’s invariance principle.

1. INTRODUCTION

In recent past, many mathematicians have been presented and developed mathematical models in order to describe the interaction between the uninfected cells and the viruses such as human immunodeficiency virus (HIV) (see e.g. [1]-[22]), hepatitis B virus (HBV) [23]-[26], hepatitis C virus (HCV) [27]-[29], human T cell leukemia HTLV [30] and dengue virus [31], etc. Mathematical models of viral infection can help for understanding the viral dynamics and developing antiviral drug therapies. The immune system has two main responses to viral infections. The first is based on the Cytotoxic T Lymphocyte (CTL) cells which are responsible to attack and kill the infected cells. The second immune response is based on the antibodies that are produced by the B cells. The function of the antibodies is to attack the viruses [4]. In some infections such as in malaria, the CTL immune response is less effective than the antibody immune response [32]. In the literature, several mathematical models have been appeared to
consider the antibody immune response into the viral infection models ([33]-[39]). The basic model of viral infection with antibody immune response has been introduced by Murase et. al. [33] and Shifi Wang [39] as:

\[
\begin{align*}
\dot{x}(t) &= s - dx(t) - \beta v(t)x(t), \\
\dot{y}(t) &= \beta v(t)x(t) - ay(t), \\
\dot{v}(t) &= ky(t) - bz(t)v(t) - cv(t), \\
\dot{z}(t) &= rz(t)v(t) - \mu z(t),
\end{align*}
\]

where \(x(t), y(t), v(t)\) and \(z(t)\) denote the populations of uninfected target cells, infected cells, free virus particles and antibody immune cells at time \(t\), respectively. Parameters \(s, k\) and \(r\) represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the proliferation rate constant of antibody immune cells. Parameters \(d, a, c\) and \(\mu\) are the natural death rate constants of the uninfected cells, infected cells, free virus particles and antibody immune cells, respectively. Parameter \(\beta\) is the infection rate constant and \(b\) is the removal rate constant of the virus due to the antibodies. All the parameters given in model (1.1)-(1.4) are positive.

In model (1.1)-(1.4), the intracellular time delays in the viral life cycle is neglected. Actually, there is a delay between the virus contact a target cell and the creation of new infectious viruses. When the time delay is considered, the interaction between the viruses and target cells will be modeled by delay differential equations [11]-[20]. We note that, the incidence rate of infection is based on the mass action principle which can not be completely describe the interaction between the uninfected target cells and viruses. Nevertheless, there are many types of improved incidence rates which are more commonly used due to their benefit for helping us gain the unification theory through passing over the unessential details (see e.g. [40] and [41]). Different forms of the incidence rate have been considered in viral infection models with antibody immune response such as saturated incidence rate, \(\frac{\beta xv}{1+\alpha v}\) where \(\alpha \geq 0\) [42], [37], Beddington-DeAngelis functional response, \(\frac{\beta xv}{1+\gamma x+\alpha v}\), \(\alpha, \gamma \geq 0\) [36], and general form, \(\psi(x, v)v\) [38]. In [38], a discrete time delay has been incorporated within the model. However, the infection rate does not depend on the infected cells \(y\). In some viral infections such as HBV, the infection rate depends on \(x, y\) and \(v\) [25], [24]. In [43]-[46], a viral infection model with general incidence rate \(\psi(x, y, v)v\) and discrete time delays has been studied, however the antibody immune response has been neglected.

Our aim in this paper is to investigate the global stability analysis of a viral infection model with antibody immune response taking into consideration two types of distributed time delays. We assume that the incidence rate is given by a general function which satisfies a set of conditions. Two threshold parameters will be derived, the basic infection reproduction number \(R_0\) and the antibody immune response activation number \(R_1\). We will show that, under a set of conditions of the incidence rate function and on the parameters \(R_0\) and \(R_1\), the global stability of equilibria of the model can be established.
2. The mathematical model

In this section, we consider the following viral infection model with general incidence rate taking into consideration the antibody immune response and two types of intracellular distributed delays.

\[
\begin{align*}
\dot{x}(t) &= s - dx(t) - \psi(x(t), y(t), v(t))v(t), \\
\dot{y}(t) &= \int_{0}^{h_1} \rho_1(\tau)e^{-\mu_1\tau}\psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)d\tau - ay(t), \\
\dot{v}(t) &= Na \int_{0}^{h_2} \rho_2(\tau)e^{-\mu_2\tau}y(t - \tau)d\tau - bz(t)v(t) - cv(t), \\
\dot{z}(t) &= rz(t)v(t) - \mu z(t),
\end{align*}
\]  

(2.1)

(2.2)

(2.3)

(2.4)

where \(N\) is the average number of virus particles generated in the lifetime of the infected cell. We assume that, the virus contacts an uninfected target cell at time \(t - \tau\), the cell becomes infected at time \(t\), where \(\tau\) is linked to a probability distribution \(\rho_1(\tau)\) over the time interval \([0, h_1]\) and \(h_1\) is limit superior of this delay period. The term \(e^{-\mu_1\tau}\) represents the probability of surviving the contacted cell during the time delay interval, where \(\mu_1\) is the death rate constant of the contacted cells. In addition, we assume that, a cell infected at time \(t\) starts to generate new infectious viruses at time \(t\), where \(\tau\) is linked to a probability distribution \(\rho_2(\tau)\) over the time interval \([0, h_2]\) and \(h_2\) is limit superior of this delay. The term \(e^{-\mu_2\tau}\) denotes the probability of surviving the infected cell during the time delay interval, where \(\mu_2\) is a constant.

The definitions of all variables and parameters are identical to those given in Section 1. The incidence rate of infection is presented by a general function in the form \(\psi(x, y, v)v\), where \(\psi\) is continuously differentiable and satisfies the following assumptions:

**Assumption A1.** \(\psi(0, y, v) = 0\) for all \(y, v \geq 0\), \(\psi(x, y, v) > 0\) for all \(x > 0, y \geq 0, v \geq 0\).

**Assumption A2.** \(\frac{\partial \psi(x, y, v)}{\partial x} > 0\) for all \(x > 0, y \geq 0\) and \(v \geq 0\).

**Assumption A3.** \(\frac{\partial \psi(x, y, v)}{\partial y} < 0\), \(\frac{\partial \psi(x, y, v)}{\partial v} < 0\) for all \(x, y, v > 0\).

**Assumption A4.** \(\frac{\partial (\psi(x, y, v)v)}{\partial v} > 0\) for all \(x > 0, y > 0\) and \(v > 0\).

Let us assume that the probability distribution functions \(\rho_1(\tau)\) and \(\rho_2(\tau)\) satisfy \(\rho_1(\tau) > 0\) and \(\rho_2(\tau) > 0\), and

\[
\int_{0}^{h_1} \rho_1(\tau)d\tau = \int_{0}^{h_2} \rho_2(\tau)d\tau = 1, \quad \int_{0}^{h_1} \rho_1(u)e^{\ell u}du < \infty, \quad \int_{0}^{h_2} \rho_2(u)e^{\ell u}du < \infty,
\]
where \( \ell > 0 \). Let us denote:

\[
F = \int_{0}^{h_1} \rho_1(\tau)e^{-\mu_1 \tau} d\tau, \quad G = \int_{0}^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} d\tau.
\]

Thus

\[0 < F \leq 1, \quad 0 < G \leq 1.\]

Let the initial conditions for system (2.1)-(2.4) be given as:

\[
x(\eta) = \xi_1(\eta), \quad y(\eta) = \xi_2(\eta), \quad v(\eta) = \xi_3(\eta), \quad z(\eta) = \xi_4(\eta),
\]

\[
\xi_j(\eta) \geq 0, \quad \eta \in [-\omega, 0), \quad j = 1, ..., 4,
\]

\[
\xi_j(0) > 0, \quad j = 1, ..., 4; \quad (2.5)
\]

where \( \omega = \max\{h_1, h_2\}, (\xi_1(\eta), \xi_2(\eta), \xi_3(\eta), \xi_4(\eta)) \in C([-\omega, 0], \mathbb{R}^4_{\geq 0}). \) We denote by \( C = C([-\omega, 0], \mathbb{R}^4_{\geq 0}) \) the Banach space of continuous functions mapping the interval \([-\omega, 0]\) into \( \mathbb{R}^4_{\geq 0} \) with norm

\[||\xi|| = \sup_{-\omega \leq \eta \leq 0} |\xi(\eta)| \text{ for } \xi \in C.\]

We note that the system (2.1)-(2.4) with initial states (2.5) has a unique solution [47].

3. NON-NEGATIVITY AND BOUNDEDNESS OF SOLUTIONS

In this section, we show that the solutions of (2.1)-(2.4) with initial states (2.5) are non-negative and bounded.

**Proposition 3.1.** Assume that Assumption A1 is satisfied. Then all solutions of (2.1)-(2.4) with initial conditions (2.5), are non-negative and ultimately bounded.

**Proof.** The solution \((x(t), y(t), v(t), z(t))\) of system (2.1)-(2.4) with initial states (2.5) exists and is unique on its maximal interval of existence \([0, \gamma]\) for some \(\gamma > 0\) [47]. We see that \(x(t) > 0\) for all \(t \in [0, \gamma]\). Indeed this follows from equation (2.1) that \(\dot{x} |_{x=0} = s > 0\), for any \(t \in [0, \gamma]\). Now from Eqs. (2.2)-(2.4) we get

\[
y(t) = y(0)e^{-at} + \int_{0}^{t} e^{-a(t-\eta)} \int_{0}^{h_1} \rho_1(\tau)e^{-\mu_1 \tau} \psi(x(\eta - \tau), y(\eta - \tau), v(\eta - \tau)) v(\eta - \tau) d\tau d\eta,
\]

\[
v(t) = v(0)e^{-\int_{0}^{t}(e^{bz}\xi) d\xi} + Na \int_{0}^{t} e^{-\int_{0}^{\tau}(e^{bz}\xi) d\xi} \int_{0}^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} y(\eta - \tau) d\tau d\eta,
\]

\[
z(t) = z(0)e^{-\int_{0}^{t}(\mu-rv) d\xi},
\]

which yield \(y(t), v(t), z(t) \geq 0\) for all \(t \in [0, \omega]\). By a recursive argument, we get that \(y(t), v(t), z(t) \geq 0\) for all \(t \geq 0\).
Next we prove the ultimate bound of the solutions of system (2.1)-(2.4). From equation (2.1) we get \( \dot{x}(t) \leq s - dx(t) \) and thus \( \limsup_{t \to \infty} x(t) \leq \frac{s}{d} \). Let 

\[
T_1(t) = \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}x(t-\tau)d\tau + y(t),
\]

then

\[
\dot{T}_1(t) = \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}(s - dx(t-\tau) - \psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau))d\tau
\]

\[+ \int_0^{h_2} \rho_1(\tau)e^{-\mu_1 \tau}\psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)d\tau - ay(t)
\]

\[= \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}d\tau - d \int_0^{h_2} \rho_1(\tau)e^{-\mu_1 \tau}x(t-\tau)d\tau - ay(t)
\]

\[\leq \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}d\tau - \sigma_1 \left( \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}x(t-\tau)d\tau + y(t) \right)
\]

\[= \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}d\tau - \sigma_1 T_1(t) \leq s - \sigma_1 T_1(t),
\]

where \( \sigma_1 = \min\{d, a\} \). Hence \( \limsup_{t \to \infty} T_1(t) \leq L_1 \), where \( L_1 = \frac{s}{\sigma_1} \). Since \( \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau}x(t-\tau)d\tau > 0 \) and \( y(t) \geq 0 \), then \( \limsup_{t \to \infty} y(t) \leq L_1 \). Moreover, let \( T_2(t) = v(t) + \frac{b}{r}z(t) \),

\[
\dot{T}_2(t) = Na \int_0^{h_2} p_2(\tau)e^{-\mu_2 \tau}y(t-\tau)d\tau - cv(t) - \frac{b}{r}z(t)
\]

\[\leq NaL_1 \int_0^{h_2} p_2(\tau)e^{-\mu_2 \tau}d\tau - \sigma_2(v(t) + \frac{b}{r}z(t))
\]

\[= NaL_1 \int_0^{h_2} p_2(\tau)e^{-\mu_2 \tau}d\tau - \sigma_2 T_2(t) \leq NaL_1 - \sigma_2 T_2(t),
\]

where \( \sigma_2 = \min\{c, \mu\} \). It follows that, \( \limsup_{t \to \infty} T_2(t) \leq L_2 \), where \( L_2 = \frac{NaL_1}{\sigma_2} \). Since \( v(t) \) and \( z(t) \) are non-negative, \( \limsup_{t \to \infty} v(t) \leq L_2 \) and \( \limsup_{t \to \infty} z(t) \leq L_3 \), where \( L_3 = \frac{b}{r}L_2 \). Therefore, all the state variables of the system are ultimately bounded. \( \square \)
4. THE EQUILIBRIA AND THRESHOLD PARAMETERS

At any equilibrium we have

\begin{align*}
    s - dx - \psi(x, y, v)v &= 0, \quad (4.1) \\
    F\psi(x, y, v)v - ay &= 0, \quad (4.2) \\
    NaGy - bv - cv &= 0, \quad (4.3) \\
    (rv - \mu)z &= 0. \quad (4.4)
\end{align*}

From equation (4.4), either \( z = 0 \) or \( z \neq 0 \). If \( z = 0 \), then from Eqs. (4.1)-(4.3) we get

\begin{align*}
    y &= \frac{F(s - dx)}{a} = \frac{c}{NaG}v, \quad v = \frac{NFG(s - dx)}{c}. \quad (4.5)
\end{align*}

Substituting from equation (4.5) into equation (4.2) we get:

\begin{align*}
    \left[ \psi \left( x, \frac{F(s - dx)}{a}, \frac{NFG(s - dx)}{c} \right) - \frac{c}{NFG} \right] v &= 0. \quad (4.6)
\end{align*}

Equation (4.6) has two possible solutions \( v = 0 \) or \( v \neq 0 \). If \( v = 0 \), then from Eqs. (4.1) and (4.2), we get \( x = s/d \) and \( y = 0 \) which leads to the infection-free equilibrium \( E_0(x_0, 0, 0, 0) \) where \( x_0 = s/d \). If \( v \neq 0 \), then we have

\begin{align*}
    \psi \left( x, \frac{F(s - dx)}{a}, \frac{NFG(s - dx)}{c} \right) - \frac{c}{NFG} &= 0.
\end{align*}

Let

\begin{align*}
    \Phi_1(x) &= \psi \left( x, \frac{F(s - dx)}{a}, \frac{NFG(s - dx)}{c} \right) - \frac{c}{NFG} = 0,
\end{align*}

then, we have

\begin{align*}
    \Phi'_1(x) &= \frac{\partial \psi}{\partial x} - \frac{F d \partial \psi}{a \partial y} - \frac{NFGd \partial \psi}{c \partial v}.
\end{align*}

Because of Assumptions A2 and A3, we have \( \Phi'_1(x) > 0 \) which implies that function \( \Phi_1(x) \) is strictly increasing function of \( x \). Moreover,

\begin{align*}
    \Phi_1(0) &= \psi \left( 0, \frac{Fs}{a}, \frac{NFGs}{c} \right) - \frac{c}{NFG} = \frac{c}{NFG} < 0, \\
    \Phi_1(x_0) &= \psi(x_0, 0, 0) - \frac{c}{NFG} = \frac{c}{NFG} \left( \frac{NFG\psi(x_0, 0, 0)}{c} - 1 \right).
\end{align*}

Therefore, if \( \frac{NFG\psi(x_0, 0, 0)}{c} > 1 \), then there exists a unique \( x_1 \in (0, x_0) \) such that \( \Phi_1(x_1) = 0 \). It follows from equation (4.5) that \( y_1 = \frac{Fd(x_0 - x_1)}{a} > 0 \) and \( v_1 = \frac{NFGd(x_0 - x_1)}{c} > 0 \). Therefore, a chronic-infection equilibrium without antibody immune response \( E_1^c(x_1, y_1, v_1, 0) \) exists when \( \frac{NFG\psi(x_0, 0, 0)}{c} > 1 \). Let us define the basic infection reproduction number as:
The parameter $R_0$ determines whether a chronic-infection can be established. The other possibility of equation (4.4) is $z \neq 0$ which leads to $v_2 = \frac{\mu}{r}$. From equation (4.1) we let

$$\Phi_2(x) = s - dx - \psi(x, \frac{F(s - dx)}{a}, v_2)v_2 = 0.$$ 

According to Assumptions A2 and A3, we know that $\Phi_2$ is a strictly decreasing function of $x$. Clearly, $\Phi_2(0) = s > 0$ and $\Phi_2(x_0) = -\psi(x_0, 0, v_2)v_2 < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. It follows from Eqs. (4.3) and (4.5) that, $y_2 = \frac{Fd(x_0 - x_2)}{a} > 0$ and

$$z_2 = \frac{NFG\psi(x_2, y_2, v_2)}{b} - \frac{c}{b} = \frac{c}{b} \left( \frac{NFG\psi(x_2, y_2, v_2)}{c} - 1 \right).$$

Then, if $\frac{NFG\psi(x_2, y_2, v_2)}{c} > 1$ then $z_2 > 0$. Now we define the antibody immune response activation number as:

$$R_1 = \frac{NFG\psi(x_2, y_2, v_2)}{c},$$

which determines whether a persistent antibody immune response can be established. Hence, $z_2$ can be rewritten as $z_2 = \frac{c}{b}(R_1 - 1)$. It follows that, there exists a chronic-infection equilibrium with antibody immune response $E_2(x_2, y_2, v_2, z_2)$ when $R_1 > 1$.

Clearly from Assumptions A2 and A3, we have

$$R_1 = \frac{NFG\psi(x_2, y_2, v_2)}{c} < \frac{NFG\psi(x_0, y_2, v_2)}{c} < \frac{NFG\psi(x_0, 0, 0)}{c} = R_0.$$ 

5. Global stability analysis

In this subsection, we give proofs of the global asymptotic stability of the three equilibria of model (2.1)-(2.4) by using direct Lyapunov method and applying LaSalle’s invariance principle. Let us define the function $H : (0, \infty) \rightarrow [0, \infty)$ as

$$H(u) = u - 1 - \ln u.$$ 

**Theorem 5.1.** Let Assumptions A1-A3 be hold true and $R_0 \leq 1$, then the infection-free equilibrium $E_0$ is globally asymptotically stable (GAS).
Proof. We construct a Lyapunov functional as:

\[ U_0 = NFG \left[ x - x_0 - \int_{x_0}^{x} \frac{\psi(x_0, 0, 0)}{\psi(\eta, 0, 0)} d\eta + \frac{1}{F} y \right. \]

\[ + \frac{1}{F} \int_{0}^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} \int_{0}^{\tau} \psi(x(t - \eta), y(t - \eta), v(t - \eta)) v(t - \eta) d\eta d\tau \]

\[ + \frac{a}{FG} \int_{0}^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} \int_{0}^{\tau} y(t - \eta) d\eta d\tau \right] + v + \frac{b}{r} z. \]  

(5.1)

We calculate \( \frac{dU_0}{dt} \) along the solutions of model (2.1)-(2.4) as:

\[ \frac{dU_0}{dt} = NFG \left[ \left( 1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} \right) (s - dx - \psi(x, y, v) v) \right. \]

\[ + \frac{1}{F} \int_{0}^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} \psi(x(t - \tau), y(t - \tau), v(t - \tau)) v(t - \tau) d\tau - \frac{a}{F} y \]

\[ + \frac{1}{F} \int_{0}^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} (\psi(x, y, v) v - \psi(x(t - \tau), y(t - \tau), v(t - \tau)) v(t - \tau)) d\tau \]

\[ + \frac{a}{FG} \int_{0}^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} (y(t - \tau)) d\tau \right] + Na \int_{0}^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} y(t - \tau) d\tau \]

\[ - b z v - c v + b z v - \frac{b \mu}{r} z \]

\[ = NFGs \left( 1 - \frac{x}{x_0} \right) \left( 1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} \right) + \left( NFG \psi(x, y, v) \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} - c \right) v - \frac{b \mu}{r} z \]

\[ = NFGs \left( 1 - \frac{x}{x_0} \right) \left( 1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} \right) + c \left( R_0 \frac{\psi(x, y, v)}{\psi(x, 0, 0)} - 1 \right) v - \frac{b \mu}{r} z. \]  

(5.2)

From Assumptions A2-A3 we know that \( \psi(x, y, v) \) is a strictly increasing function of \( x \) and a strictly decreasing function of \( y \) and \( v \). Then, the first term of equation (5.2) is less than or equal zero and

\[ \psi(x, y, v) < \psi(x, 0, 0), \quad x, y, v > 0 \]

It follows that

\[ \frac{dU_0}{dt} \leq NFGs \left( 1 - \frac{x}{x_0} \right) \left( 1 - \frac{\psi(x_0, 0, 0)}{\psi(x, 0, 0)} \right) + c (R_0 - 1) v - \frac{b \mu}{r} z. \]  

(5.3)
Therefore, if $R_0 \leq 1$, then $\frac{dU}{dt} \leq 0$ for all $x, y, v, z > 0$. Hence, solutions of system (2.1)-(2.4) with (2.5) limited to $M$, the largest invariant subset of $\left\{ \frac{dU}{dt} = 0 \right\}$ [47]. We see that $\frac{dU}{dt} = 0$ if and only if $x(t) = x_0$, $v(t) = 0$ and $z(t) = 0$ for all $t$. By the above discussion each element of $M$ satisfies $v(t) = 0$ and $z(t) = 0$. Then, from equation (2.3)

$$\dot{v}(t) = 0 = Na \int_{0}^{h_2} \rho_2(\tau)e^{-\mu_2 \tau}y(t-\tau)d\tau.$$  

It follows that, $y(t) = 0$ for all $t$. Using LaSalle’s invariance principle, we derive that $E_0$ is GAS.

To proof the global stability of the two equilibria $E_1$ and $E_2$, we need the following condition on the incidence rate function $\psi$.

**Assumption A5.**

$$\left(1 - \frac{\psi(x, y, v)}{\psi(x, y_i, v_i)}\right) \left(\frac{\psi(x, y_i, v_i)}{\psi(x, y, v)} - \frac{v}{v_i}\right) \leq 0, \ i = 1, 2 \text{ for all } x, y, v > 0.$$

**Theorem 5.2.** Let Assumptions A1-A5 be hold true and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without antibody immune response $E_1$ is GAS.

**Proof.** Define

$$U_1 = NFG \left[ x - x_1 - \int_{x_1}^{x} \frac{\psi(x_1, y_1, v_1)}{\psi(\eta, y_1, v_1)} d\eta + \frac{1}{F}y_1H \left( \frac{y}{y_1} \right) \right.\left. + \frac{\psi(x_1, y_1, v_1)v_1}{F} \int_{0}^{h_1} \rho_1(\tau)e^{-\mu_1 \tau} \int_{0}^{\tau} H \left( \frac{\psi(x(t-\eta), y(t-\eta), v(t-\eta))}{\psi(x_1, y_1, v_1)v_1} \right) d\eta d\tau \right.\left. + \frac{ag_1}{FG} \int_{0}^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} \int_{0}^{\tau} H \left( \frac{y(t-\eta)}{y_1} \right) d\eta d\tau \right] + v_1H \left( \frac{v}{v_1} \right) + \frac{b}{r}z.$$  

(5.4)
Calculating the time derivative of $U_1$ along the trajectories of system (2.1)-(2.4), we obtain

$$\frac{dU_1}{dt} = NFG \left[ \left( 1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) \left( s - dx - \psi(x, y, v) v \right) + \frac{1}{F} \left( 1 - \frac{y_1}{y} \right) \left( \int_0^{h_1} \rho_1(\tau)e^{-\mu_1\tau}\psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)d\tau - ay \right) + \frac{1}{F} \int_0^{h_1} \rho_1(\tau)e^{-\mu_1\tau}(\psi(x, y, v) - \psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau))d\tau \right]$$

$$+ \psi(x_1, y_1, v_1)v_1 \ln \left( \frac{\psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)}{\psi(x, y, v)v} \right) d\tau + \frac{a}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau}(y - y(t-\tau) + y_1 \ln \left( \frac{y(t-\tau)}{y} \right)) d\tau$$

$$+ \frac{B}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau}_y y(t-\tau) + \frac{y_1}{y} \left( \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau} \psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)d\tau + \frac{a}{F} y_1 \right)$$

$$- \frac{1}{F} \int_0^{h_1} \rho_1(\tau)e^{-\mu_1\tau}\psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)d\tau + \frac{a}{F} y_1$$

$$- \frac{B}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau} y(t-\tau) + \frac{y_1}{y} \left( \int_0^{h_1} \rho_1(\tau)e^{-\mu_1\tau} \ln \left( \frac{\psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)}{\psi(x, y, v)v} \right) d\tau \right)$$

$$- \frac{B}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau} \ln \left( \frac{y(t-\tau)}{y} \right) d\tau - cv \right] \right] - cv$$

$$= \frac{B}{FG} \left( \psi(x_1, y_1, v_1)v_1 \sqrt{B} = a \right)$$

$$- \frac{y_1}{y} \left( \int_0^{h_1} \rho_1(\tau)e^{-\mu_1\tau} \psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)d\tau + \frac{a}{F} y_1 \right)$$

$$- \frac{B}{FG} \left( \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau} y(t-\tau)d\tau + cv + \frac{y_1}{y} \right)$$

$$- \frac{B}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau} \ln \left( \frac{y(t-\tau)}{y} \right) d\tau - cv$$

$$- \frac{y_1}{y} \left( \int_0^{h_1} \rho_1(\tau)e^{-\mu_1\tau} \psi(x(t-\tau), y(t-\tau), v(t-\tau))v(t-\tau)d\tau + \frac{a}{F} y_1 \right)$$

$$- \frac{B}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2\tau} \ln \left( \frac{y(t-\tau)}{y} \right) d\tau - cv$$

$$= \frac{B}{FG} \left( \psi(x_1, y_1, v_1)v_1 \sqrt{B} = a \right)$$

Using the equilibrium conditions for $E_1$:

$$s = dx_1 + \frac{a}{F} y_1, \quad F\psi(x_1, y_1, v_1)v_1 = ay_1, \quad cv_1 = NaGy_1,$$
we obtain

\[ \frac{dU_1}{dt} = NG \left[ dx_1 \left(1 - \frac{x}{x_1}\right) \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)}\right) + \frac{a}{F} y_1 \right. \]

\[ - \frac{a}{F} y_1 \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} + \frac{a}{F} y_1 \frac{\psi(x, y, v)}{\psi(x, y_1, v_1)} \]

\[ - \frac{a}{F} y_1 \int_0^{h_1} \rho_1(\tau)e^{-\mu_1 \tau} \frac{y_1 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y\psi(x_1, y_1, v_1)v_1} d\tau + 2 \frac{a}{F} y_1 \]

\[ + \frac{a}{F} y_1 \int_0^{h_1} \rho_2(\tau)e^{-\mu_2 \tau} \ln \left( \frac{\psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{\psi(x, y, v)} \right) d\tau \]

\[ + \frac{ay_1}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} \ln \left( \frac{y(t - \tau)}{y} \right) d\tau \]

\[ - \frac{ay_1}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} \frac{v_1 y(t - \tau)}{vy_1} d\tau - \frac{a}{F} y_1 \frac{v}{v_1} \]

\[ + b \left( v_1 - \frac{\mu}{r} \right) z. \]  

(5.6)

Using the following equalities:

\[ \ln \left( \frac{\psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{\psi(x, y, v)} \right) \]

\[ = \ln \left( \frac{y_1 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y\psi(x_1, y_1, v_1)v_1} \right) \]

\[ + \ln \left( \frac{\psi(x_1, y_1, v_1)}{\psi(x_1, y_1, v_1)} \right) + \ln \left( \frac{\psi(x, y_1, v_1)}{\psi(x, y_1, v_1)} \right) + \ln \left( \frac{v_1 y}{vy_1} \right), \]

\[ \ln \left( \frac{y(t - \tau)}{y} \right) = \ln \left( \frac{vy_1}{vy_1} \right) + \ln \left( \frac{vy_1(t - \tau)}{vy_1} \right), \]
we get

\[
\frac{dU_1}{dt} = NFG \left[ \frac{dx_1}{1 - x_1} \left( 1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) - \frac{ay_1}{F} \left( \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} - 1 \right) \right]
\]

From Assumptions A1 and A5, we get that the first and second terms of equation (5.8) are less than or equal zero. Now we show that if \( R_1 \leq 1 \) then \( v_1 \leq \frac{\mu}{r} = v_2 \). Let \( R_0 > 1 \), then we want to show that

\[
\text{sgn}(x_2 - x_1) = \text{sgn}(v_1 - v_2) = \text{sgn}(y_1 - y_2) = \text{sgn}(R_1 - 1).
\]

From Assumptions A2-A4, for \( x_1, x_2, y_1, y_2, v_1, v_2 > 0 \), we have

\[
(\psi(x_2, y_2, v_2) - \psi(x_1, y_2, v_2))(x_2 - x_1) > 0,
\]
\[ (\psi(x_1, y_1, v_1) - \psi(x_1, y_2, v_1))(y_2 - y_1) > 0, \quad (5.10) \]
\[ (\psi(x_1, y_1, v_1) - \psi(x_1, y_1, v_2))(v_2 - v_1) > 0, \quad (5.11) \]
\[ (\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_1))(v_2 - v_1) > 0. \quad (5.12) \]

First, we claim \( \text{sgn}(x_2 - x_1) = \text{sgn}(v_1 - v_2) \). Suppose this is not true, i.e., \( \text{sgn}(x_2 - x_1) = \text{sgn}(v_2 - v_1) \). Using the conditions of the equilibria \( E_1 \) and \( E_2 \) we have
\[ (s - dx_2) - (s - dx_1) = \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1 \]
\[ = \frac{a}{F}(y_2 - y_1). \quad (5.13) \]

Then
\[ \text{sgn}(x_2 - x_1) = \text{sgn}(y_1 - y_2). \quad (5.14) \]

Moreover,
\[ (s - dx_2) - (s - dx_1) = \psi(x_2, y_2, v_2)v_2 - \psi(x_1, y_1, v_1)v_1 \]
\[ = (\psi(x_2, y_2, v_2)v_2 - \psi(x_2, y_2, v_1)v_1) + (\psi(x_2, y_2, v_1)v_1 \]
\[ - \psi(x_1, y_2, v_1)v_1 + (\psi(x_1, y_2, v_1)v_1 - \psi(x_1, y_1, v_1)v_1). \]

Therefore, from Eqs. (5.9)-(5.14) we get:
\[ \text{sgn}(x_1 - x_2) = \text{sgn}(x_2 - x_1), \]

which leads to contradiction. Thus, \( \text{sgn}(x_2 - x_1) = \text{sgn}(v_1 - v_2) \). Using the equilibrium conditions for \( E_1 \) we have \( \frac{NFG\psi(x_1, y_1, v_1)}{c} = 1 \), then
\[ R_1 - 1 = \frac{NFG\psi(x_2, y_2, v_2)}{c} - \frac{NFG\psi(x_1, y_1, v_1)}{c} \]
\[ = \frac{NFG}{c} [\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_1) + \psi(x_2, y_2, v_1) \]
\[ - \psi(x_1, y_2, v_1) + \psi(x_1, y_2, v_1) - \psi(x_1, y_1, v_1)]. \]

We get \( \text{sgn}(R_1 - 1) = \text{sgn}(v_1 - v_2) \). Hence, if \( R_0 > 1 \), then \( x_1, y_1, v_1 > 0 \), and if \( R_1 \leq 1 \), then \( v_1 \leq v_2 = \frac{\mu}{r} \). It follows from the above discussion that \( \frac{dV_1}{dt} \leq 0 \) for all \( x, y, v, z > 0 \). The solutions of model (2.1)-(2.4) converge to \( \Omega \), the largest invariant subset of \( \{ (x, y, v, z) : \frac{dV_1}{dt} = 0 \} \). We have \( \frac{dV_1}{dt} = 0 \) iff \( x = x_1, v = v_1, z = 0 \) and \( H = 0 \) i.e.
\[ \frac{y_1\psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y\psi(x_1, y_1, v_1)v_1} = \frac{v_1y(t - \tau)}{v_1y_1} = 1 \text{ for all } \tau \in [0, \omega]. \quad (5.15) \]

From equation (5.15), if \( v = v_1 \) then \( y = y_1 \) and hence \( \frac{dV_1}{dt} = 0 \) at \( E_1 \). So \( \Omega \) contains a unique point, that is \( E_1 \). Thus, the global asymptotic stability of the chronic-infection equilibrium without antibody immune response \( E_1 \) follows from LaSalle’s invariance principle.

In the following we consider the global asymptotic stability of the chronic-infection equilibrium with antibody immune response \( E_2 \).

**Theorem 5.3.** Let Assumptions A1-A5 be hold true and \( R_1 > 1 \), then \( E_2 \) is GAS.
Proof. We construct a Lyapunov functional in the form:

\[
U_2 = NFG \left[ x - x_2 - \int_{x_2}^{x} \frac{\psi(x_2, y_2, v_2)}{\psi(y_2, v_2)} \, d\eta + \frac{1}{F} y_2 H \left( \frac{y}{y_2} \right) \right. \\
+ \frac{\psi(x_2, y_2, v_2)}{F} \int_{0}^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} \int_{0}^{\tau} H \left( \frac{\psi(x(t - \eta), y(t - \eta), v(t - \eta)) v(t - \eta)}{\psi(x_2, y_2, v_2)} \right) \, d\eta d\tau \\
+ \frac{\rho_2(\tau) e^{-\mu_2 \tau}}{FG} \int_{0}^{h_2} \frac{\rho_2(\tau) e^{-\mu_2 \tau}}{y_2} \int_{0}^{\tau} H \left( \frac{y(t - \eta)}{y_2} \right) \, d\eta d\tau \\
\left. + v_2 H \left( \frac{v}{v_2} \right) + \frac{b}{r} z_2 H \left( \frac{z}{z_2} \right) \right]. 
\]  

(5.16)

Function \( U_2 \) satisfies:

\[
\frac{dU_2}{dt} = NFG \left[ \left( 1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y, v)} \right) (s - dx - \psi(x, y, v)) \right. \\
\left. + \frac{1}{F} \left( 1 - \frac{y_2}{y} \right) \left( \int_{0}^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} \psi(x(t - \tau), y(t - \tau), v(t - \tau)) v(t - \tau) d\tau - a y \right) \right. \\
\left. + \frac{1}{F} \int_{0}^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} (\psi(x, y, v) v - \psi(x(t - \tau), y(t - \tau), v(t - \tau)) v(t - \tau)) v(t - \tau) d\tau \right. \\
\left. + \psi(x_2, y_2, v_2) v_2 \ln \left( \frac{\psi(x(t - \tau), y(t - \tau), v(t - \tau)) v(t - \tau)}{\psi(x, y, v) v} \right) \right] \, d\tau \\
\left. + \frac{a}{FG} \int_{0}^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} \left( y - y(t - \tau) + y_2 \ln \left( \frac{y(t - \tau)}{y} \right) \right) \, d\tau \right] \\
\left. + \left( 1 - \frac{v_2}{v} \right) \left( Na \int_{0}^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} y(t - \tau) d\tau - b z v - c v \right) \right. \\
\left. + \left( 1 - \frac{z_2}{z} \right) \left( b z v - \frac{b \mu}{r} z \right) \right]. 
\]  

(5.17)
Collecting terms of equation (5.17) and applying \( s = dx_2 + \frac{a}{F} y_2 \), we get

\[
\frac{dU_2}{dt} = NFG \left[ d(x_2 - x) \left( 1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) + \frac{a}{F} y_2 - \frac{a}{F} y_2 \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right.
\]
\[+ \psi(x, y, v) \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \frac{1}{F} \int_0^{h_1} (\rho_1(\tau)e^{-\mu_1 \tau}) \frac{y_2 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y \psi(x_2, y_2, v_2)v_2} d\tau + \frac{a}{F} y_2 \]
\[+ \frac{a y_2}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} \ln \left( \frac{y(t - \tau)}{y} \right) d\tau - \frac{a y_2}{FG} \int_0^{h_2} \rho_2(\tau)e^{-\mu_2 \tau} \frac{v_2 y_2(t - \tau)}{v y_2} d\tau \]
\[= -cv + cv_2 + bv_2 z - bv z_2 - \frac{b \mu}{r} z + \frac{b \mu}{r} z_2. \]  

(5.18)

By using the equilibrium conditions of \( E_2 \)

\[F \psi(x_2, y_2, v_2)v_2 = ay_2, \quad cv_2 = NaGy_2 - bv z_2, \quad \mu = rv_2, \]

and the following equalities

\[cv = cv_2 \frac{v}{v_2} = (NaGy_2 - bv z_2) \frac{v}{v_2}, \]
\[
\ln \left( \frac{\psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{\psi(x, y, v)} \right)
= \ln \left( \frac{y_2 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y \psi(x_2, y_2, v_2)v_2} \right)
+ \ln \left( \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) + \ln \left( \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)} \right)
+ \ln \left( \frac{v_2 y_2}{v y_2} \right), \]
\[
\ln \left( \frac{y(t - \tau)}{y} \right) = \ln \left( \frac{v y_2}{v_2 y} \right) + \ln \left( \frac{v_2 y(t - \tau)}{v y_2} \right), \]
we obtain
\[
\frac{dU_2}{dt} = NFG \left[ d(x_2 - x) \left( 1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x_2, y_2, v_2)} \right) + \frac{ay_2}{F} \left( -\frac{\psi(x_2, y_2, v_2)}{\psi(x_2, y_2, v_2)} \right) \right]
\]

\[
+ \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)} \frac{v}{v_2} - 1 \right) \right) \right) \] - \frac{ay_2}{F} \left( \psi(x, y_2, v_2) \right) - 1 - \ln \left( \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)} \right) \right) \right) \]

\[- \frac{ay_2}{F^2} \int_0^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} \left( \frac{y_2 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y\psi(x_2, y_2, v_2)} \right) - 1 \right) \right) \right) \]

\[- \frac{ay_2}{FG} \int_0^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} \left( \frac{v_2 y(t - \tau)}{v y_2} \right) - 1 - \ln \left( \frac{v_2 y(t - \tau)}{v y_2} \right) \right) \right) \right) \]

(5.19)

We can rewrite equation (5.19) as
\[
\frac{dU_2}{dt} = NFG \left[ dx_2 \left( 1 - \frac{x_2}{x} \right) \left( 1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x_2, y_2, v_2)} \right) + \frac{ay_2}{F} \left( 1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x_2, y_2, v_2)} \right) \right]
\]

\[- \frac{ay_2}{F^2} \int_0^{h_1} \rho_1(\tau) e^{-\mu_1 \tau} \left( \frac{y_2 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y\psi(x_2, y_2, v_2)} \right) - 1 \right) \right) \]

\[- \frac{ay_2}{FG} \int_0^{h_2} \rho_2(\tau) e^{-\mu_2 \tau} \left( \frac{v_2 y(t - \tau)}{v y_2} \right) - 1 - \ln \left( \frac{v_2 y(t - \tau)}{v y_2} \right) \right) \right) \]

(5.20)

We note that from Assumptions A2 and A5, the first and second terms of equation (5.20) are less than or equal zero. Noting that \(x, y, v, z > 0\), we have that \(\frac{dU_2}{dt} \leq 0\). The solutions of model (2.1)-(2.4) converge to \(\Omega\), the largest invariant subset of \(\{(x, y, v, z) : \frac{dU_2}{dt} = 0\}\)[47].

We have \(\frac{dU_2}{dt} = 0\) if and only if \(x = x_2, v = v_2\) and \(H = 0\) i.e.,
\[
\frac{y_2 \psi(x(t - \tau), y(t - \tau), v(t - \tau))v(t - \tau)}{y\psi(x_2, y_2, v_2)v_2} = \frac{v_2 y(t - \tau)}{v y_2} = 1 \text{ for all } \tau \in [0, \omega].
\]

(5.21)

This yields that \(y = y_2\) for all \(\tau \in [0, \omega]\). Therefore, if \(v = v_2\) and \(y = y_2\), then from equation (2.3) we have \(0 = ky_2 - b v_2 - cv_2\) which gives \(z = z_2\). Therefore, \(\frac{dU_2}{dt} = 0\) at \(E_2\). The global
asymptotic stability of the chronic-infection equilibrium with antibody immune response $E_2$ follows from LaSalle’s invariance principle.

6. Conclusion

In this paper, we have proposed a viral infection model with general incidence rate function and antibody immune response. Two types of distributed time delays have been incorporated into the model to describe the time needed for the virus enters the target cell and the emission of new infectious viruses. We have derived a set of conditions on the general functional response and have determined two threshold parameters $R_0$ and $R_1$ to prove the existence and the global stability of the model’s equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without antibody immune response and chronic-infection with antibody immune response has been proven by using direct Lyapunov method and LaSalle’s invariance principle.

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References


HIGHER ORDER DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS
FOR NONLINEAR PARABOLIC PROBLEMS

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ABSTRACT. In this paper, we consider discontinuous Galerkin finite element methods with interior penalty term to approximate the solution of nonlinear parabolic problems with mixed boundary conditions. We construct the finite element spaces of the piecewise polynomials on which we define fully discrete discontinuous Galerkin approximations using the Crank-Nicolson method. To analyze the error estimates, we construct an appropriate projection which allows us to obtain the optimal order of a priori $\ell^\infty(L^2)$ error estimates of discontinuous Galerkin approximations in both spatial and temporal directions.

1. INTRODUCTION

Discontinuous Galerkin finite element methods (DGM) have recently received a lot of interest. The advantage of DGM is the flexible decomposition of the spatial domain and the construction of the spaces of finite elements consisting of different order of polynomials without continuity requirement. Since the classical DGM was first introduced by Nitsche [8] as a method which enforced the Dirichlet boundary conditions weakly, various types of DGMs are applied to solve time dependent problems as well as elliptic problems. And the DGM was applied to solve the interface problems in [5, 6]. In [3], the authors introduced the local DGM for time-dependent convection-diffusion systems and analyzed the convergence of the approximations.

In [10], Rievière and Wheeler formulated and analyzed a family of discontinuous methods to approximate the solution of the transport problem with nonlinear reaction. They constructed...
semidiscrete approximations which converge optimally in $h$ and suboptimally in $p$ for the energy norm and suboptimally for the $L^2$ norm. They also constructed fully discrete approximations and proved the optimal convergence in the temporal direction. To solve reactive transport problems, Sun and Wheeler in [14] analyzed three discontinuous Galerkin methods, which were symmetric interior penalty Galerkin method, nonsymmetric interior penalty Galerkin method, and incomplete interior penalty Galerkin method. They obtained error estimates in $L^2(H^1)$ which are optimal in $h$ and nearly optimal in $p$ and they developed a parabolic lift-technique for SIPG which leads to $h$-optimal and nearly $p$-optimal error estimates in $L^2(L^2)$ and negative norms.

Rievière and Wheeler [13] formulated semidiscrete and a family of time-discrete locally conservative discontinuous Galerkin procedures for approximations to nonlinear parabolic equations and obtained the optimal spatial rates in $H^1$ and time truncation errors in $L^2$. In [9], the authors constructed discontinuous Galerkin semidiscrete approximations of the nonlinear parabolic differential problems and proved the optimal order of convergence in $L^2$ normed space. Furthermore the authors in [7] applied the hp-version discontinuous Galerkin finite element method with interior penalty to semilinear parabolic problems with locally Lipschitz continuous nonlinearity and analyzed the error bound of the spatially semidiscrete hp-DGM.

Rievière and Shaw [11] developed the discontinuous Galerkin finite element approximation of a nonlinear model of non-fickian diffusion in viscoelastic polymers and proved optimal orders of convergence. In [12], the authors considered dynamic linear solid viscoelasticity problems, defined a fully discrete approximation based on a spatially discontinuous Galerkin finite element method and provided an a priori error estimate. We may refer other references [4, 15] concerning DGM applied to time-dependent problems, for example, the Camassa-Holm equation or the Keller-Segel chemitoxis model.

In this paper, we approximate the solution of nonlinear parabolic problems using a discontinuous Galerkin method with interior penalties for the spatial discretization and Crank-Nicolson method for the time stepping. The main object of this paper is to obtain the optimal $\ell^\infty(L^2)$ error estimates in both spatial and temporal directions by adopting an appropriate elliptic-type projection. The rest of this paper is organized as follows: In section 2, we introduce our problem and some notations. In section 3, we construct appropriate finite element spaces, define an elliptic-type projection, and prove its approximation properties. In section 4, by applying the Crank-Nicolson method, we construct discontinuous Galerkin fully discrete approximations which yield optimal order convergence in the temporal direction as well as the spatial direction.
2. MODEL PROBLEM AND NOTATIONS

In this paper, we consider the following nonlinear parabolic equation:

\[
\begin{align*}
  u_t - \nabla \cdot \{a(x,u) \nabla u\} &= f(x,t,u), & \text{in } \Omega \times (0,T], \\
  u &= g_D, & \text{on } \partial \Omega_D \times (0,T], \\
  (a(x,u) \nabla u) \cdot n &= g_N, & \text{on } \partial \Omega_N \times (0,T], \\
  u(x,0) &= u_0(x), & \text{in } \Omega,
\end{align*}
\]  

(2.1)

where \( \Omega \) denotes an open convex polygonal domain in \( \mathbb{R}^d \), \( d = 1, 2, 3 \) with its boundary \( \partial \Omega \), \( \partial \Omega = \partial \Omega_D \cup \partial \Omega_N \), \( \partial \Omega_D \cap \partial \Omega_N = \emptyset \), \( T \) is a given positive real number, \( n \) denotes the unit outward normal vector to \( \partial \Omega \), and \( u_0(x) \) and \( f(x,t,u) \) are given functions. We assume that \( u_0(x) \in H^s(\Omega) \) and \( f \) satisfies the locally Lipschitz continuous condition in \( u \) and assume that there exist positive constants \( a_s \) and \( a^* \) such that \( a_s \leq a(x,u) \leq a^* \) for all \( (x,u) \), and \( a_u, a_uu, \) and \( a_{uuu} \) are bounded. Let \( \Omega_h = \{K_i\}_{i=1}^{N_h} \) be a regular quasi-uniform subdivision of \( \Omega \) where \( K_i \) is an interval if \( d = 1 \), \( K_i \) is a triangle or a quadrilateral if \( d = 2 \), and \( K_i \) is a 3-simplex or parallelogram if \( d = 3 \). Let \( h_j = \text{diam}(K_j) \) and \( h = \max_{1 \leq j \leq N_h} h_j \). The regular subdivision requires that there exists a constant \( \rho > 0 \) such that each \( K_j \) contains a ball of radius \( \rho h_j \). The quasi-uniformity requires the existence of a constant \( \gamma > 0 \) such that

\[
h_j / h \leq \gamma \quad \text{for } j = 1, 2, \cdots, N_h.
\]

If \( d = 2 \) (or 3), then we denote the set of the edges (resp., faces for \( d = 3 \)) of \( K_i, 1 \leq i \leq N_h \) by \( \{e_1, e_2, \cdots, e_{M_h}\} \) where \( e_k \) has positive \( d-1 \) dimensional Lebesgue measure, \( e_k \subset \Omega \) for \( 1 \leq k \leq P_h \), \( e_k \subset \partial \Omega_D \) for \( P_h + 1 \leq k \leq L_h \), and \( e_k \subset \partial \Omega_N \) for \( L_h + 1 \leq k \leq M_h \). With each edge (or face) \( e_k = \partial K_i \cap \partial K_j \) and \( i < j \), we associate a unit normal vector \( n_k \) to \( E_i \). For \( k \geq P_h + 1 \), \( n_k \) is taken to be the unit outward normal vector to \( \partial \Omega \).

For an \( s \geq 0 \), \( 1 \leq p \leq \infty \), and a domain \( K \subset \mathbb{R}^d \), we denote by \( \mathcal{W}^{s,p}(K) \) the Sobolev space of order \( s \) equipped with the usual Sobolev norm \( \| \cdot \|_{\mathcal{W}^{s,p}(K)} \). As usual we simply use the notation \( H^s(K) \) instead of \( \mathcal{W}^{s,2}(K) \), \( \| \cdot \|_s \) instead of \( \| \cdot \|_{\mathcal{W}^{s,2}(K)} \), and \( \| \cdot \|_K \) instead of \( \| \cdot \|_{L^p(K)} \) if \( p = 2 \). And we also define the usual norm and seminorm on \( H^s(K) \) denoted by \( \| \cdot \|_s, K \) and \( \| \cdot \|_{s,K} \), respectively. We denote \((\cdot, \cdot)\) for the usual inner product of two functions.

Now for an \( s \geq 0 \) and a given subdivision \( \Omega_h \), let

\[
H^s(\Omega_h) = \{v \in L^2(\Omega) \mid v|_{K_i} \in H^s(K_i), \ i = 1, 2, \cdots, N_h\}.
\]

For a \( v \in H^s(\Omega_h) \) with \( s > 1/2 \), we define the average function \( \{v\} \) and the jump function \([v]\) such that

\[
\begin{align*}
  \{v\} &= \frac{1}{2}(v|_{K_i})|_{e_k} + \frac{1}{2}(v|_{K_j})|_{e_k}, \quad \forall x \in e_k, \ 1 \leq k \leq P_h, \\
  [v] &= (v|_{K_i})|_{e_k} - (v|_{K_j})|_{e_k}, \quad \forall x \in e_k, \ 1 \leq k \leq P_h.
\end{align*}
\]  

(2.2)
where $e_k = \partial K_i \cap \partial K_j$ with $i < j$. If $e_k \in \partial \Omega \cap K_i$, then
\[
\{v\} = v|_{K_i}, \quad \forall x \in e_k, \ P_h + 1 \leq k \leq M_h,
\]
\[
[v] = v|_{K_i}, \quad \forall x \in e_k, \ P_h + 1 \leq k \leq M_h.
\]

We define the following broken norms on $H^s(\Omega_h)$
\[
\|v\|_0^2 = \sum_{i=1}^{N_h} \|v\|_{0,K_i}^2,
\]
\[
\|v\|_1^2 = \sum_{i=1}^{N_h} \|v\|_{1,K_i}^2 + \sum_{k=1}^{L_h} h \|\{\nabla v \cdot n_k\}\|_{e_k}^2 + J^\sigma(v, v),
\]
where
\[
J^\sigma(v, w) = \sum_{k=1}^{L_h} \sigma h^{-1} \int_{e_k} [v][w] ds
\]
is an interior penalty term and $\sigma$ is a positive constant.

3. APPROXIMATION PROPERTIES AND AN AUXILIARY PROJECTION

For a positive integer $r$, we construct the following finite element spaces
\[
D_r(\Omega_h) = \{v \in L^2(\Omega) \mid v|_{K_i} \in P_r(K_i), \ i = 1, 2, \ldots, N_h\}
\]
where $P_r(K_i)$ denotes the set of polynomials of total degree less than or equal to $r$ on $K_i$.

Now we state the following approximation properties and trace inequalities whose proofs are provided in [1,2]. Hereafter $C$ denotes a positive generic constant depending on $u, \Omega, \gamma$ and $\rho$ but independent of $h$ and $\Delta t$ defined in Section 4 and any two $C$s in different places don’t need to be equal.

**Lemma 3.1.** Let $K_j \in \Omega_h$ and $v \in H^s(K_j)$. Then there exist a positive constant $C$ depending on $s, \gamma$, and $\rho$ but independent of $v, r$ and $h$ and a sequence $\{z_r^h\}_{r \geq 1} \in P_r(K_j)$ such that for any $0 \leq q \leq s$ and $1 \leq p \leq \infty$
\[
\|v - z_r^h\|_{W^{q,p}(K_j)} \leq Ch_j^{r-q}\|v\|_{W^{q,p}(K_j)}, \quad s \geq 0,
\]
\[
\|v - z_r^h\|_{e_j} \leq Ch_j^{r-\frac{3}{2}}\|v\|_{s,K_j}, \quad s > \frac{1}{2},
\]
\[
\|v - z_r^h\|_{1,e_j} \leq Ch_j^{r-\frac{3}{2}}\|v\|_{s,K_j}, \quad s > \frac{3}{2},
\]
where $\mu = \min(r + 1, s)$ and $e_j$ is an edge or a face of $K_j$. 
Lemma 3.2. For each $K_j \in \Omega_h$, there exists a positive constant $C$ depending only on $\gamma$ and $\rho$ such that the following trace inequalities hold:

$$
\|v\|_{e_j}^2 \leq C \left( \frac{1}{h_j} |v|_{0,K_j}^2 + h_j |v|_{1,K_j}^2 \right), \quad \forall v \in H^1(K_j),
$$

and

$$
\left\| \frac{\partial v}{\partial n_j} \right\|_{e_j}^2 \leq C \left( \frac{1}{h_j} |v|_{1,0,K_j}^2 + h_j |v|_{2,0,K_j}^2 \right), \quad \forall v \in H^2(K_j),
$$

where $e_j$ is an edge or a face of $K_j$ and $n_j$ is the unit outward normal vector to $K_j$.

Now we define the bilinear mapping $A(u : \cdot, \cdot)$ on $H^s(\Omega_h) \times H^s(\Omega_h)$ as follows:

$$
A(u : v, w) = (a(x, u) \nabla v, \nabla w) - \sum_{k=1}^{L_h} \int_{e_k} \{a(x, u) \nabla v \cdot n_k\} [w] \, dx
$$

$$
- \sum_{k=1}^{L_h} \int_{e_k} \{a(x, u) \nabla w \cdot n_k\} [v] \, dx + J^\sigma(v, w).
$$

Then the weak formulation of the problem (2.1) is given as follows:

$$
(u_t, v) + A(u : u, v) = (f(x, t, u), v) + I(v), \quad \forall v \in H^s(\Omega_h),
$$

$$
I(v) = \sum_{k=L_h+1}^{M_h} (g_N, [v])_{e_k} + \sum_{k=P_h+1}^{L_h} (g_D, \sigma h^{-1}[v])_{e_k}, \quad \forall v \in H^s(\Omega_h).
$$

For a given $\lambda > 0$, we define the bilinear form $A_\lambda(u : \cdot, \cdot)$ on $H^s(\Omega_h) \times H^s(\Omega_h)$ as follows:

$$
A_\lambda(u : v, w) = A(u : v, w) + \lambda(v, w).
$$

Lemma 3.3. For a given $\lambda > 0$, there exists a constant $C > 0$, independent of $u$, such that

$$
|A_\lambda(u : v, w)| \leq C \|v\|_1 \|w\|_1, \quad \forall v, w \in H^s(\Omega_h).
$$
Proof. Let \( v, w \in H^s(\Omega_h) \). Then we have
\[
|A_\lambda(u : v, w)| \leq \sum_{i=1}^{N_h} (a(x, u) \nabla v, \nabla w) + \sum_{k=1}^{L_h} \int_{e_k} \{|a(x, u) \nabla v \cdot n_k\}[v]dx \\
+ \sum_{k=1}^{P_h} \int_{e_k} \{|a(x, u) \nabla w \cdot n_k\}[v]dx + J^\sigma(v, w) + \lambda(v, w)
\]
\[
\leq a^* \sum_{i=1}^{N_h} \|\nabla v\|_{K_i} \|\nabla w\|_{K_i} \\
+ a^* \left( \sum_{k=1}^{L_h} \sigma h \|\nabla v\|^2_{e_k} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{L_h} \sigma h \|\nabla v \cdot n_k\|^2_{e_k} \right)^{\frac{1}{2}} \\
+ a^* \left( \sum_{k=1}^{P_h} \sigma h \|\nabla w\|^2_{e_k} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{P_h} \sigma h \|\nabla w \cdot n_k\|^2_{e_k} \right)^{\frac{1}{2}} \\
+ \left( \sum_{k=1}^{L_h} \sigma h^{-1} \|\nabla v\|^2_{e_k} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{L_h} \sigma h^{-1} \|\nabla w\|^2_{e_k} \right)^{\frac{1}{2}} \\
+ \lambda(v, w)
\]
\[
\leq C \|v\|_1 \|w\|_1.
\]
This completes the proof. □

Lemma 3.4. For a given \( \lambda > 0 \), there exists a constant \( \tilde{c} > 0 \), independent of \( u \), such that
\[
A_\lambda(u : v, v) \geq \tilde{c} \|v\|_1^2, \quad \forall v \in D_r(\Omega_h).
\] (3.8)

Proof. Let \( v \in D_r(\Omega_h) \). Then we have
\[
A_\lambda(u : v, v) = \sum_{i=1}^{N_h} (a(x, u) \nabla v, \nabla v)_{K_i} - \sum_{k=1}^{L_h} \int_{e_k} \{|a(x, u) \nabla v \cdot n_k\}[v]dx \\
- \sum_{k=1}^{P_h} \int_{e_k} \{|a(x, u) \nabla v \cdot n_k\}[v]dx + J^\sigma(v, v) + \lambda(v, v)
\]
\[
\geq a^* \sum_{i=1}^{N_h} \|\nabla v\|^2_{K_i} - 2 \sum_{k=1}^{L_h} \left( \frac{a^*}{c} h \|\nabla v \cdot n_k\|^2_{e_k} + \epsilon h^{-1} \|\nabla v\|^2_{e_k} \right) \\
+ \sum_{k=1}^{L_h} \sigma h^{-1} \|\nabla v\|^2_{e_k} + \lambda \|v\|^2.
\]
By Lemma 3.2, the following estimation can be obtained
\[
A_\lambda(u : v, v) \geq (a_s - \frac{a^* C}{E}) \sum_{i=1}^{N_h} \|\nabla v\|_{K_i}^2 + \sum_{k=1}^{L_h} (\sigma - 2\mathcal{E}) h^{-1} \|v\|_{e_k}^2 + \lambda \|v\|^2 \\
\geq C \left( \sum_{i=1}^{N_h} \|\nabla v\|_{K_i}^2 + \sum_{k=1}^{L_h} h \|\{\nabla v \cdot n_k\}\|_{e_k}^2 \right) + C' h^{-1} \|v\|_{e_k}^2 + \lambda \|v\|^2 \\
\geq \tilde{c} \|v\|_{1}^2,
\]
for sufficiently large $\mathcal{E}$ and $\sigma > 2\mathcal{E}$. This completes the proof. \hfill \Box

Now for a given $u$, we construct a projection $\tilde{u} \in D_r(\Omega_h)$ satisfying
\[
A_\lambda(u : \tilde{u}, v) = 0, \quad \forall v \in D_r(\Omega_h),
\]
Then, by Lemma 3.3 and Lemma 3.4, $\tilde{u}$ is obviously well-defined. We denote
\[
\eta(x, t) = u(x, t) - \tilde{u}(x, t), \quad \theta(x, t) = \tilde{u}(x, t) - \tilde{u}(x, t),
\]
where $\tilde{u}(x, t)$ is the approximation of $u(x, t)$ satisfying the approximation properties of Lemma 3.1.

**Lemma 3.5.** If $\mu \geq \frac{d}{2} + 1$ and $u \in H^s(\Omega)$, then there exist constants $C > 0$ and $C^* > 0$ such that
\[
\|\theta\|_1 \leq C h^{\mu-1} \|u\|_s, \quad \|\theta\|_{L^\infty} \leq \frac{C^*}{3},
\]
where $\mu = \min(r + 1, s)$. 

**Proof.** By Lemma 3.4, we have
\[
\tilde{c} \|\theta\|_{1}^2 \leq A_\lambda(u : \theta, \theta) = A_\lambda(u : \theta - \eta, \theta) = A_\lambda(u : \tilde{u} - u, \theta) \leq C \|\tilde{u} - u\|_1 \|\theta\|_1
\]
and hence $\|\theta\|_1 \leq C \|\tilde{u} - u\|_1$. By Lemma 3.1 we get
\[
\|\theta\|_1 \leq C \left( \sum_{i=1}^{N_h} \|\tilde{u} - u\|_{1,K_i}^2 + \sum_{k=1}^{L_h} h^2 \|\{\nabla(\tilde{u} - u) \cdot n_k\}\|_{e_k}^2 + \sum_{k=1}^{L_h} h^{-1} \|\tilde{u} - u\|_{e_k}^2 \right)^{\frac{1}{2}} \\
\leq C(h^{\mu-1} + h^\frac{1}{2} h^{\mu-\frac{3}{2}} + h^{-\frac{1}{2}} h^{\mu-\frac{1}{2}}) \|u\|_s \leq C h^{\mu-1} \|u\|_s.
\]
If $\mu \geq \frac{d}{2} + 1$, then by the inverse inequality
\[
\|\theta\|_{L^\infty} \leq C h^{-\frac{3}{2}} h^{\mu-1} \|u\|_s \leq \frac{C^*}{3}
\]
for some constant $C^*$. This completes the proof. \hfill \Box

Now we state the following approximation results for $\eta$ and $\eta_h$ whose proofs can be found in [9].
Theorem 3.1. If \( u \in H^s(\Omega) \) and \( u_t \in H^s(\Omega) \), then there exists a constant \( C \), independent of \( h \), such that

1. \( \| \eta \| + h \| \eta \|_1 \leq Ch^\mu \| u \|_s \);
2. \( \| \eta_t \| + h \| \eta_t \|_1 \leq Ch^\mu (\| u \|_s + \| u_t \|_s) \),

where \( \mu = \min(r+1, s) \).

And by following the ideas in the proofs of Theorem 3.1, we obtain the following results for \( \eta_{tt} \) and \( \eta_{ttt} \).

Theorem 3.2. If \( u \in H^s(\Omega) \), \( u_t \in H^s(\Omega) \), \( u_{tt} \in H^s(\Omega) \) and \( u_{ttt} \in H^s(\Omega) \), then there exists a constant \( C \), independent of \( h \), such that

1. \( \| \eta_{tt} \| + h \| \eta_{tt} \|_1 \leq Ch^\mu \{ \| u \|_s + \| u_t \|_s + \| u_{tt} \|_s \} \);
2. \( \| \eta_{ttt} \| + h \| \eta_{ttt} \|_1 \leq Ch^\mu \{ \| u \|_s + \| u_t \|_s + \| u_{tt} \|_s + \| u_{ttt} \|_s \} \),

where \( \mu = \min(r+1, s) \).

4. The optimal \( \ell^\infty(L^2) \) error estimates of fully discrete approximations

Now using Crank-Nicolson method, we construct the fully discrete discontinuous Galerkin approximations for nonlinear parabolic problems and prove the optimal convergence in \( L^2 \) normed space. For a positive integer \( N \), let \( \Delta t = \frac{T}{N} \), \( t^j = j(\Delta t) \) for \( j = 0, 1, \ldots, N \), and \( t^{j+\frac{1}{2}} = \frac{1}{2}(t^j + t^{j+1}) \) for \( j = 0, 1, \ldots, N - 1 \). For a function \( g(x, t) \) defined on \( \Omega \times [0, T] \), let

\[
g^j = g(t^j) = g(x, t^j) \quad \text{for} \quad j = 0, 1, \ldots, N \quad \text{and} \quad \partial_t g^j = \frac{g^{j+1} - g^j}{\Delta t} \quad \text{and} \quad g^{j+\frac{1}{2}} = \frac{1}{2}(g^j + g^{j+1})
\]

for \( j = 0, 1, \ldots, N - 1 \).

Then the extrapolated Crank-Nicolson discontinuous Galerkin approximation \( \{U^j\}_{j=0}^N \subset D_r(\Omega_h) \) is defined as follows: for \( j = 1, 2, \ldots, N - 1 \)

\[
(\partial_t U^j, v) + A(EU^j : U^{j+\frac{1}{2}}, v) = (f(x, t^{j+\frac{1}{2}}, EU^j), v) + l(v), \quad \forall v \in D_\ell(\Omega_h)
\]

(4.1)

and

\[
(\partial_t U^0, v) + A(U_0 : U^\frac{1}{2}, v) = \left( f(x, t^{\frac{1}{2}}, U^\frac{1}{2}), v \right) + l(v), \quad \forall v \in D_\ell(\Omega_h),
\]

(4.2)

where \( EU^j = \frac{3}{2}U^j - \frac{1}{2}U^{j-1} \). To prove the optimal convergence of \( u^j - U^j \) in \( L^2 \) normed space, we denote

\[
\xi^j = \tilde{u}^j - U^j, \quad j = 0, 1, \ldots, N.
\]

By simple computations and the applications of Theorem 3.2, we obtain the following Lemma 4.1 and Lemma 4.2.
Lemma 4.1. If $u \in L^\infty(H^s(\Omega))$, $u_t \in L^\infty(H^s(\Omega))$, $u_{tt} \in L^\infty(H^s(\Omega))$ and $u_{ttt} \in L^\infty(H^s(\Omega))$ and if
\[
\rho^{j+\frac{1}{2}} = \frac{\partial_t \tilde{u}^j - \tilde{u}_t(t^{j+\frac{1}{2}})}{\Delta t},
\]
then there exists a constant $C$, independent of $h$ and $\Delta t$, such that
\[
\|\rho^{j+\frac{1}{2}}\|_0 \leq C \Delta t (\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}) + \|u_{ttt}\|_{L^\infty(H^s)}; \\
\|\rho^{j+\frac{1}{2}}\|_1 \leq C \Delta t (\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}) + \|u_{ttt}\|_{L^\infty(H^s)}).
\]

Lemma 4.2. If $u \in L^\infty(H^s(\Omega))$, $u_t \in L^\infty(H^s(\Omega))$ and $u_{tt} \in L^\infty(H^s(\Omega))$ and if $r^{j+\frac{1}{2}} = \tilde{u}(t^{j+\frac{1}{2}}) - \tilde{u}_t(t^{j+\frac{1}{2}})$, then there exists a constant $C$, independent of $h$ and $\Delta t$, such that
\[
\|r^{j+\frac{1}{2}}\|_0 \leq C (\Delta t)^2 (\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}); \\
\|r^{j+\frac{1}{2}}\|_1 \leq C (\Delta t)^2 (\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}).
\]

Lemma 4.3. If $u \in L^\infty(H^s(\Omega))$, $u_t \in L^\infty(H^s(\Omega))$ and $u_{tt} \in L^\infty(H^s(\Omega))$ and if $\mu > \frac{d}{2} + 1$ and $\varphi^{j+\frac{1}{2}} = \tilde{u}(t^{j+\frac{1}{2}}) - E\tilde{u}(t^j)$, then the following statements hold:

(i) $\|\varphi^{j+\frac{1}{2}}\| \leq C (\Delta t)^2$;

(ii) $\|\nabla \tilde{u}^j\|_\infty$ is bounded.

Proof. By the simple calculation, we obtain
\[
\|\varphi^{j+\frac{1}{2}}\| = \|\tilde{u}(t^{j+\frac{1}{2}}) - \tilde{u}_t(t^{j+\frac{1}{2}})\| + \frac{1}{2} \tilde{u}(t^{j+\frac{1}{2}}) - \frac{3}{2} \tilde{u}(t^j) + \frac{1}{2} (\tilde{u}(t^j) - \Delta t \tilde{u}_t(t^j))\| + C (\Delta t)^2 \\
\leq C (\Delta t)^2.
\]

Therefore the bound in (i) holds. And we get the statement (ii) in the following way
\[
\|\nabla \tilde{u}^j\|_\infty \leq \|\nabla \tilde{u}^j - \nabla u^j\|_\infty + \|\nabla u^j\|_\infty \\
\leq \|\nabla \tilde{u}^j - \nabla u^j\|_\infty + \|\nabla \tilde{u}^j - \nabla u^j\|_\infty + \|\nabla u^j\|_\infty \\
\leq C h^{-\frac{d}{2}} \|\nabla \theta^j\| + h^{\mu-1-\frac{d}{2}} \|u^j\|_\infty + \|\nabla u^j\|_\infty \\
\leq C h^{\mu-1-\frac{d}{2}} \|u\|_\infty + \|\nabla u^j\|_\infty \leq C
\]
if $\mu > \frac{d}{2} + 1$. This completes the proof.

\[\Box\]

Theorem 4.1. For $0 < \lambda < 1$, if $u \in L^\infty(H^s(\Omega))$, $u_t \in L^\infty(H^s(\Omega))$, $u_{tt} \in L^\infty(H^s(\Omega))$ and $u_{ttt} \in L^\infty(H^s(\Omega))$, then there exists a constant $C > 0$, independent on $h$ and $\Delta t$, such that
\[
\|e^1\|_0 \leq C (h^\mu + (\Delta t)^2), \\
\|e^1\|_1 \leq C (h^\mu + (\Delta t)^2),
\]
where $\mu = \min(r + 1, s)$ and $\mu \geq \frac{d}{2} + 1$. 
Proof. The proof of Theorem 4.1 is very similar to one of Theorem 4.2 which will be given later, in detail. So we will skip the proof of Theorem 4.1. This completes the proof. □

Theorem 4.2. Suppose that the assumptions of Lemma 4.1 and 4.2 hold and that

\[ |f(x, t, u) - f(x, t, \alpha)| \leq C(C^*, u)|u - \alpha|, \]
\[ |a(x, u) - a(x, \alpha)| \leq C(C^*, u)|u - \alpha| \]

for \(|u - \alpha| < C^*. Then for \(0 < \lambda < 1\), there exists a constant \(C > 0\), independent on \(h\) and \(\Delta t\), such that for \(j = 0, 1, \cdots, N\)

\[ \|u(t^j) - U^j\|_0 \leq C(h^\mu + (\Delta t)^2). \]  

(4.3)

hold where \(\mu = \min(r + 1, s)\) and \(\mu \geq \frac{d}{2} + 1\).

Proof. To prove this theorem, we will prove, by mathematical induction, that

\[ \|\xi^n\|_0 \leq C(h^\mu + (\Delta t)^2), \quad n = 0, 1, \cdots, N. \]

By (4.2) and Theorem 4.1, \(\|\xi^0\|_0 \leq C h^\mu\) and \(\|\xi^1\|_0 \leq C(h^\mu + (\Delta t)^2)\), respectively. Now we suppose that for all \(j\), \(0 \leq j \leq N - 1\) we have

\[ \|\xi^j\|_0 \leq C(h^\mu + (\Delta t)^2). \]  

(4.4)

From (4.1) and (3.5), we have

\[ (u_t(t^{j+\frac{1}{2}}) - \partial_t U^j, v) + A_\lambda(u(t^{j+\frac{1}{2}}) : u(t^{j+\frac{1}{2}}), v) - A_\lambda(EU^j : U^{j+\frac{1}{2}}, v) \]
\[ = (f(x, t^{j+\frac{1}{2}}, u(t^{j+\frac{1}{2}})) - f(x, t^{j+\frac{1}{2}}, EU^j), v) + \lambda(u(t^{j+\frac{1}{2}}) - U^{j+\frac{1}{2}}, v). \]  

(4.5)

By the notations of \(\eta\) and \(\xi\), we get

\[ u_t(t^{j+\frac{1}{2}}) - \partial_t U^j = u_t(t^{j+\frac{1}{2}}) - \partial_t u^j + \partial_t \tilde{u}^j - \partial_t U^j = \eta_t(t^{j+\frac{1}{2}}) - \Delta t \rho^{j+\frac{1}{2}} + \partial_t \xi^j. \]  

(4.6)

From the definition of \(\eta\) and \(\xi\), we obtain

\[ A_\lambda(u(t^{j+\frac{1}{2}}) : u(t^{j+\frac{1}{2}}), v) - A_\lambda(EU^j : U^{j+\frac{1}{2}}, v) \]
\[ = A_\lambda(EU^j : \xi^{j+\frac{1}{2}}, v) + A_\lambda(u(t^{j+\frac{1}{2}}) : \eta(t^{j+\frac{1}{2}}), v) \]
\[ + A_\lambda(u(t^{j+\frac{1}{2}}) : \tilde{u}(t^{j+\frac{1}{2}}) - \tilde{u}^{j+\frac{1}{2}}, v) \]
\[ + A_\lambda(u(t^{j+\frac{1}{2}}) : \tilde{u}^{j+\frac{1}{2}}, v) - A_\lambda(EU^j : \tilde{u}^{j+\frac{1}{2}}, v). \]  

(4.7)

Substituting (4.6) and (4.7) in (4.5) and choosing \(v = \xi^{j+\frac{1}{2}}\) in (4.5), we have

\[ (\partial_t \xi^j, \xi^{j+\frac{1}{2}}) + A_\lambda(EU^j : \xi^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \]
\[ = -(\eta_t(t^{j+\frac{1}{2}}) - \Delta t \rho^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) - A_\lambda(u(t^{j+\frac{1}{2}}) : \eta(t^{j+\frac{1}{2}}), \xi^{j+\frac{1}{2}}) \]
\[ - A_\lambda(u(t^{j+\frac{1}{2}}) : \rho^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) - A_\lambda(u(t^{j+\frac{1}{2}}) : \tilde{u}^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \]
\[ + A_\lambda(EU^j : \tilde{u}^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) + \lambda(u(t^{j+\frac{1}{2}}) - U^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \]
\[ + (f(x, t^{j+\frac{1}{2}}, u(t^{j+\frac{1}{2}})) - f(x, t^{j+\frac{1}{2}}, EU^j), \xi^{j+\frac{1}{2}}). \]  

(4.8)
Notice that

\[ (\partial_t \xi^j, \xi^{j+\frac{1}{2}}) = \frac{1}{2\Delta t} (\|\xi^{j+1}\|_0^2 - \|\xi^j\|_0^2). \] \quad (4.9)

Applying (4.9) in (4.8) and using Lemma 3.4, we obtain

\[
\begin{align*}
\frac{1}{2\Delta t} (\|\xi^{j+1}\|_0^2 - \|\xi^j\|_0^2) + c\|\xi^{j+\frac{1}{2}}\|_2^2 \\
\leq - (\eta(t^{j+\frac{1}{2}}) - \Delta t \rho^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \\
+ \lambda(u(t^{j+\frac{1}{2}}) - U^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \\
- A_\lambda(u(t^{j+\frac{1}{2}}) : \tau^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \\
- \left( A_\lambda(u(t^{j+\frac{1}{2}}) : \tilde{u}^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) - A_\lambda(EU^j : \tilde{u}^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}}) \right) \\
+ \left( f(x,t^{j+\frac{1}{2}}, u^{j+\frac{1}{2}}) - f(x,t^{j+\frac{1}{2}}, EU^j), \xi^{j+\frac{1}{2}} \right) \\
= \sum_{i=1}^5 I_i.
\end{align*}
\] \quad (4.10)

By applying Lemma 4.1 there exists a constant \( C > 0 \) such that

\[
|I_1| \leq (\|\eta(t^{j+\frac{1}{2}})\|_0 + \|\Delta t \rho^{j+\frac{1}{2}}\|_0) \|\xi^{j+\frac{1}{2}}\|_0
\leq C(\|\eta(t^{j+\frac{1}{2}})\|_0^2 + (\Delta t)^2 \|\rho^{j+\frac{1}{2}}\|_0^2 + \|\xi^{j+\frac{1}{2}}\|_0 + \|\xi^j\|_0^2)
\leq C(h^{2\mu} + (\Delta t)^4 + \|\xi^{j+1}\|_0^2 + \|\xi^j\|_0^2).
\]

For sufficiently small \( \epsilon > 0 \) we obtain the following estimates of \( I_2 \) and \( I_3 \):

\[
|I_2| \leq \lambda(\|\eta(t^{j+\frac{1}{2}})\|_0 + \|\rho^{j+\frac{1}{2}}\|_0 + \|\xi^{j+\frac{1}{2}}\|_0) \|\xi^{j+\frac{1}{2}}\|_0
\leq C(h^{2\mu} + (\Delta t)^4 + \|\xi^{j+1}\|_0^2 + \|\xi^j\|_0^2),
\]

\[
|I_3| \leq C\|\rho^{j+\frac{1}{2}}\|_1 \|\xi^{j+\frac{1}{2}}\|_1 \leq C(\Delta t)^4 + \epsilon \|\xi^{j+\frac{1}{2}}\|_1^2.
\]

Now to calculate the bound for \( I_4 \), we split it into 3 terms as follows:

\[
I_4 = \left( (a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j)) \nabla \tilde{u}^{j+\frac{1}{2}}, \nabla \xi^{j+\frac{1}{2}} \right)
- \sum_{k=1}^{L_h} \int_{e_k} \{(a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j)) \nabla \tilde{u}^{j+\frac{1}{2}} \cdot n_k \}[\xi^{j+\frac{1}{2}}]
- \sum_{k=1}^{P_h} \int_{e_k} \{(a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j)) \nabla \xi^{j+\frac{1}{2}} \cdot n_k \}[\tilde{u}^{j+\frac{1}{2}}] = \sum_{i=1}^3 I_{4i}.
\]
Note that by Taylor’s expansion, Lemma 3.1, Lemma 3.5 and the assumption (4.4), we get
\[
\|u(t^{j+\frac{1}{2}}) - EU^j\|_{L^\infty} \leq \|u(t^{j+\frac{1}{2}}) - EU^j\|_{L^\infty} + \|EU^j - E\hat{u}^j\|_{L^\infty} \\
+ \|E\hat{u}^j - EU^j\|_{L^\infty} + \|EU^j - E\hat{u}^j\|_{L^\infty}
\]
(4.11)
\[
\leq C((\Delta t)^2 + h^\mu) + \frac{C^*}{3} + \frac{C^*}{3} \leq C^*
\]
for sufficiently small \( h \) and \( \Delta t \). By (4.11) we obtain
\[
|a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j)| \leq C(C^*)|u(t^{j+\frac{1}{2}}) - EU^j|
\]
and by Lemma 4.3
\[
I_{41} = \left| \left( (a(x, u(t^{j+\frac{1}{2}})) - a(x, EU^j)) \nabla \hat{u}^{j+\frac{1}{2}}, \nabla \xi^{j+\frac{1}{2}} \right) \right|
\leq C\|\nabla \hat{u}^{j+\frac{1}{2}}\|_{L^\infty} \left( \|\eta(t^{j+\frac{1}{2}})\| + \Delta t^2 + \|\xi^j\|_0 + \|\xi^{j-1}\|_0 \right) \|\nabla \xi^{j+\frac{1}{2}}\|_0
\leq C(h^{2\mu} + (\Delta t)^4 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2) + \epsilon\|\xi^{j+\frac{1}{2}}\|_1^2.
\]
Similarly there exists a constant \( C > 0 \) such that
\[
|I_{42}| \leq \sum_{k=1}^{L_h} C\|\nabla \hat{u}^{j+\frac{1}{2}}\|_{L^\infty}(e_k) \left( \|\eta(t^{j+\frac{1}{2}})\|_{e_k} + \|\varphi^{j+\frac{1}{2}}\|_{e_k} + \|\xi^j\|_{e_k} + \|\xi^{j-1}\|_{e_k} \right) \|\xi^{j+\frac{1}{2}}\|_{e_k}
\leq C \sum_{i=1}^{N_h} \|\nabla \hat{u}^{j+\frac{1}{2}}\|_{L^\infty(K_i)} \left( h^{-\frac{1}{2}} \|\eta(t^{j+\frac{1}{2}})\|_{K_i} + h^{\frac{1}{2}} \|\nabla \eta(t^{j+\frac{1}{2}})\|_{K_i} + h^{-\frac{1}{2}} \|\varphi^{j+\frac{1}{2}}\|_{K_i}
\right)
+ h^{-\frac{1}{2}} \|\xi^j\|_{K_i} + h^{\frac{1}{2}} \|\xi^{j-1}\|_{K_i}) h^{\frac{1}{2}} \|\xi^{j+\frac{1}{2}}\|_1
\leq C(\|\eta(t^{j+\frac{1}{2}})\|_0 + \|\varphi^{j+\frac{1}{2}}\|_0) (\Delta t)^4 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2 + \epsilon\|\xi^{j+\frac{1}{2}}\|_1^2
\leq C(h^{2\mu} + (\Delta t)^4 + \|\xi^j\|_0^2 + \|\xi^{j-1}\|_0^2) + \epsilon\|\xi^{j+\frac{1}{2}}\|_1^2.
\]
Since \( |u| = 0 \) on \( e_k \in P_h \), we have
\[
|I_{43}| \leq C \sum_{k=1}^{P_h} \|\nabla \xi^{j+\frac{1}{2}}\|_{L^\infty}(e_k) \left( \|\eta(t^{j+\frac{1}{2}})\|_{e_k} + \|\varphi^{j+\frac{1}{2}}\|_{e_k} + \|\xi^j\|_{e_k} + \|\xi^{j-1}\|_{e_k} \right) \|\eta^{j+\frac{1}{2}}\|_{e_k}
\leq C \sum_{i=1}^{N_h} \|\nabla \xi^{j+\frac{1}{2}}\|_{L^\infty(K_i)} h^{-\frac{1}{2}} (\|\eta(t^{j+\frac{1}{2}})\|_{K_i} + h \|\nabla \eta(t^{j+\frac{1}{2}})\|_{K_i} + \|\varphi^{j+\frac{1}{2}}\|_{K_i})
+ \|\xi^j\|_{K_i} + \|\xi^{j-1}\|_{K_i}) h^{-\frac{1}{2}} (\|\eta^{j+\frac{1}{2}}\|_{K_i} + h \|\nabla \eta^{j+\frac{1}{2}}\|_{K_i})
\leq C \sum_{i=1}^{N_h} \|\nabla \xi^{j+\frac{1}{2}}\|_{K_i} h^{-\frac{1}{2}} (h^\mu + (\Delta t)^2 + \|\xi^j\|_{K_i} + \|\xi^{j-1}\|_{K_i}) h^\mu.
\]
Since $\mu \geq \frac{d}{2} + 1$, we obtain
\[ |I_{33}| \leq C(\varepsilon \mu^2 + (\Delta t)^4 + \|\xi_0\|^2 + \|\xi_{j-1}\|^2_0) + \epsilon \|\xi_j\|^2 + \|\xi_j\|^2_0. \]

From the bounds of $|I_{3i}|$, $1 \leq i \leq 3$, we get
\[ |I_4| \leq C(\varepsilon \mu^2 + (\Delta t)^4 + \|\xi_0\|^2 + \|\xi_{j-1}\|^2_0) + 3\epsilon \|\xi_j\|^2_1. \]

Now we compute the bound of $I_5$
\[ |I_5| = |(f(x, t^{j+\frac{1}{2}}, u(t^{j+\frac{1}{2}})) - f(x, t^{j+\frac{1}{2}}, EU^j), \xi_{j+\frac{1}{2}})| \]
\[ \leq C\|u(t^{j+\frac{1}{2}}) - EU^j\|^2_0 + \|\xi_{j+\frac{1}{2}}\|^2_0 \]
\[ \leq C(\|u(t^{j+\frac{1}{2}}) - \bar{u}(t^{j+\frac{1}{2}})\|^2_0 + \|\bar{u}(t^{j+\frac{1}{2}}) - E(\bar{u}^j)\|^2_0 + \|E\xi_0\|^2_0) \]
\[ + C(\|\xi_{j+1}\|^2_0 + \|\xi_j\|^2_0) \]
\[ \leq C((\Delta t)^4 + \varepsilon^2 + \|\xi_{j+1}\|^2_0 + \|\xi_j\|^2_0 + \|\xi_{j-1}\|^2_0). \]

Substituting the bounds of $I_i$, $1 \leq i \leq 5$, into (4.10), we get
\[ \frac{1}{2\Delta t}(\|\xi_{j+1}\|^2_0 - \|\xi_j\|^2_0) + \frac{\epsilon}{2} \|\xi_{j+\frac{1}{2}}\|^2_1 \]
\[ \leq C\left(\varepsilon \mu^2 + (\Delta t)^4 + \|\xi_{j+1}\|^2_0 + \|\xi_j\|^2_0 + \|\xi_{j-1}\|^2_0\right), \tag{4.12} \]

for sufficiently small $\epsilon$. If we sum both sides of (4.12) from $j = 1$ to $N - 1$, then we obtain
\[ \|\xi_N\|^2_0 - \|\xi_1\|^2_0 \leq C \left\{ (\Delta t) \sum_{j=1}^{N-1} \left( \varepsilon \mu^2 + (\Delta t)^4 \right) + (\Delta t) \sum_{j=0}^{N} \|\xi_j\|^2_0 \right\}, \]

which implies
\[ \|\xi_N\|^2_0 \leq \|\xi_1\|^2_0 + C(\Delta t) \sum_{j=1}^{N-1} \left( \varepsilon \mu^2 + (\Delta t)^4 \right) + C(\Delta t) \sum_{j=0}^{N} \|\xi_j\|^2_0, \]

where $\Delta t$ is sufficiently small. By applying the discrete version of Gronwall’s inequality, we have
\[ \|\xi_N\|^2_0 \leq C \left( \|\xi_1\|^2_0 + \Delta t \sum_{j=1}^{N-1} \left( \varepsilon \mu^2 + (\Delta t)^4 \right) \right). \]

Therefore we prove by mathematical induction that
\[ \|\xi_n\|^2_0 \leq C(h^\mu + (\Delta t)^2), \quad n = 0, 1, \ldots, N, \]

which implies that
\[ \|e\|_{L^2(L^2)} := \max_{0 \leq n \leq N} \|e^n\|_0 \leq C(h^\mu + (\Delta t)^2), \]

that is, we obtain the optimal $L^\infty(L^2)$ error estimation of the fully discrete solutions. This completes the proof. \qed
REFERENCES

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