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Volume 20, Number 1, March 2016

Contents

DIVIDED DIFFERENCES AND POLYNOMIAL CONVERGENCES
SUK BONG PARK, GANG JOON YOON, AND SEOK-MIN LEE ........................................ 1

DYNAMIC CHARACTERISTICS OF A ROTATING TIMOSHENKO BEAM
SUBJECTED TO A VARIABLE MAGNITUDE LOAD TRAVELLING AT VARYING SPEED
BABATOPE OMOLOFE AND SEGUN NATHANIEL OGUNYEBI ........................................ 17

LARGE EDDY SIMULATION OF TURBULENT CHANNEL FLOW USING
ALGEBRAIC WALL MODEL
MUHAMMAD SAIFUL ISLAM MALLIK AND MD. ASHRAF UDDIN ................................... 37

EXISTENCE AND CONTROLLABILITY OF FRACTIONAL NEUTRAL
INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY IN
BANACH SPACES
SUBRAMANIAN KAILASAVALLI, SELVARAJ SUGANYA, AND MANI MALLIKA ARJUNAN... 51

COMPARISON OF NUMERICAL METHODS FOR TERNARY FLUID FLOWS:
IMMERSED BOUNDARY, LEVEL-SET, AND PHASE-FIELD METHODS
SEUNGGYU LEE, DARAE JEONG, YONGHO CHOI, AND JUNSEOK KIM
DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 136-713, KOREA ........ 83
DIVIDED DIFFERENCES AND POLYNOMIAL CONVERGENCES

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ABSTRACT. The continuous analysis, such as smoothness and uniform convergence, for polynomials and polynomial-like functions using differential operators have been studied considerably, parallel to the study of discrete analysis for these functions, using difference operators. In this work, for the difference operator $\nabla_h$ with size $h > 0$, we verify that for an integer $m \geq 0$ and a strictly decreasing sequence $h_n$ converging to zero, a continuous function $f(x)$ satisfying

$$\nabla_{h_n}^{m+1} f(kh_n) = 0,$$

for every $n \geq 1$ and $k \in \mathbb{Z}$,

turns to be a polynomial of degree $\leq m$. The proof used the polynomial convergence, and additionally, we investigated several conditions on convergence to polynomials.

1. INTRODUCTION

In recent decades, the problem of approximation by a linear combination of integer translates of one or more basis functions has arisen, especially in the study of Wavelet and Computer Aided Geometric Design (CAGD). In CAGD, polynomials and polynomial-like functions are used as basis functions, for example, Bernstein polynomials, B-spline polynomials, Box splines and so on. In drawing curves and surfaces using computers, polynomials and polynomial-like functions are playing much important roles. In turn, the continuous analysis, such as smoothness and uniform convergence, for these functions have been studied considerably, in harmony with the study of discrete analogue for these functions. For more details, we refer to [1] and [2].

In polynomial applications, we come across the difference operators and the polynomial (point-wise) convergence in analyzing the smoothness of the curves or surfaces. The difference operators and the convergence of polynomial sequences provide efficient implements in

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estimating whether a subdivision scheme generates polynomials. Just as the differential operator is to the continuous analysis, so is the difference operator to the discrete analysis.

In this work, we consider the divided difference equations and several conditions on the convergence to polynomial. The backward and forward (divided) difference operators with size \( h > 0 \) are defined as

\[
\nabla_h f(x) := f(x) - f(x - h) \quad \text{and} \quad \Delta_h f(x) := f(x + h) - f(x)
\]

for a function \( f(x) \). The iterated operators are written as

\[
\nabla^m_h f(x) := \nabla_h (\nabla^{m-1}_h f(x)) \quad \text{and} \quad \Delta^m_h f(x) := \Delta_h (\Delta^{m-1}_h f(x)).
\]

Note that we have

\[
\nabla^m_h f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x - kh) \quad \text{and} \quad \Delta^m_h f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + kh).
\]

Then for a polynomial \( p(x) \) of degree \( \leq m \), we easily see, like differential equations, that \( \nabla^{m+1}_h p \equiv 0 \) and \( \Delta^{m+1}_h p \equiv 0 \) for any \( h \).

Conversely, we verify that for a strictly decreasing sequence \( \{h_n\}_{n=1}^{\infty} \) convergent to zero, a continuous function \( f \) satisfying the divided difference equations of order \( m + 1 \)

\[
\nabla^{m+1}_{h_n} f(kh_n) = 0, \quad \text{for every } n \geq 1 \text{ and } k \in \mathbb{Z},
\]

is also a polynomial of degree \( \leq m \). However, the continuity condition on solutions of the divided difference equations is necessary. For example, a real valued function \( f \) defined on the whole real line \( \mathbb{R} \) satisfying the linearity:

\[
f(x + y) = f(x) + f(y), \quad \text{for all } x, y \in \mathbb{R},
\]

which yields \( f \) to satisfy the differences

\[
\nabla^2 f \equiv 0 \quad \text{and} \quad \Delta^2 f \equiv 0 \quad \text{for every } h > 0.
\]

Such a function \( f \) seems to be a line. Surprisingly, there are functions which satisfy the linearity but are not continuous at any point. We construct a function satisfying the linearity but are not continuous at any point (see Theorem 2.1). And we study several conditions on the convergence to polynomial; we investigate the convergence in the dyadic set (Theorem 3.1 and Corollary 3.2), and we consider general cases (Theorem 3.3 and Remark 3.4). In Section 4, we generalize Theorem 2.1 to Theorem 4.2 and Corollary 4.3 using the polynomial convergence based on the results in Section 3, and give some analogy of a differential equation with respect to a linear combination of discrete difference operators. Even though the results have been widely used in Approximation theory and Functional Analysis, their rigorous proofs have not been given yet as far as we know.
2. Characterization of Solutions to Difference Equations

In the section, we investigate the solution to the backward difference equations

$$\nabla^{m+1}_h f \equiv 0, \quad \text{for every } h = 1/2^\ell, \ \ell = 0, 1, 2, \cdots. \quad (2.1)$$

To this end, we introduce some notations and terminologies. By $\mathbb{N}_0$ we denote the set of all non-negative integers and by $K$ the dyadic set defined as

$$K = \{i/2^j : i \in \mathbb{Z}, j \in \mathbb{N}_0\},$$

where $\mathbb{Z}$ stands for the set of all integers. For given $m + 1$ distinct (real or complex) numbers $\{x_k\}_{k=1}^{m+1}$, we introduce the Lagrange polynomials $\{\ell_k(x)\}_{k=1}^{m+1}$ defined by

$$\ell_k(x) = \frac{w(x)}{w'(x_k)(x-x_k)}, \quad k = 1, 2, \cdots, m+1,$$

where $w(x) = (x-x_1)(x-x_2)\cdots(x-x_{m+1})$. Then it is well-known that $\ell_k(x)$ is a unique polynomial of degree $\leq m$ having the properties:

$$\ell_k(x_j) = \delta_{kj} = \begin{cases} 0, & \text{if } k \neq j, \\ 1, & \text{if } k = j. \end{cases}$$

For a given function $f$, the interpolation polynomial $p_m(f; x)$ of degree $\leq m$ satisfying the conditions:

$$p_m(f; x_k) = f(x_k), \quad k = 1, 2, \cdots, m + 1$$

is written as

$$p_m(f; x) = \sum_{k=1}^{m+1} f(x_k)\ell_k(x). \quad (2.2)$$

In particular, if $f$ is a polynomial of degree $\leq m$, then Equation (2.2) implies that $f(x)$ is identically equal to $p_m(f; x)$.

**Theorem 2.1.** Let $f$ be a function defined on the whole real line $\mathbb{R}$. Assume that there is an integer $m \geq 0$ such that

$$\nabla^{m+1}_h f(jh) = 0, \quad \text{for every } h = 1/2^\ell, \ \ell \in \mathbb{N}_0, \text{ and } j \in \mathbb{Z}. \quad (2.3)$$

If $f$ is continuous, then $f$ is a polynomial of degree $\leq m$. Moreover, there is a discontinuous function that satisfies

$$\nabla^{m+1}_h f \equiv 0$$

for any size $h > 0$.

**Proof.** Assume that $f$ is continuous on $\mathbb{R}$. Fix $\ell$ to be a non-negative integer. Then Equation (2.3) implies that for every $j \in \mathbb{Z}$, we have

$$\nabla^{m+1}_h f(j/2^\ell) = \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} f\left(\frac{j - k}{2^\ell}\right) = 0, \quad (2.4)$$
where $h = 1/2^\ell$. Let $p_\ell(x)$ be the polynomial of degree $\leq m$ interpolating to $f(x)$ at
\[ x = j/2^\ell, \quad j = 0, 1, 2, \ldots, m. \]
Then $p_\ell(x)$ satisfies the difference equation (2.3). Since
\[
\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} y_{i-k} = 0, \quad i \in \mathbb{Z}
\]
has a unique solution $\{y_i\}_{i \in \mathbb{Z}}$ with initial values $y_0, y_1, \ldots, y_m$, we induce that
\[
f \left( \frac{i}{2^\ell} \right) = p_\ell \left( \frac{i}{2^\ell} \right), \quad \text{for every } i \in \mathbb{Z}.
\] (2.5)
By the same argument with $h = 1/2^{\ell+1}$, we can see that there is a polynomial $p_{\ell+1}(x)$ of
degree $\leq m$ such that
\[
f \left( \frac{i}{2^{\ell+1}} \right) = p_{\ell+1} \left( \frac{i}{2^{\ell+1}} \right), \quad \text{for every } i \in \mathbb{Z}.
\] (2.6)
Equations (2.5) and (2.6) show that the polynomials $p_\ell(x)$ and $p_{\ell+1}(x)$ have the same values
at all the points $x = i/2^\ell$, $i \in \mathbb{Z}$. Thus $p_\ell$ and $p_{\ell+1}$ are the same. Continuing this process, we
have that there is a polynomial $p(x)$ of degree $\leq m$ such that
\[
f(x) = p(x), \quad \text{for every } x \in K.
\]
Since the set $K$ is dense in $\mathbb{R}$, for every $x \in \mathbb{R}$, there is a sequence $\{y_k\}_{k=0}^\infty$ in $K$
converging to $x$. The continuity of $f$ at $x$ implies that
\[
f(x) = \lim_{k \to \infty} f(y_k) = \lim_{k \to \infty} p(y_k) = p(x).
\]
Thus we have that $f \equiv p$, which proves the first claim.

Now, we shall construct a discontinuous function satisfying (2.3), which is somewhat in the
abstract.

We regard the space of real numbers $\mathbb{R}$ as an infinite dimensional vector space over the rational
field $\mathbb{Q}$, the set of all rational numbers. Using Zorn’s lemma ([3, Chapter 4]), we deduce that
$\mathbb{R}$ has a (Hamel) basis $\{e_\alpha\}_{\alpha \in I}$ including 1, $\sqrt{2}$. Let $B$ denote the basis. Then every real
number $x$ has a unique representation as a linear combination of finitely many elements of $B$,
\[
x = c_1 + c_2 \sqrt{2} + \sum_{k=1}^n c_k e_{\alpha_k},
\] (2.7)
where $c_1, c_2$, and $c_k$ are nonzero rational numbers depending on $x$. Now, define a function $f$
on $\mathbb{R}$ as follows. First, define the value of $f$ on $B$ as
\[
f(1) = 0, \quad f(\sqrt{2}) = 1, \quad \text{and} \quad f(e_\alpha) = 0, \quad \forall \alpha \in I,
\]
and then for every $x \in \mathbb{R}$ in the form (2.7), $f(x)$ is defined as
\[
f(x) = f \left( c_1 + c_2 \sqrt{2} + \sum_{k=1}^n c_k e_{\alpha_k} \right) := c_1 f(1) + c_2 f(\sqrt{2}) + \sum_{k=1}^n c_k f(e_{\alpha_k}).
\]
By the definition of \( f \), we find that \( f \) satisfies the linearity:

(i) \( f(rx) := rf(x) \) for every \( r \in \mathbb{Q} \) and \( x \in \mathbb{R} \);

(ii) \( f(x + y) := f(x) + f(y) \) for every \( x, y \in \mathbb{R} \).

In other words, \( f(x) \) is the \( \mathbb{Q} \)-linear function from \( \mathbb{R} \) to \( \mathbb{Q} \) sending a real number to its \( \sqrt{2} \)-coordinate with respect to the basis. Let \( y \) be a fixed real number. We can also find sequences \( \{x_n\}_{n=1}^{\infty} \) and \( \{y_n\}_{n=1}^{\infty} \) of rational numbers converging to \( \sqrt{2} \) and \( y \), respectively. We have from the definition of \( f \) that

\[
f(y_n) = 0, \quad \text{for all } n \geq 1.
\]

On the other hand, the sequence \( \{x_n^{-1}y_n\sqrt{2}\}_{n=1}^{\infty} \) converges to \( y \) and

\[
\lim_{n \to \infty} f\left(\frac{y_n}{x_n}\sqrt{2}\right) = \lim_{n \to \infty} \frac{y_n}{x_n} = \frac{y}{\sqrt{2}},
\]

which shows the discontinuity of \( f \) at \( y \neq 0 \). Also, we see that sequence \( \{\sqrt{2}/x_n - 1\} \) converges to zero as \( n \) tends to \( \infty \), but

\[
\lim_{n \to \infty} f\left(\frac{\sqrt{2}}{x_n} - 1\right) = \lim_{n \to \infty} \frac{1}{x_n} = \frac{1}{\sqrt{2}} \neq f(0).
\]

Hence, we have shown that \( f \) is discontinuous at every point in \( \mathbb{R} \).

Note that the function \( f \) satisfies

\[
\nabla_h^2 f \equiv 0, \quad \forall h > 0 \quad \text{and} \quad \nabla_h f \equiv 0, \quad \forall h = \frac{1}{2^\ell}, \ell \in \mathbb{N}_0.
\]

We define \( f_m \) by \( f_m(x) := (f(x))^m \). Using the identities

\[
\sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} k^j = \begin{cases} 
0 & \text{if } 0 \leq j < m+1, \\
(-1)^{m+1}(m+1)! & \text{if } j = m+1,
\end{cases}
\]

we can see that

\[
\nabla_h^{m+1} f_m \equiv 0 \quad \forall h > 0
\]

but for any real number \( x \), there exists at least one (and therefore infinitely many) \( h > 0 \) such that

\[
\nabla_h^m f_m(x) \neq 0.
\]

\[ \square \]

**Definition 2.2.** We call the function \( f \) in the proof of the second argument of Theorem 2.1 as the \( \sqrt{2} \)-coordinate function, and \( f_m \) as the \( m \)th-power \( \sqrt{2} \)-coordinate function.

A subtle change of expression of Theorem 2.1 makes the following corollary.

**Corollary 2.3.** Let \( f \) be a continuous function on the set of real numbers. Then there is an integer \( m \geq 0 \) such that

\[
\nabla_h^{m+1} f \equiv 0, \quad \text{for every } h = 1/2^\ell, \ell \in \mathbb{N}_0
\]

if and only if \( f \) is a polynomial of degree \( \leq m \).
The property for the forward difference follows from the same argument of proof of Theorem 2.1:

**Corollary 2.4.** Let \( f \) be a function defined on the whole real line. Assume that there is an integer \( m \geq 0 \) such that

\[
\Delta_h^{m+1} f(jh) = 0, \quad \text{for every } h = 1/2^\ell, \ n \in \mathbb{N}_0, \text{ and } j \in \mathbb{Z}.
\]

If \( f \) is continuous, then \( f \) is a polynomial of degree \( \leq m \).

Now we give some generalization for Theorem 2.1. We write the condition (2.3) for Theorem 2.1 as follows:

\[
\Delta_h^{m+1} f = 0 \text{ in } D \text{ where } h = 1/2^\ell \text{ for infinitely many } \ell \in \mathbb{N}_0.
\] (2.8)

By changing \( K \) into some dense subgroup \( D \) of \( \mathbb{R} \), we have the following.

**Theorem 2.5.** Let \( D \) be a subgroup of the additive group \( \mathbb{R} \) which is dense in \( \mathbb{R} \). Suppose \( D \) has a set of generators \( B = \{h_1, h_2, h_3, \ldots \} \) such that for all \( i \in \mathbb{N} \), \( h_{i-1} = n_i h_i \) for some integer \( n_i \). Assume that for a real valued function \( f \) defined on \( \mathbb{R} \), there is an integer \( m \geq 0 \) such that

\[
\Delta_h^{m+1} f = 0 \text{ in } D, \quad \text{for all } n \in \mathbb{N}.
\] (2.9)

Then \( f \) is a polynomial of degree \( \leq m \) if and only if \( f \) is continuous.

**Proof.** The proof is the same as that of the first part of Theorem 2.1 by replacing \( 1/2^\ell \) into \( h \), with \( n = \ell + 1 \). \( \square \)

**Remark 2.6.** Let \( D \) be a dense subgroup of \( \mathbb{R} \). The followings are equivalent:

1. \( D \) has a set of generators \( B = \{b_1, b_2, b_3, \ldots \} \) such that any two elements \( b_i, b_j \) in \( B \) has a rational ratio, that is, \( b_i = r_{i,j} b_j \) for some rational number \( r_{i,j} \).

2. \( D \) has a set of generators \( C = \{h_1, h_2, h_3, \ldots \} \) with the property \( h_{i-1} = n_i h_i \) for some integer \( n_i \).

**Proof.** (2) trivially implies (1). Suppose that (1) holds. First we notice that there is a real number \( \alpha \) such that any element of \( D \) can be written as \( q \alpha \) for some rational number \( q \). Take \( h_1 = b_1 \). Inductively, suppose we have elements \( h_1, h_2, \ldots, h_i \) of \( D \) such that \( h_{j-1} = n_j h_j \) for \( j = 2, 3, \ldots, i \), and the subgroup \( \langle h_1, h_2, \ldots, h_i \rangle = \langle h_i \rangle = \langle kh_i : k \in \mathbb{Z} \rangle \) generated by those elements is equal to the subgroup \( \langle b_1, b_2, \ldots, b_i \rangle \). The subgroup generated by \( h_i \) and \( b_{i+1} \) has one generator. Indeed, if we denote \( h_i = (k_1/k_2) \alpha \) and \( b_{i+1} = (k_3/k_4) \alpha \) for integers \( k_1, k_2, k_3, \) and \( k_4 \), then

\[
\langle h_i, b_{i+1} \rangle = \left\langle \frac{\gcd(k_1 k_4, k_3 k_2)}{k_2 k_4} \alpha \right\rangle.
\]

Take \( h_{i+1} \in D \) the generator. Then \( \langle h_{i+1} \rangle = \langle h_1, h_2, \ldots, h_{i+1} \rangle = \langle b_1, b_2, \ldots, b_{i+1} \rangle \) and there is an integer \( n_{i+1} \) such that \( h_i = n_{i+1} h_{i+1} \). \( \square \)

If a dense subgroup is not of type in Remark 2.6, then there exist at least two elements with irrational ratio. But a subgroup containing two elements with irrational ratio is dense in \( \mathbb{R} \), e.g.
Theorem 2.1 will be generalized further (Theorem 4.2 and Corollary 4.3). The union of subgroups generated by each \( h \) by each \( f \) to \( p \) may be seen in Theorem 2.5. For any decreasing sequence \( \{ n \} \) of \( \alpha, \beta \) to be positive and suppose \( b \neq 0 \) where \( a, b \) are integers. We may assume \( 0 < \gamma < \alpha \) by subtracting \( k \alpha \) for a suitable integer \( k \). Take a positive integer \( m \) such that \( m \gamma < \alpha < (m + 1) \gamma \), that is, \( m = \lfloor \frac{\alpha}{\gamma} \rfloor \). Then one of the positive differences \( (m + 1) \gamma - \alpha \) or \( \alpha - m \gamma \) should be less than \( \gamma / 2 \), and it is in the subgroup \( \langle \alpha, \beta \rangle \), with coefficient of \( \beta \) equal to \( (m + 1) b \) or \( -m b \), any of which is not zero. For the case \( \gamma = a \alpha \), change the roles of \( \alpha \) and \( \beta \). Hence the subgroup \( \langle \alpha, \beta \rangle \) is dense subgroup of \( \mathbb{R} \).

Now, set \( h_1 = \beta - k \alpha \) where the integer \( k \) satisfying \( 0 < h_1 < \alpha \) and make the sequence \( \{ h_n \}_{n \in \mathbb{N}} \) by the above process: for example, we get a decreasing sequence \( \{ \sqrt{2} - 1 \approx 0.414, 3 - 2\sqrt{2} \approx 0.172, 17 - 12\sqrt{2} \approx 0.0294, \ldots \} \) for the group \( \{ 1, \sqrt{2} \} = \mathbb{Z}[\sqrt{2}] \). The obtained sequence generates the subgroup \( \langle \alpha, \beta \rangle \). Indeed, write \( h_n = a_n \alpha + b_n \beta \) for \( a_n, b_n \in \mathbb{Z} \). Since \( h_{n+1} = (m_n + 1) h_n - \alpha \) or \( h_{n+1} = \alpha - m_n h_n \) for the chosen integer \( m_n \), we have \( \alpha \in \langle h_n, h_{n+1} \rangle \) and the coefficient \( b_{n+1} \) of \( \beta \) in \( h_{n+1} \) is a multiple of \( b_n \). Therefore, we have

\[
\langle h_n, h_{n+1} \rangle = \langle \alpha, b_n \beta \rangle,
\]

and, since \( b_1 = 1 \), the decreasing sequence \( \{ h_n \} \) is a generating set of the group \( \langle \alpha, \beta \rangle \).

In summary, for every dense group we can find a dense subgroup with a constructible set of generators as a sequence \( \{ h_n \}_{n \in \mathbb{N}} \) of positive numbers decreasing and converging to zero. Then, with such a sequence \( \{ h_n \}_{n=1}^{\infty} \), it is enough to consider the union of subgroups generated by each \( h_n \): \( \{ kh_n : k \in \mathbb{Z}, n \in \mathbb{N} \} \), in order to obtain the results given here in the work, as may be seen in Theorem 2.5. For any decreasing sequence \( \{ h_n \}_{n \in \mathbb{N}} \) converging to zero, the union of subgroups generated by each \( h_n \) is dense in \( \mathbb{R} \). With such a decreasing sequence, Theorem 2.1 will be generalized further (Theorem 4.2 and Corollary 4.3).

3. Point-wise Convergence to Polynomial

In this section, we investigate conditions on the point-wise convergence to a polynomial of degree \( \leq m \) on the dyadic set \( K \). And we consider general cases: the set of randomly chosen points (see Theorem 3.3 and Remark 3.4.) So that Theorem 3.1 is improved to Theorem 3.3.

**Theorem 3.1.** Let \( f \) be a continuous function defined on the whole real line and let \( \{ p_n \}_{n=1}^{\infty} \) be a sequence of polynomials \( p_n \) of degree \( \leq m \) for an integer \( m \geq 0 \). If \( \{ p_n(x) \}_{n=1}^{\infty} \) converges to \( f(x) \) in \( K \) point-wise:

\[
\lim_{n \to \infty} p_n(x) = f(x), \quad \text{for all } x \in K.
\]

Then \( f \) is a polynomial of degree \( \leq m \) and for each \( j \geq 0 \),

\[
\lim_{n \to \infty} p_n^{(j)}(x) = f^{(j)}(x)
\]
uniformly on every bounded subset of the real line.

**Proof.** For each \( n \geq 0 \), let

\[
p_n(x) = a_{n0} + a_{n1}x + \cdots + a_{nm}x^m.
\]

We shall show that if \( \{p_n(x)\}_{n=1}^\infty \) converges to \( f(x) \) at every \( x \in K \), then for each \( j = 0, 1, 2, \cdots \), \( \{a_{nj}\}_{n=1}^\infty \) converges to a real number, say \( a_j \):

\[
\lim_{n \to \infty} a_{nj} = a_j, \quad j = 0, 1, \cdots, m.
\]

By linearity of the difference operator and the assumption that \( \{p_n(x)\} \) converges to \( f(x) \) for every \( x \in K \) which contains all integers, we have that

\[
\lim_{n \to \infty} \nabla^m p_n(0) = \lim_{n \to \infty} \sum_{k=0}^{m} (-1)^k \binom{m}{k} p_n(-k)
\]

\[
= \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(-k)
\]

\[
= \nabla^m f(0),
\]

where \( \nabla \) is the backward difference with size \( h = 1 \). On the other hand, using the identities

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} k^j = \begin{cases} 
0 & \text{if } 0 \leq j < m, \\
(-1)^m m! & \text{if } j = m
\end{cases}
\]

we have that

\[
\nabla^m p_n(0) = \sum_{k=0}^{m} a_{nk} \nabla^m x^k |_{x=0} = a_{nm} m!.
\]

Thus the sequence \( \{a_{nm}\}_{n=0}^\infty \) converges to \( a_m \),

\[
a_m := \lim_{n \to \infty} a_{nm} = \nabla^m f(0)/m!.
\]

By induction on \( j = m, m - 1, \cdots, 0 \), we assume that there exists a positive integer \( k \leq m \) such that for \( j = k, k + 1, \cdots, m \), the sequence \( \{a_{nj}\}_{n=0}^\infty \) converges to \( a_j \), that is,

\[
a_j := \lim_{n \to \infty} a_{nj}, \quad \text{for } j = k, k + 1, \cdots, m.
\]

Then by the assumption and linearity, we have that \( \lim_{n \to \infty} \nabla^{k-1} p_n(0) = \nabla^{k-1} f(0) \) and

\[
\lim_{n \to \infty} \nabla^{k-1} p_n(0) = \lim_{n \to \infty} \sum_{i=0}^{m} a_{ni} \nabla^{k-1} x^i |_{x=0}
\]

\[
= \lim_{n \to \infty} \sum_{i=k-1}^{m} a_{ni} \nabla^{k-1} x^i |_{x=0}
\]

\[
= \lim_{n \to \infty} \left[ (k-1)! a_{n,k-1} + \sum_{i=k}^{m} a_{ni} \nabla^{k-1} x^i |_{x=0} \right]
\]
which implies that

\[ \lim_{n \to \infty} a_{n,k-1} := (\nabla^{k-1} f(0) - \sum_{i=k}^{m} a_i \nabla^{k-1} x^i|_{x=0})/(k - 1)!. \]

So we have proved that there are numbers \( \{a_i\}_{i=0}^{m} \) such that

\[ \lim_{n \to \infty} a_{ni} = a_i, \quad i = 0, 1, 2, \ldots, m. \]

Define a polynomial \( p(x) \) by

\[ p(x) = a_0 + a_1 x + \cdots + a_m x^m, \]

then from the convergence of coefficients \( \{a_{ni}\}_{n=0, i=0}^{\infty} \), we can see that for each \( j \geq 0 \),

\[ \lim_{n \to \infty} p_n^{(j)}(x) = p^{(j)}(x) \]

point-wisely for every real number \( x \in \mathbb{R} \) and uniformly on every bounded subset of \( \mathbb{R} \). Also since the set \( K \) is dense in \( \mathbb{R} \) and \( f \) is continuous on \( \mathbb{R} \), \( f(x) = p(x) \) for all the real number \( x \), which proves the theorem. \( \square \)

**Corollary 3.2.** Let \( f \) be a function defined on the whole real line and let \( \{p_n\} \) be a sequence of polynomials \( p_n \) of degree \( \leq m \) for an integer \( m \geq 0 \). If \( \{p_n(x)\} \) converges point-wisely to \( f(x) \) at every real number \( x \) :

\[ \lim_{n \to \infty} p_n(x) = f(x), \quad \text{for all } x \in \mathbb{R}, \]

then \( f \) is a polynomial of degree \( \leq m \) and for each \( j \geq 0 \),

\[ \lim_{n \to \infty} p_n^{(j)}(x) = p^{(j)}(x), \]

uniformly on every bounded subset of the real line.

Now we consider a more general case of polynomial convergence.

**Theorem 3.3.** Let \( \{p_n(x)\}_{n=1}^{\infty} \) be a sequence of polynomials of degree \( \leq m \) for some non-negative integer \( m \). Suppose that there are \( m+1 \) distinct numbers \( \{x_k\}_{k=1}^{m+1} \) at which \( \{p_n(x)\}_{n=1}^{\infty} \) converges, say,

\[ \lim_{n \to \infty} p_n(x_k) = y_k, \quad k = 1, 2, \ldots, m + 1, \]

then for each \( j \geq 0 \) and any bounded subset \( \Omega \) of \( \mathbb{R} \) or \( \mathbb{C} \):

\[ \lim_{n \to \infty} p_n^{(j)}(x) = p^{(j)}(x) \quad \text{uniformly on } \Omega, \quad (3.2) \]

where \( p(x) \) is the polynomial of degree \( \leq m \) given by

\[ p(x) = \sum_{k=1}^{m+1} y_k \ell_k(x) \quad (3.3) \]

for the Lagrange polynomials \( \{\ell_k(x)\}_{k=1}^{m+1} \) for the numbers \( \{x_k\} \).
Proof. The uniqueness of interpolation polynomial guarantees that each polynomial \( p_n(x) \) is written as
\[
p_n(x) = \sum_{k=1}^{m+1} p_n(x_k) \ell_k(x), \quad n = 1, 2, \ldots,
\]
for the Lagrange polynomials \( \{ \ell_k(x) \}_{k=1}^{m+1} \). Let \( j \geq 0 \) be an integer. Differentiating \( j \) times the both sides of Equation (3.4) with respect to the variable of \( x \), we have
\[
p_n^{(j)}(x) = \sum_{k=1}^{m+1} p_n(x_k) \ell_k^{(j)}(x), \quad n = 1, 2, \ldots,
\]
Since \( \{ \ell_k(x) \}_{k=1}^{m+1} \) are uniformly bounded on any bounded subset \( \Omega \), it is easily shown that
\[
\lim_{n \to \infty} p_n^{(j)}(x) = \lim_{n \to \infty} \sum_{k=1}^{m+1} p_n(x_k) \ell_k^{(j)}(x)
= \sum_{k=1}^{m+1} y_k \ell_k^{(j)}(x)
= p^{(j)}(x) \quad \text{uniformly on } \Omega
\]
where \( p(x) \) is the polynomial of degree \( \leq m \) as in (3.3), which completes the proof. 

Note that relations (3.5) show that for \( j = 0, 1, \ldots, m \), the sequences of the coefficients of \( x^j \) in \( p_n(x) \) also converge.

Remark 3.4. The convergence holds even if we replace the point-wise convergence by the condition that there is a linearly independent set \( \{ L_i \}_{i=1}^{m+1} \) of linear functionals on \( \mathcal{P}_m \), the space of all polynomials of degree \( \leq m \), such that for each \( k \), \( \{ L_k(P_n) \}_{n=1}^{\infty} \) converges to \( y_k \). Then we have
\[
\lim_{n \to \infty} p_n^{(j)}(x) = \sum_{k=1}^{m+1} y_k \tilde{\ell}_k^{(j)}(x) \quad \text{uniformly on } \Omega,
\]
where \( \{ \tilde{\ell}_k(x) \}_{k=1}^{m+1} \) is a linearly independent set in \( \mathcal{P}_m \) which is biorthonormal with respect to \( \{ L_i \}_{i=1}^{m+1} \):
\[
L_i(\tilde{\ell}_j) = \delta_{ij}.
\]
We may consult [4] for more details.

Note that \( m + 1 \) is the possible least number on the conditions which guarantee the results obtained in Theorem 3.3 and Remark 3.4.

Corollary 3.5. Let \( \Omega \) be a bounded subset of \( \mathbb{R} \) or \( \mathbb{C} \) consisting of at least \( m + 1 \) elements. Then the space of all polynomials of degree \( \leq m \) is a Banach space with the norm \( \| \cdot \| \) defined by
\[
\| p \| = \sup_{x \in \Omega} | p(x) |.
\]
4. Generalizations

Now, we consider more general cases for divided difference operators treated in Section 2, using the polynomial convergence. From now on, let $I = (a, b)$ be an interval of positive measure and we denote $\{h_n\}_{n=1}^{\infty}$ to be a decreasing sequence of positive numbers $h_n$ converging to zero; for each $n \geq 1, h_n > h_{n+1}$ and

$$\lim_{n \to \infty} h_n = 0.$$  

First, we characterize continuous functions $f$ satisfying

$$\nabla_{h_n} f(kh_n) = 0,$$  \hspace{1cm} (4.1)

for every $n \geq 1$ and $k \in \mathbb{Z}$ such that $kh_n \in I$.

**Lemma 4.1.** Let $I$ be an interval of positive measure and $x_1 < x_2 < \cdots < x_{m+1}$ distinct points in $I$. For $i = 1, \ldots, m+1$, let $\{x_{i,k}\}_{k=1}^{\infty} \subset I$ and $\{y_{i,k}\}_{k=1}^{\infty}$ be sequences such that $\lim_{k \to \infty} x_{i,k} = x_i$ and $\lim_{k \to \infty} y_{i,k} = y_i$ for each $i = 1, 2, \ldots, m+1$ for some constants $y_k$. Then the polynomials $p_{m,k}(x)$ of degree $m$ interpolating to $y_{i,k}$ at $x_{i,k}$ for $i = 1, \ldots, m+1$ converges uniformly on $I$ to $p_m(x)$ of degree $m$,

$$\lim_{k \to \infty} p_{m,k}(x) = p_m(x) \quad \text{uniformly on } I$$

where $p_m(x)$ is the interpolation polynomial satisfying $p_m(x_j) = y_j$ for $j = 1, 2, \ldots, m+1$.

**Proof.** For each $k \geq 1$, we may write the interpolation polynomials $p_{m,k}(x)$ as

$$p_{m,k}(x) = \sum_{j=1}^{m+1} y_{k,j} \ell_{k,j}(x)$$

for the Lagrange polynomials $\ell_{k,j}(x)$ satisfying

$$\ell_{k,j}(x_{k,i}) = \delta_{i,j} \quad \text{for } i, j = 1, 2, \ldots, m+1.$$  

In Section 2, we have shown that $\ell_{k,j}$ is given by

$$\ell_{k,j}(x) = \frac{w_k(x)}{w_k'(x_{k,j})(x - x_{k,j})}, \quad w_k(x) = (x - x_{k,1}) \cdots (x - x_{k,m+1}).$$  \hspace{1cm} (4.2)

Fix $j$, we write the polynomials $\ell_{k,j}$ by

$$\ell_{k,j}(x) = \sum_{r=0}^{m} a_{k,r} x^r.$$  

Now we consider the polynomials $w_k'(x_{k,j})\ell_{k,j}$ by

$$w_k'(x_{k,j})\ell_{k,j}(x) = \sum_{r=0}^{m} b_{k,r} x^r.$$
Since the coefficients $b_{k,r}$ are linear combinations of multiplications of $x_{k,1}, x_{k,2}, \ldots, x_{m+1},$ as we may see in (4.2), and the constants $x_{k,i}$ converge to $x_i$ as $k$ tends to $\infty$, we have
\[
\lim_{k \to \infty} b_{k,r} = b_r, \quad r = 0, 1, \ldots, m,
\]
for some constants $b_r$. Consequently, with the limit
\[
\lim_{k \to \infty} w'_k(x_{k,j}) = \lim_{k \to \infty} \prod_{i \neq j} (x_{k,j} - x_{k,i}) = \prod_{i \neq j} (x_j - x_i),
\]
we can see that
\[
\lim_{k \to \infty} a_{k,r} = a_r, \quad r = 0, 1, \ldots, m,
\]
here $a_r$ is the coefficient of $x^r$ in the Lagrange polynomial $\ell_j(x)$
\[
\ell_j(x) = \frac{w(x)}{w'(x_j)(x - x_j)} , \quad w(x) = (x - x_1) \cdots (x - x_{m+1}).
\]
Since the interval $I$ is bounded, we show that the interpolation polynomials $p_{m,k}(x)$ of degree $m$ interpolating to $y_{i,k}$ at $x_{i,k}$ for $i = 1, \ldots, m+1$,
\[
p_{m,k}(x) = \sum_{j=1}^{m+1} y_{k,j} \ell_{k,j}(x),
\]
converges uniformly on $I$ to $p_m(x)$ of degree $m$,
\[
\lim_{k \to \infty} p_{m,k}(x) = \sum_{j=1}^{m+1} y_j \ell_j(x), \quad \text{uniformly on } I
\]
where $p_m(x) := \sum_{j=1}^{m+1} y_j \ell_j(x)$ is the interpolation polynomial satisfying $p_m(x_j) = y_j$ for $j = 1, 2, \ldots, m + 1$. This completes the proof. 

\[ \square \]

Now we are ready to characterize continuous functions satisfying the equation (4.1)

**Theorem 4.2.** Let $I$ be an interval of positive measure and let \( \{h_n\}_{n=1}^{\infty} \) be a decreasing sequence of positive numbers which converges to zero. Let $f$ be a function defined on a set containing $I$ such that for all $x \in I$ and $n \geq 1$, $\nabla_{h_n}^{m+1} f(x)$ is well-defined. Assume that $f$ satisfies the equations
\[
\nabla_{h_n}^{m+1} f(kh_n) = 0
\]
for every $n \geq 1$ and $k \in \mathbb{Z}$ such that $kh_n \in I$. If $f$ is continuous, then $f$ is a polynomial of degree $\leq m$ in $I$.

**Proof.** For each $n \geq 1$, let $I_n$ be the subset of $I$ given by
\[
I_n = \{kh_n : kh_n \in I \text{ for some integer } k\} = I \cap \langle h_n \rangle.
\]
Here, we may assume without lose of generality that each $I_n$ contains at least $m+1$ elements. Now choose $m+1$ points $x_1 < x_2 < \cdots < x_{m+1}$ in $I$ such that $x_1$ and $x_{m+1}$ are not boundary
points of \( I \), and define \( \xi := \min_{i=1,2,\ldots,m} |x_i - x_{i+1}| \) and \( N_0 \) be an integer such that \( 2h_{N_0} < \xi \). Then, for each \( n \geq N_0 \), there exists \( m+1 \) points \( x_{n,1} < x_{n,2} < \cdots < x_{n,m+1} \) in \( I_n \) such that
\[
|x_{n,i} - x_i| < h_n, \quad i = 1,\ldots,m + 1.
\]
This choice shows that for each \( i = 1,\ldots,m + 1, x_{n,i} \) converges to \( x_i \) as \( n \) tends to \( \infty \). Since \( \nabla^{m+1} f \equiv 0 \) in \( I_n \), we can see that there exists a polynomial \( p_n(x) \) of degree \( \leq m \) such that
\[
p_n(x) = f(x), \quad \text{for all } x \in I_n.
\]
In particular, \( p_n \) is the polynomial interpolating to \( f(x_{n,i}) \) at \( x_{n,i} \) for \( i = 1,\ldots,m + 1 \). The continuity of \( f \) shows that for each \( i = 1,\ldots,m + 1, \lim_{n \to \infty} f(x_{n,i}) = f(x_i) \). Now, Lemma 4.1 implies that \( p_n(x) \) converges uniformly to \( p(x) \), the polynomial of degree \( \leq m \) interpolating to \( f(x_i) \) for \( i = 1,\ldots,m + 1, \)
\[
\lim_{n \to \infty} p_n(x) = p(x), \quad \text{uniformly for } x \in I.
\]
Let \( t \in I \) be an arbitrary point. If \( t = x_i \) for some \( i = 1,\ldots,m + 1 \), then we have shown that \( f(t) = p(t) \).

Now, we consider the case where \( t \neq x_i \) for any \( i = 1,\ldots,m + 1 \). Let \( \varepsilon > 0 \) be arbitrarily given. Since \( f \) and \( p \) are continuous at \( x = t \), there exists \( \delta > 0 \) such that for every \( x \in I \) satisfying \( |x - t| \leq \delta \), we have
\[
|f(x) - f(t)| \leq \varepsilon/3 \quad \text{and} \quad |p(x) - p(t)| \leq \varepsilon/3.
\]
And the uniform convergence of \( p_n \) to \( p \) in \( I \) implies that there exists an integer \( N_1 \) such that\[
\sup_{x \in I} |p_n(x) - p(x)| \leq \frac{\varepsilon}{3}, \quad \text{for } n \geq N_1.
\]
Then for a sufficiently large \( n \geq N_1 \), there exists a point \( t_n \in I_n \) such that \( |t - t_n| \leq \delta \). In this case, we have \( f(t_n) = p_n(t_n) \) (see (4.3)) so that we obtain
\[
|f(t) - p(t)| \leq |f(t) - f(t_n)| + |p_n(t_n) - p(t_n)| + |p(t_n) - p(t)|
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
= \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, therefore, we have that \( f(t) = p(t) \). This completes the proof. \( \square \)

**Corollary 4.3.** Let \( \{h_n\}_{n=1}^{\infty} \) be a decreasing sequence of positive numbers which is converging to zero. Let \( f \) be a continuous function defined on \( \mathbb{R} \) and assume that \( f \) satisfies the equations
\[
\nabla_{h_n} f(kh_n) = 0 \quad \text{for every } n \geq 1 \text{ and } k \in \mathbb{Z}.
\]
Then \( f \) is a polynomial of degree \( \leq m \).

**Proof.** We cover \( \mathbb{R} \) by a chain of intersecting intervals \( \{J_k\}_{k \in \mathbb{Z}} \) such that \( \nabla_{h_n}^{m+1} f(x) \) is well-defined in \( J_k \) for all \( k \in \mathbb{Z} \). In each interval, \( f \) is equal to a polynomial \( p_k(x) \) of degree \( \leq m \) by Theorem 4.2. Since polynomials \( p_{k-1}(x) \) and \( p_k(x) \) of degree \( \leq m \) have the same values at all the points in the intersection of \( J_{k-1} \) and \( J_k \), the two polynomials are equal. As a consequence, \( f(x) \) is equal to one polynomial on the set of all real numbers. \( \square \)
In the following, we relate divided differences to derivatives, which is a mean value theorem for equally spaced points.

**Theorem 4.4.** Let \( f(x) \in C[a, b] \) and suppose that \( f^{(n)}(x) \) exists at each point of \((a, b)\), then for \( b \geq x_0 > x_1 = x_0 - h > \cdots > x_n = x_0 - nh \geq a \), there exists a point \( \xi \in (x_n, x_0) \) satisfying

\[
\nabla_h^n f(x_0) = h^n f^{(n)}(\xi), \quad x_n < \xi < x_0.
\]

**Proof.** From the definition of the forward and backward operators, we can see that

\[
\nabla_h^n f = \Delta_h^n f(\cdot - nh).
\]

Then the theorem follows directly from Corollary 3.4.4 in [4].

A polynomial \( p(x) \) of degree \( \leq m \) satisfies the difference equation

\[
\nabla_h^{m+1} p(x) = 0
\]

for any \( h > 0 \) and the differential equation

\[
\frac{d^{m+1} p(x)}{dx^{m+1}} = 0
\]

as well. That is, we may regard the difference operator \( \nabla_h^n \) as the discretization of the differential operator \( D^n (D = d/dx) \). In the following we extend to differential operators of the form

\[
\sum_{k=0}^{m} a_k D^k,
\]

where \( a_k \) are real numbers.

**Theorem 4.5.** Let \( \{h_n\}_{n=1}^{\infty} \) be a decreasing sequence of positive numbers which converges to zero and let \( f \in C^m(I) \) be a function such that for all \( x \in I \) and \( n \geq 1 \), \( \nabla_h^{m+1} f(x) \) is well-defined. Assume that

\[
\lim_{n \to \infty} \sum_{k=0}^{m} a_k h_n^{-k} \nabla_h^k f(x) = 0, \quad \text{for all } x \in I, \quad (4.4)
\]

then

\[
\sum_{k=0}^{m} a_k f^{(k)}(x) = 0, \quad \text{for all } x \in I. \quad (4.5)
\]

**Proof.** Let \( x \) be chosen arbitrarily in \((a, b)\) and fixed. Since \( \{h_n\}_{n=1}^{\infty} \) is a decreasing sequence convergent to zero, we may assume that for all \( n \geq 1 \), \( x_n > a \). From Theorem 4.4, we have the relation

\[
\nabla_h^k f(x) = h_n^k f^{(k)}(\xi_{k,n}), \quad x - kh_n < \xi_{k,n} < x.
\]
By the assumption that $f \in C^m(I)$ and $f$ satisfies the condition (4.4), we have that $\lim_{n \to \infty} f^{(k)}(\xi_{k,n}) = f^{(k)}(x)$ and

$$0 = \lim_{n \to \infty} \sum_{k=0}^{m} a_k h_n^{-k} \nabla^k_{h_n} f(x)$$

$$= \lim_{n \to \infty} \sum_{k=0}^{m} a_k h_n^{-k} \nabla^k_{h_n} f(\xi_{k,n}) (x - kh_n < \xi_{k,n} < x)$$

$$= \sum_{k=0}^{m} a_k (\lim_{n \to \infty} f^{(k)}(\xi_{k,n}))$$

$$= \sum_{k=0}^{m} a_k f^{(k)}(x),$$

which implies that

$$\sum_{k=0}^{m} a_k f^{(k)}(x) = 0.$$ 

Since $x$ is chosen arbitrarily, $f$ satisfies the equation (4.5) for all $x \in I$. The converse of Theorem 4.5 is obvious. But we can see that a function $f$ satisfying (4.5) may not satisfy the difference equation

$$\sum_{k=0}^{m} a_k h_n^{-k} \nabla^k_{h_n} f(x) = 0$$

in general. The functions satisfying the differential equation (4.5) play fundamental roles for the construction of a subdivision scheme for $C^{m-2}$-exponential B-splines, whose pieces are solutions to the differential equation (4.5). For details, we refer to [5] and [2].

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REFERENCES

DYNAMIC CHARACTERISTICS OF A ROTATING TIMOSHENKO BEAM SUBJECTED TO A VARIABLE MAGNITUDE LOAD TRAVELLING AT VARYING SPEED

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\textbf{ABSTRACT.} In this study, the dynamic behaviour of a rotating Timoshenko beam when under the actions of a variable magnitude load moving at non-uniform speed is carried out. The effect of cross-sectional dimension and damping on the flexural motions of the elastic beam was neglected. The coupled second order partial differential equations incorporating the effects of rotary and gyroscopic moment describing the motions of the beam was scrutinized in order to obtain the expression for the dynamic deflection and rotation of the vibrating system using an elegant technique called Galerkin’s Method. Analyses of the solutions obtained were carried out and various results were displayed in plotted curve. It was found that the response amplitude of the simply supported beam increases with an increase in the value of the foundation reaction modulus. Effects of other vital structural parameters were also established.

\section{1. INTRODUCTION}

Studies concerning vibrating bodies resting on an elastic foundation carrying moving loads are of considerable practical importance and have been a subject of numerous scientific investigations by different authors in past few years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In most of the studies available in literature, such as the work of Sadiku and Leipolz [21], Oni and Awodola [22], the scope of the problem of assessing the dynamic response of a structural member under the passage of moving load has been limited to that of thin beam or thick beam. Kolousek et al. [23] studied works on uniform thin beam. In their analysis the adopted a normal modes method, effects of damping and foundation were not
included in their analysis. Kenny [24] worked on the problem of infinite elastic beam resting on elastic foundation and under the influence of a dynamic load moving with constant speed. Unlike Bernoulli-Euler beam or Rayleigh beam, forced vibrations of deep beam have received little attention for the past few years. Among few authors who have published scholarly article on the dynamic characteristics of a Timoshenko beam subjected moving loads are Oni [25] who considered the problem of a harmonic time-dependent concentrated force moving at constant velocity. The method of integral transform method was adopted. Series solution which converges is obtained for the deflection of simply supported beam. Travelling force on a Timoshenko beam has also been studied by Florence [26], Huang [27] studied the effect of rotary inertia and of shear deformation on the frequency and normal mode equations of uniform with simple end conditions. Deterministic and random vibration of an axially loaded Timoshenko beam resting on an elastic foundation has been considered by Chang [28]. Vibration and reliability of a rotating beam with random properties under random excitation was presented by Hosseini and Khadem [29]. More recently, Omolofe et al. [30] studied the transverse motion of non-prismatic deep beam under the actions of variable magnitude moving loads.

In all these aforementioned studies, investigations are limited to the cases where the velocity of the travelling masses is held constant throughout its motion on the structure. However, situation arises when a travelling mass may accelerate by a forward force or decelerate, reduce speed and come to rest at any desired position on the beam thereby causing the friction between the mass and the beam to increase considerably. Wang [31] studied the dynamical analysis of a finite inextensible beam with an attached accelerating mass. He employed Galerkin procedure in conjunction with the method of numerical integration to tackle the partial differential equation which describes the transient vibration of the beam mass system. He concluded that the applied forward force amplifies the speed of the mass and the displacement of the beam. To the authors best of knowledge, studies concerning structural members where the effects of the rotary inertia correction factor and shear deformation are incorporated into the governing equation of motions are not common in literature and where they rarely exist the traversing load is assumed to travel with constant speed.

This present study therefore concerns the dynamic characteristics of a uniform beam resting on elastic foundation of the Winkler type and incorporating the effects of rotary inertia correction factor and shear deformation into the governing equation of motion. It is assumed that the speed at which the travelling load traverses the structural elements is time varying.

2. MATHEMATICAL FORMULATION

Consider a deep elastic beam having length $L$ and resting on an elastic foundation with subgrade reaction modulus $K$ which is directly proportional to beam deflection. If the system does not experience friction as the beam maintain contact with the subgrade, the deflection $\phi(x,t)$ and rotation $\psi(x,t)$ is aptly described by the system of partial differential equations

$$
\frac{m}{\partial^2 \phi(x,t)} - K^*GF \left[ \frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{\partial \psi(x,t)}{\partial x} \right] + K \phi(x,t) = P(x,t) \quad (2.1)
$$
Where \( m \) is the constant mass per unit length of the beam, \( K^* \) is a constant dependent on the shape of the cross section, \( G \) is the modulus of elasticity in shear, \( F \) is the cross sectional area, \( P(x,t) \) is the harmonic force, \( E \) is the Young modulus of the beam, \( J \) is the constant moment of inertia of the beam cross section and \( D \) is the mass per unit volume.

Furthermore, at each of the boundary points there are two boundary conditions of the type

\[
\phi(0,t) = 0; \quad \psi(0,t) = 0 \quad (2.3)
\]

\[
\frac{\partial \phi(0,t)}{\partial x} = 0; \quad \frac{\partial \psi(L,t)}{\partial x} = 0 \quad (2.4)
\]

and the initial conditions are

\[
\phi(x,0) = 0 = \frac{\partial \phi(x,0)}{\partial t} \quad (2.5)
\]

\[
\psi(x,0) = 0 = \frac{\partial \psi(x,0)}{\partial t} \quad (2.6)
\]

The effect of gyroscopic moment \( \dot{\psi}(x,t) \) is incorporated into the governing equations (2.1) and (2.2) to induce a displacement component perpendicular to the direction of the load. While \( \ddot{\psi}(x,t) \) represents the effect of rotary inertia. Since the velocity of our moving force is non-uniform, the moving force \( P(x,t) \) acting on the beam is chosen as

\[
P(x,t) = P \cos \omega t \delta(x - f(t)) \quad (2.7)
\]

where \( \omega \) is the circular frequency of the harmonic load, \( \delta(\cdot) \) and \( f(t) \) is the distance covered by the moving load at any time \( t \) and is given by

\[
f(t) = x_0 + \gamma \sin \beta t \quad (2.8)
\]

where \( x_0 \) is the equilibrium position of the longitudinal oscillating load, \( \gamma \) is the longitudinal amplitude of oscillation of the load and \( \beta \) is the longitudinal frequency of the load. Furthermore, for the variable elastic foundation function \( K(x) \) we adopt the example in [30] and we have

\[
K(x) = K_0 \left(4x - 3x^2 + x^3\right) \quad (2.9)
\]

3. Solution Technique and Procedure

To solve the beam problem stated above, we shall use an elegant solution technique called Galerkin’s method. This method requires that the solutions of the deflection and the rotation of the coupled beam problems (2.1) and (2.2) be written as

\[
\phi_i(x,t) = \sum_{i=1}^{n} V_i(t) U_i(x) \quad (3.1)
\]

and

\[
\psi_i(x,t) = \sum_{i=1}^{n} Y_i(t) X_i(x) \quad (3.2)
\]
The function $U_i(x)$ and $X_i(x)$ are usually chosen to satisfy the pertinent boundary conditions. Thus, substituting equations (3.1) and (3.2) into the system of equations (2.1) and (2.2), yields

$$m \sum_{i=1}^{n} \dddot{V}_i(t) U_i(x) - K^*GF \left[ \sum_{i=1}^{n} V_i(t) U_i''(x) - \sum_{i=1}^{n} Y_i(t) X_i'(x) \right] + K(x) \sum_{i=1}^{n} V_i(t) U_i(x) = P \cos \omega t \delta [x - (x_0 + \gamma \sin \beta t)]$$

(3.3)

and

$$EJ \sum_{i=1}^{n} Y_i(t) X_i''(x) + K^*GF \left[ \sum_{i=1}^{n} V_i(t) U_i'(x) - \sum_{i=1}^{n} Y_i(t) X_i(x) \right] - JD \sum_{i=1}^{n} \dddot{Y}_i(t) X_i(x) = 0$$

(3.4)

To determine the expression for $V_i(t)$ and $Y_i(t)$, the expression on the left hand side of the equations (3.3) and (3.4) are required to be orthogonal to functions $U_j(x)$ and $X_j(x)$ respectively. Thus,

$$\int_{0}^{L} \left[ \sum_{i=1}^{n} \left( m \sum_{i=1}^{n} \dddot{V}_i(t) U_i(x) - K^*GF \left[ \sum_{i=1}^{n} V_i(t) U_i''(x) - \sum_{i=1}^{n} Y_i(t) X_i'(x) \right] + K(x) \sum_{i=1}^{n} V_i(t) U_i(x) \right) \cdot U_j(x) \right] dx = 0$$

(3.5)

and

$$\int_{0}^{L} \left[ \sum_{i=1}^{n} EJY_i(t) X_i''(x) + K^*GF \left[ \sum_{i=1}^{n} V_i(t) U_i'(x) - \sum_{i=1}^{n} Y_i(t) X_i(x) \right] - JD \dddot{Y}_i(t) X_i(x) \right] \cdot X_j(x) dx = 0$$

(3.6)

Equations (3.5) and (3.6) after some rearrangements yields,

$$\sum_{i=1}^{n} \left\{ P_1(i,j) \dddot{V}_i(t) + P_2(i,j) V_i(t) + P_3(i,j) Y_i(t) \right\} = P \cos \omega t U_j(x_0 + \gamma \sin \beta t)$$

(3.7)

and

$$\sum_{i=1}^{n} \left( Q_1(i,j) \dddot{Y}_i(t) + Q_2(i,j) V_i(t) + Q_3(i,j) Y_i(t) \right) = 0$$

(3.8)
where

\[
P_1 (i,j) = \bar{m} \int_0^L U_i (x) U_j (x) \, dx,
\]
\[
P_2 (i,j) = \int_0^L \left[ -K^* GFU_i' (x) U_j (x) + K (x) U_i (x) U_j (x) \right] \, dx
\]
\[
P_3 (i,j) = \int_0^L K^* GFX_i' (x) U_j (x) \, dx,
\]
\[
Q_1 (i,j) = JD \int_0^L X_i (x) U_j (x) \, dx
\]
\[
Q_2 (i,j) = \int_0^L K^* GFU_i' (x) U_j (x) \, dx,
\]
\[
Q_3 (i,j) = \int \left[ EJX_i'' (x) U_j (x) - K^* GFX_i (x) U_j (x) \right] \, dx
\]

(3.9)

Since our beam is assumed to have simple supports at both ends \( x = 0 \) and \( L = 0 \), the mode functions \( U_i (x) \) and \( X_i (x) \), are chosen to be \( \sin \frac{i\pi x}{L} \) and \( \cos \frac{i\pi x}{L} \) respectively. Thus substituting these into integrals (3.9), one obtains

\[
P_1 (i,j) = \frac{mL^2}{2}, \quad P_2 (i,j) = K_0 L^2 \left( 1 - \frac{L^2}{2} + \frac{L^2}{8} \right) + \frac{i^2 \pi^2}{L^2} K^* GF \cdot \frac{L}{2},
\]
\[
P_3 (i,j) = \frac{i\pi}{L} \frac{K^* GF \cdot L}{2}, \quad Q_1 (i,j) = -JD \frac{L}{2},
\]
\[
Q_2 (i,j) = \frac{i\pi}{L} \frac{K^* GF \cdot L}{2}, \quad Q_3 (i,j) = \left[ \frac{EJ\pi^2}{L^2} + K^* GF \right] \cdot \frac{L}{2}
\]

(3.10)

Now considering only the \( i \)th concentrated moving force, equation (3.7) and (3.8) can be simplified further to give

\[
P_1 (i,j) \ddot{V}_i (t) + P_2 (i,j) V_i (t) + P_3 (i,j) Y_i (t) = P_0 \cos \omega t \cos (\gamma \sin \beta t) + P_1 \cos \omega t \sin (\gamma \sin \beta t)
\]

(3.11)

and

\[
Q_1 (i,j) \ddot{Y}_i (t) + Q_2 (i,j) V_i (t) + Q_3 (i,j) = 0
\]

(3.12)

where

\[
P_0 = P \sin \frac{j\pi x_0}{L}, \quad P_1 = P \cos \frac{j\pi x_0}{L}, \quad \text{and} \quad \gamma = \frac{j\pi x}{L}
\]

(3.13)

In order to further simplify equation (3.11), use is made of the following Bessel relations.

\[
\sin (\gamma \sin \beta t) = 2 \sum_{i=0}^{\infty} J_{2k-1} (\gamma) \sin ([2k+] \beta t)
\]

(3.14)
\[ \cos(\gamma \sin \beta t) = J_0(\gamma) + 2 \sum_{k=0}^{\infty} J_{2k}(\gamma) \cos(2k\beta t) \quad (3.15) \]

where
\[ J_k(\gamma) = \sum_{m=0}^{\infty} (-1)^m \left( \frac{\gamma}{m} \right)^{k+2m} \frac{1}{m!(k+1)!} \quad (3.16) \]
is the modified Bessel function of the first kind of order \( k \).

In view of the equation (3.14) and (3.15), equation (3.11) becomes
\[ P_1(i,j) \ddot{V}_i(t) + P_2(i,j) V_i(t) + P_3(i,j) Y_i(t) = P_0 J_0(\gamma) \cos \omega t + 2 P_0 \sum_{k=1}^{\infty} J_{2k}(\gamma) \cos(2k\beta t) \]
\[ + 2 P_1 \sum_{k=0}^{\infty} J_{2k-1}(\gamma) \cos \omega t \sin[(2k+1)\beta t] \quad (3.17) \]

and
\[ Q_1(i,j) \ddot{Y}_i(t) + Q_2(i,j) V_i(t) + Q_3(i,j) Y_i(t) = 0 \quad (3.18) \]

which can further be simplified to take the form
\[ P_1(i,j) \ddot{V}_i(t) + P_2(i,j) V_i(t) + P_3(i,j) Y_i(t) = P_0 J_0(\gamma) \cos \omega t + 2 P_0 \sum_{k=1}^{\infty} J_{2k}(\gamma) [\cos \eta_1 t - \cos \eta_2 t] \]
\[ + 2 P_1 \sum_{k=0}^{\infty} J_{2k-1}(\gamma) [\sin \eta_3 t - \sin \eta_4 t] \quad (3.19) \]

and
\[ Q_1(i,j) \ddot{Y}_i(t) + Q_2(i,j) V_i(t) + Q_3(i,j) Y_i(t) = 0 \quad (3.20) \]

where
\[ \eta_1 = \omega + 2k\beta, \quad \eta_2 = \omega - 2k\beta, \quad \eta_3 = \omega + (2k+1)\beta, \quad \eta_4 = \omega - (2k+1)\beta \quad (3.21) \]

Subjecting the system of ordinary differential equations (3.19) and (3.20), to a Laplace transform
\[ (\hat{\cdot}) = \int_0^{\infty} e^{-st} dt \quad (3.22) \]

In conjunction with the initial conditions defined in (2.5) and (2.6), one obtains the following algebraic simultaneous equations
\[ \left[ P_1(i,j) S^2 + P_2(i,j) \right] V_i(S) + P_3(i,j) Y_i(S) = P_0 J_0(\gamma) \left[ \frac{S}{S^2 + \omega^2} \right] + P_0 \sum_{k=0}^{\infty} J_{2k}(\gamma) \left[ \frac{S}{S^2 + \eta_1^2} - \frac{S}{S^2 + \eta_2^2} \right] \]
Furthermore, equation (3.28) can be re-written in the form

\[ Y_i (S) + Q_2 (i, j) V_i (S) = 0 \]  

(3.24)

In order to solve the above system, the following representations are made

\[
\begin{align*}
\Omega_0 &= \begin{bmatrix} P_1 (i, j) S^2 + P_2 (i, j) & P_3 (i, j) \\ Q_2 (i, j) & Q_1 (i, j) + Q_3 (i, j) \end{bmatrix} \\
\Omega_1 &= \begin{bmatrix} \Omega_1 (1, 1) \\ 0 \end{bmatrix}
\end{align*}
\]

(3.25)

where

\[
\begin{align*}
\Omega_1 (1, 1) &= \left( P_0 J_0 (\gamma) \left[ \frac{S}{S^2 + \omega^2} \right] + P_0 \sum_{k=0}^{\infty} J_{2k} (\gamma) \left[ \frac{S}{S^2 + \eta_3^2} - \frac{S}{S^2 + \eta_4^2} \right] \\
&\quad + P_1 \sum_{k=0}^{\infty} J_{2k-1} (\gamma) \left[ \frac{\eta_3}{S^2 + \eta_3^2} - \frac{\eta_4}{S^2 + \eta_4^2} \right] \right) \\
\end{align*}
\]

(3.26)

and

\[
\begin{align*}
\Omega_2 &= \begin{bmatrix} (P_1 (i, j) S^2 + P_2 (i, j)) & \Omega_2 (1, 2) \\ Q_2 (i, j) & 0 \end{bmatrix}
\end{align*}
\]

(3.27)

where

\[
\begin{align*}
\Omega_2 (1, 2) &= \left( P_0 J_0 (\gamma) \left[ \frac{S}{S^2 + \omega^2} \right] + P_0 \sum_{k=0}^{\infty} J_{2k} (\gamma) \left[ \frac{S}{S^2 + \eta_3^2} - \frac{S}{S^2 + \eta_4^2} \right] \\
&\quad + P_1 \sum_{k=0}^{\infty} J_{2k-1} (\gamma) \left[ \frac{\eta_3}{S^2 + \eta_3^2} - \frac{\eta_4}{S^2 + \eta_4^2} \right] \right)
\end{align*}
\]

thus

\[
\begin{align*}
V_i (S) &= \frac{(Q_1 (i, j) S^2 + Q_3 (i, j)) \Omega_2 (1, 2)}{P_1 (i, j) Q_1 (i, j) S^4 + (P_1 (i, j) Q_3 (i, j) + P_2 (i, j) Q_1 (i, j)) S^2 - P_3 (i, j) Q_2 (i, j)} \\
\end{align*}
\]

(3.28)

and

\[
\begin{align*}
Y_i (S) &= \frac{Q_2 (i, j) \Omega_1 (1, 1)}{P_1 (i, j) Q_1 (i, j) S^4 + (P_1 (i, j) Q_3 (i, j) + P_2 (i, j) Q_1 (i, j)) S^2 - P_3 (i, j) Q_2 (i, j)} \\
\end{align*}
\]

(3.29)

Furthermore, equation (3.28) can be re-written in the form

\[
V_i (S) = \frac{(Q_1 (i, j) S^2 + Q_3 (i, j)) \Omega_2 (1, 2)}{B_1 [(S^2 + \phi^2) (S^2 + \varphi^2)]} \\
\]

(3.30)
and

\[ Y_1(S) = \frac{Q_2(i,j)\Omega_1(1,1)}{B_1\left[(S^2 + \phi^2)\left(S^2 + \varphi^2\right)\right]} \]  

(3.31)

where

\[ \phi = \sqrt{\frac{B_2}{2B_1} - \left(\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}\right)^{\frac{1}{2}}} \quad \varphi = \sqrt{\frac{B_2}{2B_1} + \left(\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}\right)^{\frac{1}{2}}} \]

\[ B_1 = P_1(i,j)Q_1(i,j), \quad B_2 = P_1(i)Q_3(i,j) + P_2(i,j)Q_1(i,j), \quad B_3 = -P_3(i,j)Q_2(i,j) \]  

(3.32)

Using partial fractions technique, equations (3.30) and (3.31) can further be rewritten as

\[
 V_i(S) = \frac{1}{B_1} \left( \left[ \frac{Q_1(i,j)\phi^2 - Q_3(i,j)}{(\phi^2 - \varphi^2)} \right] - \left[ \frac{Q_1(i,j)\phi^2 - Q_3(i,j)}{(\varphi^2 - \varphi^2)} \right] \right) \cdot \left( P_0J_0(\gamma) \left[ \frac{S}{S^2 + \omega^2} + P_1 \sum_{k=0}^{\infty} J_{2k}(\gamma) \left[ \frac{S}{S^2 + \eta_1^2} - \frac{S}{S^2 + \eta_2^2} \right] \right) \right) 

(3.33)

\[ + P_1 \sum_{k=0}^{\infty} J_{2k-1}(\gamma) \left[ \frac{\eta_3}{S^2 + \eta_3^2} - \frac{\eta_4}{S^2 + \eta_4^2} \right) \right] \]

and

\[
 V_i(S) = \frac{1}{B_1} \left( \frac{Q_2(i,j)}{(\phi^2 - \phi^2)} \left( \frac{1}{S^2 + \phi^2} - \frac{1}{S^2 + \varphi^2} \right) \right) \cdot \left( P_0J_0(\gamma) \left[ \frac{S}{S^2 + \omega^2} \right) \right) \right) 

(3.34)

\[ + P_0 \sum_{k=0}^{\infty} J_{2k}(\gamma) \left[ \frac{S}{S^2 + \eta_1^2} - \frac{S}{S^2 + \eta_2^2} \right] \right) + P_1 \sum_{k=0}^{\infty} J_{2k-1}(\gamma) \left[ \frac{\eta_3}{S^2 + \eta_3^2} - \frac{\eta_4}{S^2 + \eta_4^2} \right] \right) \]
which after some simplifications and rearrangements gives,

\[
V_i (S) = \frac{P_0 J_0 (\gamma)}{B_1} \left[ \left( \frac{Q_1 (i, j)}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \phi^2} \right) \cdot \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \omega^2} \right.
\]
\[
- \left[ \frac{Q_1 (i, j)}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \phi^2} \right] \cdot \frac{1}{S^2 + \phi^2} \cdot \frac{S}{S^2 + \omega^2} \right)
\]
\[
+ \frac{P_0}{B_1} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left( \left[ \frac{Q_1 (i, j)}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \phi^2} \right] \cdot \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_1^2} \right.
\]
\[
- \left[ \frac{Q_1 (i, j)}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \phi^2} \right] \cdot \frac{1}{S^2 + \phi^2} \cdot \frac{S}{S^2 + \eta_1^2} \right)
\]
\[
+ \frac{P_1}{B_1} \sum_{k=0}^{\infty} J_{2k+1} (\gamma) \left( \left[ \frac{Q_1 (i, j)}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \phi^2} \right] \cdot \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_3^2} \right.
\]
\[
- \left[ \frac{Q_1 (i, j)}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \phi^2} \right] \cdot \frac{1}{S^2 + \phi^2} \cdot \frac{S}{S^2 + \eta_3^2} \right)
\]
\[
\left. + \frac{P_1}{B_1} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left[ \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_1^2} - \left( \frac{1}{S^2 + \phi^2} \cdot \frac{S}{S^2 + \eta_1^2} - \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_1^2} \right) \right] \right)
\]

(3.35)

and

\[
Y_i (S) = \frac{P_0 J_0 (\gamma)}{B_1 (\varphi^2 - \phi^2)} \left[ \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \omega^2} - \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \omega^2} \right]
\]
\[
+ \frac{P_0 Q_2 (i, j)}{B_1 (\varphi^2 - \phi^2)} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left[ \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_1^2} \right.
\]
\[
- \left( \frac{1}{S^2 + \phi^2} \cdot \frac{S}{S^2 + \eta_1^2} - \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_1^2} \right) \right]
\]
\[
+ \frac{P_1 Q_2 (i, j)}{B_1 (\varphi^2 - \phi^2)} \sum_{k=0}^{\infty} J_{2k+1} (\gamma) \left[ \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_3^2} - \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_3^2} \right]
\]
\[
- \left( \frac{1}{S^2 + \phi^2} \cdot \frac{S}{S^2 + \eta_3^2} - \frac{1}{S^2 + \varphi^2} \cdot \frac{S}{S^2 + \eta_3^2} \right) \right]
\]

(3.36)
In order to obtain the Laplace inversion of equations (35) and (36), the following representations are employed

\[
g_1 (S) = \frac{S}{S^2 + \omega^2}, \quad g_2 (S) = \frac{S}{S^2 + \eta_1^2}, \quad g_3 (S) = \frac{S}{S^2 + \eta_2^2}, \quad g_4 (S) = \frac{S}{S^2 + \eta_3^2} \tag{3.37}
\]

\[
g_5 (S) = \frac{S}{S^2 + \eta_4^2}, \quad f_1 (S) = \frac{\varphi}{S^2 + \varphi^2}, \quad f_2 (S) = \frac{\phi}{S^2 + \phi^2}
\]

so that the Laplace inversion of (35) and (36) is the convolution of \(g_j (s)\) and \(f_i (s)\) defined as

\[
f_i (S) * g_j (S) = \int_0^t f_i (t-u) g_j (u) \, du \quad i = 1, 2 \quad \text{and} \quad j = 1, 2, 3, 4, 5 \tag{3.38}
\]

Thus the Laplace inversions of (35) and (36) are respectively given as

\[
V_i (t) = \frac{p_0 J_0 (\gamma)}{B_1} \left\{ \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\phi^2} \right] \cdot H_1 - \left[ \frac{Q_1 (i, j) \phi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2} \right] \cdot H_2 \right\} + \frac{p_0}{B_1} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left\{ \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\phi^2} \right] \cdot H_3 - \left[ \frac{Q_1 (i, j) \phi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2} \right] \cdot H_4 \right\} - \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\phi^2} \right] \cdot H_5 - \left[ \frac{Q_1 (i, j) \phi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2} \right] \cdot H_6 \right\}
\]

\[
+ \frac{p_1}{B_1} \sum_{k=0}^{\infty} J_{2k+1} (\gamma) \left\{ \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\phi^2} \right] \cdot H_7 - \left[ \frac{Q_1 (i, j) \phi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2} \right] \cdot H_8 \right\} - \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\phi^2} \right] \cdot H_9 - \left[ \frac{Q_1 (i, j) \phi^2}{\varphi^2 - \phi^2} - \frac{Q_3 (i, j)}{\varphi^2} \right] \cdot H_{10}\right\} \tag{3.39}
\]

and

\[
Y_i (t) = \frac{p_0 J_0 (\gamma) Q_2 (i, j)}{B_1 (\varphi^2 - \phi^2)} \left[ H_2 - H_1 \right] + \frac{p_0 Q_2 (i, j)}{B_1 (\varphi^2 - \phi^2)} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left[ H_4 - H_3 - H_6 + H_5 \right]
\]

\[
+ \frac{p_1 Q_2 (i, j)}{B_1 (\varphi^2 - \phi^2)} \sum_{k=0}^{\infty} J_{2k+1} (\gamma) \left[ H_8 - H_7 - H_{10} + H_9 \right] \tag{3.40}
\]
where

\[
H_1 = \frac{1}{\phi} \int_0^L \sin \varphi (t - u) \cos \omega u \, du \quad H_2 = \frac{1}{\phi} \int_0^L \sin \phi (t - u) \cos \omega u \, du \\
H_3 = \frac{1}{\phi} \int_0^L \sin \varphi (t - u) \cos \eta_1 u \, du \quad H_4 = \frac{1}{\phi} \int_0^L \sin \phi (t - u) \cos \eta_1 u \, du \\
H_5 = \frac{1}{\phi} \int_0^L \sin \varphi (t - u) \cos \eta_2 u \, du \quad H_6 = \frac{1}{\phi} \int_0^L \sin \phi (t - u) \cos \eta_2 u \, du \\
H_7 = \frac{1}{\phi} \int_0^L \sin \varphi (t - u) \cos \eta_3 u \, du \quad H_8 = \frac{1}{\phi} \int_0^L \sin \phi (t - u) \cos \eta_3 u \, du \\
H_9 = \frac{1}{\phi} \int_0^L \sin \varphi (t - u) \cos \eta_4 u \, du \quad H_{10} = \frac{1}{\phi} \int_0^L \sin \phi (t - u) \cos \eta_4 u 
\]

By trigonometric identities, it can be shown that

\[
1 \int_0^t \sin B (t - u) \sin A u \, du = \frac{B \sin B t}{B^2 - A^2} \left[ \sin A t \sin B t - \frac{A}{B} \cos A t \cos B t - 1 \right] \\
+ \frac{B \cos B t}{B^2 - A^2} \left[ \sin A t \cos B t - \frac{A}{B} \sin A t \sin B t \right]
\]

and

\[
1 \int_0^t \sin B (t - u) \cos A u \, du = \frac{A \sin B t}{A^2 - B^2} \left[ \cos B t \sin A t - \frac{B}{A} \sin B t \cos A t \right] \\
+ \frac{A \cos B t}{A^2 - B^2} \left[ \sin B t \sin A t - \frac{B}{A} \cos B t \cos A t - 1 \right]
\]

Integrals (3.41), taking into account the identities (3.42) become,

\[
H_1 = \frac{\omega \sin \varphi t}{\varphi (\omega^2 - \varphi^2)} \left( \cos \varphi t \sin \omega t - \frac{\varphi}{\omega} \sin \varphi t \cos \omega t \right) \\
- \frac{\omega \cos \varphi t}{\varphi (\omega^2 - \varphi^2)} \left( \sin \varphi t \sin \omega t - \frac{\varphi}{\omega} \cos \varphi t \cos \omega t \right) \\
H_2 = \frac{\omega \sin \phi t}{\phi (\omega^2 - \phi^2)} \left( \cos \phi t \sin \omega t - \frac{\phi}{\omega} \sin \phi t \cos \omega t \right) \\
- \frac{\omega \cos \phi t}{\phi (\omega^2 - \phi^2)} \left( \sin \phi t \sin \omega t - \frac{\phi}{\omega} \cos \phi t \cos \omega t \right) \\
H_3 = \frac{\eta_1 \sin \varphi t}{\varphi (\eta_1^2 - \varphi^2)} \left( \cos \varphi t \sin \eta_1 t - \frac{\varphi}{\eta_1} \sin \varphi t \cos \eta_1 t \right) \\
- \frac{\eta_1 \cos \varphi t}{\varphi (\eta_1^2 - \varphi^2)} \left( \sin \varphi t \sin \eta_1 t - \frac{\varphi}{\eta_1} \cos \varphi t \cos \eta_1 t \right)
\]
\begin{align*}
H_4 &= \frac{\eta_1 \sin \phi t}{\phi (\eta_1^2 - \phi^2)} \left( \cos \phi t \sin \eta_1 t - \frac{\phi}{\eta_1} \sin \phi t \cos \eta_1 t \right) \\
     &\quad - \frac{\eta_1 \cos \phi t}{\phi (\eta_1^2 - \phi^2)} \left( \sin \phi t \sin \eta_1 t - \frac{\phi}{\eta_1} (\cos \phi t \cos \eta_1 t - 1) \right) \\
H_5 &= \frac{\eta_2 \sin \varphi t}{\varphi (\eta_2^2 - \varphi^2)} \left( \cos \varphi t \sin \eta_2 t - \frac{\varphi}{\eta_2} \sin \varphi t \cos \eta_2 t \right) \\
     &\quad - \frac{\eta_2 \cos \varphi t}{\varphi (\eta_2^2 - \varphi^2)} \left( \sin \varphi t \sin \eta_2 t - \frac{\varphi}{\eta_2} (\cos \varphi t \cos \eta_2 t - 1) \right) \quad (3.43) \\
H_6 &= \frac{\eta_2 \sin \phi t}{\phi (\eta_2^2 - \phi^2)} \left( \cos \phi t \sin \eta_2 t - \frac{\phi}{\eta_2} \sin \phi t \cos \eta_2 t \right) \\
     &\quad - \frac{\eta_2 \cos \phi t}{\phi (\eta_2^2 - \phi^2)} \left( \sin \phi t \sin \eta_2 t - \frac{\phi}{\eta_2} (\cos \phi t \cos \eta_2 t - 1) \right) \\
H_7 &= \frac{\eta_3 \sin \varphi t}{\varphi (\eta_3^2 - \varphi^2)} \left( \cos \varphi t \sin \eta_3 t - \frac{\varphi}{\eta_3} \sin \varphi t \cos \eta_3 t \right) \\
     &\quad - \frac{\eta_3 \cos \varphi t}{\varphi (\eta_3^2 - \varphi^2)} \left( \sin \varphi t \sin \eta_3 t - \frac{\varphi}{\eta_3} (\cos \varphi t \cos \eta_3 t - 1) \right) \\
H_8 &= \frac{\eta_3 \sin \phi t}{\phi (\eta_3^2 - \phi^2)} \left( \cos \phi t \sin \eta_3 t - \frac{\phi}{\eta_3} \sin \phi t \cos \eta_3 t \right) \\
     &\quad - \frac{\eta_3 \cos \phi t}{\phi (\eta_3^2 - \phi^2)} \left( \sin \phi t \sin \eta_3 t - \frac{\phi}{\eta_3} (\cos \phi t \cos \eta_3 t - 1) \right) \\
H_9 &= \frac{\eta_4 \sin \varphi t}{\varphi (\eta_4^2 - \varphi^2)} \left( \cos \varphi t \sin \eta_4 t - \frac{\varphi}{\eta_4} \sin \varphi t \cos \eta_4 t \right) \\
     &\quad - \frac{\eta_4 \cos \varphi t}{\varphi (\eta_4^2 - \varphi^2)} \left( \sin \varphi t \sin \eta_4 t - \frac{\varphi}{\eta_4} (\cos \varphi t \cos \eta_4 t - 1) \right) \\
H_{10} &= \frac{\eta_4 \sin \phi t}{\phi (\eta_4^2 - \phi^2)} \left( \cos \phi t \sin \eta_4 t - \frac{\phi}{\eta_4} \sin \phi t \cos \eta_4 t \right) \\
     &\quad - \frac{\eta_4 \cos \phi t}{\phi (\eta_4^2 - \phi^2)} \left( \sin \phi t \sin \eta_4 t - \frac{\phi}{\eta_4} (\cos \phi t \cos \eta_4 t - 1) \right) \quad (3.43)
\end{align*}

Thus, in view of expression (3.1), and taking into account (3.39), one obtains for the this vibrating system,
\[
\phi_i (x, t) = \sum_{i=1}^{n} \left\{ \left( \frac{P_0 J_0 (\gamma)}{B_1} \left\{ \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_1 - \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_2 \right\} \right\}
\]
\]
\[
+ \left( \frac{P_0}{B_1} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left\{ \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_3 - \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_4 \right\} \right)
\]
\[
- \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_5 - \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_6 \right) \right) \right) \sin \frac{i\pi x}{L}
\]
\[
+ \left( \frac{P_1}{B_1} \sum_{k=0}^{\infty} J_{2k+1} (\gamma) \left\{ \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_7 - \left[ \frac{Q_1 (i, j) \varphi^2}{\varphi^2 - \varphi^2} - \frac{Q_3 (i, j)}{\varphi^2 - \varphi^2} \right] \cdot H_8 \right\} \right) \sin \frac{i\pi x}{L}
\]
which represents the response amplitude of a prismatic deep beam when under the actions of harmonic variable magnitude load travelling at time dependent speed.

Similarly, in view of expression (3.2), taking into account of (3.40) one obtains
\[
\psi_i (x, t) = \sum_{i=1}^{n} \left\{ \frac{P_0 J_0 (\gamma) Q_2 (i, j)}{B_1 (\varphi^2 - \varphi^2)} \cdot H_2 - H_1 \right\} + \frac{P_0 Q_2 (i, j)}{B_1 (\varphi^2 - \varphi^2)} \sum_{k=1}^{\infty} J_{2k} (\gamma) \left[ H_4 - H_3 - H_6 + H_5 \right]
\]
\[
+ \frac{P_1 Q_2 (i, j)}{B_1 (\varphi^2 - \varphi^2)} \sum_{k=0}^{\infty} J_{2k+1} (\gamma) \left[ H_8 - H_7 - H_{10} + H_9 \right] \right) \cdot \cos \frac{i\pi x}{L}
\]
which represents the rotation of the dynamical system.

4. COMMENTS ON THE CLOSED FORM SOLUTION

It is well known that the displacement response of an engineering structure under excitation may grow without bound and when this happens it leads to the occurrence called resonance. This occurrence of resonance in structural and highway engineering is quite undesirable. This is so because, its effects on such dynamical system could be devastating. In particular, it causes cracks, permanent deformation and destruction in structures and makes the structural systems unsaved for its occupants. Thus, it is very pertinent at this juncture to establish the conditions under which this undesirable phenomenon may occur. Equations (3.42) and (3.44) clearly indicates that the vibrating system under discussion reaches a state of resonance whenever
\[
[P_1 (i, j) Q_3 (i, j) + P_2 (i, j) Q_1 (i, j)]^2 = -4 \cdot (P_1 (i, j) Q_1 (i, j)) \cdot P_3 (i, j) Q_2 (i, j),
\]
\[
\omega = -2k\beta, \quad \omega = 2k\beta, \quad \omega = -(2k + 1)\beta, \quad \omega = (2k + 1)\beta
\]
\[ \omega = \left( \frac{B_2}{2B_1} - \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \quad \text{or} \quad \omega = \left( \frac{B_2}{2B_1} + \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \]

\[ \eta_1 = \left( \frac{B_2}{2B_1} - \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \quad \text{or} \quad \eta_1 = \left( \frac{B_2}{2B_1} + \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \]

\[ \eta_2 = \left( \frac{B_2}{2B_1} - \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \quad \text{or} \quad \eta_2 = \left( \frac{B_2}{2B_1} + \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \]

\[ \eta_3 = \left( \frac{B_2}{2B_1} - \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \quad \text{or} \quad \eta_3 = \left( \frac{B_2}{2B_1} + \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \]

\[ \eta_4 = \left( \frac{B_2}{2B_1} - \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \quad \text{or} \quad \eta_4 = \left( \frac{B_2}{2B_1} + \sqrt{\frac{B_2^2}{4B_1^2} - \frac{B_3}{B_1}} \right)^{\frac{1}{2}} \]

5. Results and Discussions

For the purpose of illustration, we adopt the beam parameters and material properties defined in Eftekhar et al. [32]. These properties are Length \( L = 50 \) m. The modulus of elasticity \( E \) is \( 3.34 \times 10^{10} \) N/m\(^2\), moment of inertia \( I = 1.042 \times 10^4 \) m\(^4\), Density \( \rho = 2400 \) Kg/m\(^3\), the shear modulus is \( 1.34 \times 10^{10} \) N/m\(^2\) and cross-sectional area of \( 2 \) m\(^2\).

**Figure 1.** The transverse displacement response a uniform Timoshenko beam resting on variable elastic foundation and subjected to variable magnitude moving load for various values of foundation modulus \( K \).
Figure 1 display the transverse displacement response of a uniform Timoshenko beam resting on variable elastic foundation when subjected to harmonic variable magnitude loads traveling with a variable velocity. It is deduced from this figure that for fixed values of other vital parameters, the transverse deflection of a uniform Timoshenko beam resting on variable elastic foundation and traversed by fast traveling masses decreases as the values of foundation reaction subgrade $K_0$ increases.

In figure 2 the deflection profile of a uniform Timoshenko beam resting on variable elastic foundation and subjected to variable magnitude moving load traveling at non uniform velocity is displayed for various load positions. It is clearly shown that the larger the value of the load positions the lower the deflection of the elastic beam.

![Figure 2](image)

**Figure 2.** The deflection profile of a uniform Timoshenko beam resting on variable elastic foundation and subjected to variable magnitude moving load for various values of the load positions $x$.

The displacement response of a uniform Timoshenko beam resting on variable elastic foundation and under the actions of a variable magnitude moving load traveling at non uniform velocity is shown in figure 3. It is observed from this figure that higher values of the load longitudinal frequency $\beta$ produce more stabilizing effects on the elastic beam.

Figure 4 depicts the deflection profile of the uniform Timoshenko beam resting on elastic foundation and subjected to fast traveling variable magnitude moving load. It is shown from the figure that for fixed value of foundation reaction $K_0$ and other structural parameters, the deflection of the beam reduces as the values of the circular frequency $\omega$ increases.
Figure 3. The displacement response of a uniform Timoshenko beam resting on variable elastic foundation and subjected to variable magnitude moving load for various values of longitudinal frequency of the load.

Figure 4. The response amplitude of a uniform Timoshenko beam resting on variable elastic foundation and under the actions of variable magnitude moving load for various values of circular frequency $\omega$. 
Figure 5 depicts the response amplitude of a uniform Timoshenko beam resting on variable elastic foundation when subjected to harmonic variable magnitude loads travelling with a variable velocity. It is clearly shown from the figure that as the values of the longitudinal amplitude of the oscillation of the travelling load increases, for fixed values of other parameters the dynamic deflection of the beam increases.

In figure 6, the transverse displacement response of a Timoshenko beam under the actions of harmonic magnitude load is shown. It is deduce from this figure that as the values of shear modulus $G$ increases, the response amplitude of the beam increases.

6. CONCLUDING REMARKS

The dynamic behaviour of a simply supported Timoshenko beams resting on variable elastic foundation when carrying fast traveling concentrated loads of varying magnitudes has been investigated. The versatile analytical technique known as Galerkin’s method has been employed in conjunction with integral transform method to obtain closed-form solution of this dynamical beam-load problem. Both analytical and numerical results presented in this paper are in perfect agreement with existing results. Results show that, as the values of foundation stiffness $K_0$ increases, the deflection profile of the uniform Timoshenko beam increases.
Figure 6. The response amplitude of a uniform Timoshenko beam resting on variable elastic foundation and under the actions of variable magnitude moving load for various values of shear modulus $G$.

References

LARGE EDDY SIMULATION OF TURBULENT CHANNEL FLOW USING
ALGEBRAIC WALL MODEL

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ABSTRACT. A large eddy simulation (LES) of a turbulent channel flow is performed by using
the third order low-storage Runge–Kutta method in time and second order finite difference
formulation in space with staggered grid at a Reynolds number, \( Re_x = 590 \) based on the
channel half width, \( \delta \) and wall shear velocity, \( u_* \). To reduce the calculation cost of LES,
algebraic wall model (AWM) is applied to approximate the near-wall region. The computation
is performed in a domain of \( 2\pi \delta \times 2\delta \times \pi \delta \) with \( 32 \times 20 \times 32 \) grid points. Standard Smagorinsky
model is used for subgrid-scale (SGS) modeling. Essential turbulence statistics of the flow field
are computed and compared with Direct Numerical Simulation (DNS) data and LES data using
no wall model. Agreements as well as discrepancies are discussed. The flow structures in
the computed flow field have also been discussed and compared with LES data using no wall
model.

1. INTRODUCTION

Turbulent channel flow is an important test case for numerical simulations and validation
of turbulent models. It is commonly encountered in engineering practice. Because of the
simplicity in geometry and its wide application background in industry, the experimental and
computational studies of the turbulent channel flow have been carried out extensively [1-9].

In recent years, with the development of the technique of numerical simulation, LES has
been demonstrated to be an useful research tool for understanding the physics of turbulence
in more complex geometries than DNS. Although DNS is considered as the exact approach to
turbulence simulation, yet it is very expensive if the flow Reynolds number is very high and
computational grid is very large. LES is a method in which large-scale motions are exactly
calculated and the SGS motion is modeled [5-6].

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To conduct LES in wall bounded turbulent flows [3-10], a large number of computational nodes are generally required to resolve the boundary layers. Wall integration of turbulence models requires the first computational node above the wall to be situated within the viscous sublayer. But, wall functions do not require the excessive grids in the boundary layer. For instance, the first computational node might be positioned in the logarithmic inertial layer, which will lead to a significant reduction in the number of computational nodes in the boundary layer [11]. But, this is to be done without a significant loss in accuracy. However, the computational cost of LES can be reduced by using wall stress models.

Wall stress models provide an algebraic relationship between local wall shear stresses and tangential velocities at the wall-nearest velocity nodes. This approach was first employed by Schumann [12] for performing a plane channel flow simulation. He assumed that the streamwise and spanwise velocity fluctuations are in phase with the respective surface shear stress components. A number of improvements to Schumann’s model were suggested by, for example, Grötzbach [8] and Werner and Wengle [9], who wanted to avoid having to know the mean wall shear stress a priori and to simplify the computations. Another wall model was proposed by Piomelli et al. [13] as a modification of the previous wall models to empirically account for the effect of sweep and ejection events on the wall shear stress. To reduce the calculation cost of LES, in this study we have used AWM [14] at the near wall region.

To perform LES in turbulence, discretization method is another concern. For spatial discretization the conventional finite difference method is widely used with structured grids [15-17], and for temporal discretization the explicit Runge–Kutta methods [18-19] are a popular choice. Although in explicit Runge–Kutta methods generally the Poisson equation is solved for the pressure at each stage, these methods generally have better stability properties, do not have a start-up problem, and easily allow for adaptive time stepping [19].

To conduct LES in a turbulent channel flow it is necessary to do long time integration that need much wider computation region. In this case, the application of a low-storage Runge–Kutta scheme is significant to make sufficient utilization of computer resources. Because, low-storage Runge–Kutta schemes require minimum levels of memory locations during the time integration and efficiently comply with the modern large-scale scientific computing needs. A number of third-order low-storage explicit Runge–Kutta schemes were derived by Williamson [20].

The objective of this study is to perform LES of a plane turbulent channel flow using AWM. Spatial and temporal discretization has been done by using the second order finite difference formulation and third order low-storage explicit Runge–Kutta method respectively in a staggered grid system. For SGS modeling the Standard Smagorinsky model has been used. Essential turbulence statistics of the computed flow field are investigated and compared with DNS data of Moser et al. [2] and LES data of Uddin and Mallik [5]. Instantaneous streamwise velocity distribution at the centerline plane of the channel and instantaneous streamwise shear velocity distribution at the immediate vicinity of the wall have also been discussed by different contour plots and compared with those obtained by Uddin and Mallik [5]. Vortical structures using second invariant of velocity gradient tensor in the turbulent flow field are visualized and compared with that of Uddin and Mallik [5]. More specifically, the prime objective of our
investigation is whether our simulation is able to capture turbulence at low resolution by using AWM in LES.

2. Governing Equations

The governing equations of LES for an incompressible plane turbulent channel flow are the filtered Navier–Stokes and continuity equations for constant density in Cartesian co-ordinates given as [5, 21]:

\[
\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i \bar{u}_j + \tau_{ij}) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right],
\]

(2.1)

\[
\frac{\partial \bar{u}_i}{\partial x_i} = 0
\]

(2.2)

where the index \(i, j = 1, 2, 3\) refers to the \(x, y\) and \(z\) directions respectively. Here \(\bar{u}_x, \bar{u}_y, \bar{u}_z\) are streamwise, wall normal and spanwise filtered velocity respectively. \(\bar{p}\) is the filtered pressure, \(\rho\) represents the density of the fluid and \(\nu\) is the kinematic viscosity. \(\tau_{ij}\) is SGS Reynolds stress which is in fact the large scale momentum flux caused by the action of the small or unresolved scales. The equations are non-dimensionalized by the channel half-width \(\delta\), and the wall shear velocity \(u_\tau\). The flow Reynolds number is therefore written as \(Re_\tau = u_\tau \delta / \nu\). A schematic geometry of the plane turbulent channel flow and the co-ordinate system are shown in Fig. 1.

![Flow](image)

**Figure 1.** Schematic geometry of plane channel flow.

In LES, the velocity field \(u_i\) is decomposed into a filtered or large scale component \(\bar{u}_i\) and a SGS component \(u'_i\) by applying a spatial filtering operation. This decomposition is represented as [21]:

\[
u_i = \bar{u}_i + u'_i
\]

(2.3)
The resolved velocity component, \( \bar{u}_i \), can be expressed as

\[
\bar{u}_i (x_1, x_2, x_3, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \prod_{i=1}^{3} G_i (x_i - x_i') \right) u_i (x_1', x_2', x_3', t) \, dx_1' \, dx_2' \, dx_3' \quad (2.4)
\]

where \( G_i (x_i - x_i') \) is a general filtering function which satisfies the following relation:

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( \prod_{i=1}^{3} G_i (x_i - x_i') \right) \, dx_1' \, dx_2' \, dx_3' = 1 \quad (2.5)
\]

The SGS flow field, \( u_i' \), is obtained by subtracting the filtered field from the full field. The effect of the SGS field appears through the SGS Reynolds stress term, which is defined as

\[
\tau_{ij} = u_i u_j - \bar{u}_i \bar{u}_j \quad (2.6)
\]

LES requires a model to represent the effects of the SGS field on the filtered field. The models used to approximate the SGS Reynolds stress are called SGS models. The simplest and most popular SGS model is the Standard Smagorinsky model. In this model, \( \tau_{ij} \) is computed as \[21\]:

\[
\tau_{ij} = -2 \nu_S \bar{S}_{ij} \quad (2.7)
\]

where,

\[
\nu_s = (C_S \Delta)^2 \bar{S} \quad (2.8)
\]

is the SGS eddy viscosity. The quantity \( C_S \) is the Smagorinsky constant which is not fixed. Many authors used different values of \( C_S \) for LES in turbulent channel flows. In this study, the computation is performed for \( C_S = 0.065 \) for a channel flow \[3\]. \( \Delta = (\Delta x \Delta y \Delta z)^{1/3} \) is filter width and \( |\bar{S}| = \sqrt{2 \bar{S}_{ij} \bar{S}_{ij}} \) is the magnitude of strain rate, where \( \bar{S}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \).

For the reduced growth of the small scales at the near wall region the SGS eddy viscosity can be modified as \[5, 6\]:

\[
\nu_s = (C_S f_S \Delta)^2 \bar{S} \quad (2.9)
\]

Here \( f_S = 1 - \exp \left( -\frac{y^+}{A^+} \right) \) is the Van-Driest damping function, where \( y^+ \) is the distance from the wall in viscous wall units defined as \( y^+ = \frac{y u_*}{v} \) and \( A^+ \) is a constant usually taken to be approximately 25 \[21\].

3. Numerical methods and grid system

The governing equations of LES are solved using the third order low-storage explicit Runge–Kutta method in time \[22\] and the second order finite difference formulae in space. The coupling between continuity equation and pressure fields is performed by the simplified marker-and-cell (SMAC) method \[23\]. Poisson equation is solved iteratively by Incomplete Cholesky Decomposition Conjugated Gradient method. The Spalding equation is solved by iterative procedure based on the Newton method.

Conventional numerical algorithms based on a structured computational grid mostly fall into three classes: regular, staggered, and collocated grid systems. In this study, the staggered
grid system has been used. Staggered grids may be constructed by several methods. On the staggered grid, scalar variable pressure are stored at the nodes and velocities are stored at the middle of the two nodes. A staggered grid system in a two-dimensional plane has been given in our previous papers [5, 6]. The biggest advantage of the staggered arrangement is the strong coupling between the velocities and the pressure.

When the computational domain is discretized by the grid points, the governing equations should be discretized in this domain. This will result in a set of ordinary differential equations for each grid point in space. There are different methods to discretize the equations and the finite difference method is the most straightforward one. Finite difference method is simply the substitution of the continuous differential operators with corresponding discrete operators. There are a variety of discretization techniques available for developing discrete approximations to a set of governing partial differential equations such as Navier–Stokes equations. Let the finite difference operator with stencil size 1 acting on a discrete variable \( \varphi \) with respect to \( x \) for structural Cartesian meshes with uniform spacing be defined as [5, 6]:

\[
\left. \frac{\partial \varphi}{\partial x} \right|_{i,j,k} = \frac{\varphi_{i+1,j,k} - \varphi_{i-1,j,k}}{2\Delta x}
\]

(3.1)

where the grid spacings \( \Delta x \) are constant in \( x \) direction, and \((i, j, k)\) denotes associated mesh indices in \( x, y \) and \( z \) directions. Discrete operators in the \( y \) and \( z \) directions are similarly defined. In addition to the discrete differencing operator we also define interpolation operators given in our previous papers [5, 6].

Since the Navier–Stokes equations are unsteady, to solve them numerically both spatial and temporal discretizations are needed. For temporal discretization a low-storage explicit Runge–Kutta method is shortly described by Uddin and Mallik [5], and Mallik et al. [6]. Such a scheme requires only two levels of memory locations during the time integration.

4. Computational Parameters

The computational domain of the mesh is selected to be \( 2\pi\delta \times 2\delta \times \pi\delta \) in streamwise, wall normal and spanwise directions respectively. The computation is performed using \( 32 \times 20 \times 32 \) grid points in the corresponding directions, and the possible Reynolds number is \( Re_c = 590 \) based on the channel half width, \( \delta \) and wall shear velocity, \( u_r \). The computation is carried out with a non-dimensional time increment, \( \Delta t = 0.002 \), which maintained a CFL number [5, 6]:

\[
CFL = \Delta t \max \left( \left| \frac{\langle \bar{u}_x \rangle}{\Delta x} \right| + \left| \frac{\langle \bar{u}_y \rangle}{\Delta y} \right| + \left| \frac{\langle \bar{u}_z \rangle}{\Delta z} \right| \right) = 0.334 < 1.0
\]

(4.1)

where, \( \langle \bar{u}_i \rangle \) denotes an ensemble average of \( \bar{u}_i \).

The computation is executed up to time, \( t = n\Delta t \), where \( n \) is the number of time step. The computational domain is discretized with uniformly distributed grid in all the directions, and the grid spacing in the corresponding directions are \( \Delta x^+ \approx 116, \Delta z^+ \approx 58 \) and \( \Delta y^+ \approx 59 \) wall units respectively. The first mesh point away from the wall is at \( y^+ \approx 29.5 \) wall unit. The superscript ‘+’ indicates a non-dimensional quantity scaled by the wall variables; e.g., \( y^+ = yu_r/\nu \), where \( \nu \) is the kinematic viscosity and \( u_r = (\tau_w/\rho)^{1/2} \) is the wall shear velocity.
5. Boundary conditions

We consider fully developed incompressible viscous flow and make use of periodic boundary conditions in the streamwise and spanwise directions. For the staggered grid arrangement we set up additional nodes surrounding the physical boundary. The calculations are performed at internal nodes only. The wall boundary condition is no-slip. Just outside the solution domain the values of the velocity components are equated to the values of the nearest node just inside the solution domain [24]. The pressure boundary condition is periodic in the streamwise and spanwise directions. But in the wall normal direction the values of $p$, just outside the solution domain, are determined by assuming a zero gradient [25].

6. Algebraic wall model

To reduce the calculation cost of LES, the near wall layer is approximated by a special form of AWM given by Spalding [14]. In this model one uses Spalding’s law as an algebraic equation to calculate the wall shear stresses. Spalding’s law is a non-linear equation about the wall shear velocity $u_\tau$, whose special form is

$$y^+ = f(u^+) = u^+ + A \left[ \exp (\kappa u^+) - 1 - (\kappa u^+)^3/6 - (\kappa u^+)^4/24 \right]$$

where, $A = \exp(-\kappa B) = 0.1108$, $\kappa = 0.4$ and $B = 5.5$. The exceptional feature of Spalding’s equation is that it presents $y^+$ as a function of $u^+$ rather than $u^+$ as a function of $y^+$. Here $y$ is the distance from the wall and $y^+ = y u_\tau/\nu$ is the non-dimensional wall unit. In this equation $u$ is the instantaneous horizontal velocity and $u^+ = u/u_\tau$ is the non-dimensional velocity at the first off-wall computational cells. At the first off-wall computational cells the LES velocity, $\sqrt{\langle \bar{u}_x \rangle^2 + \langle \bar{u}_z \rangle^2}$ is substituted to $u$ of Eq. (6.1). Then Eq. (6.1) is solved by iterative procedure based on the Newton method for the wall shear velocity, $u_\tau$. After that the instantaneous wall shear stresses are calculated as follows:

$$\tau_{w,x}/\rho = (\nu y^+/u^+) \bar{u}_x/y \quad \tau_{w,z}/\rho = (\nu y^+/u^+) \bar{u}_z/y$$

These wall shear stresses are then used for the wall boundary condition of velocity fields. It is to be mentioned that the Spalding’s formulation for the law of the wall with values of constants, $A$ and $\kappa$ has an undisputed advantage that it satisfies no-slip condition at the wall. Near to wall region Spalding’s law governs the flow.

7. Results and discussions

7.1. Turbulence Statistics. In this section we discuss some statistics of the computed flow field in 3D turbulent channel flow. The computed results are compared with the DNS data obtained by Moser et al. [2] and LES data using no wall model (LES-NWM) obtained by Uddin and Mallik [5]. For comparison, the DNS data of Moser et al. [2] is represented by a solid line, the LES data of Uddin and Mallik [5] is represented by a line with arrow sign and the computed results using algebraic wall model (LES-AWM) are indicated by a line with circle in the following figures of this section. Furthermore, in this section we provide a sample of
error generation for LES-NWM and LES-AWM approaches at some positions in mean velocity profile. In error calculation, the DNS data are considered as the true value. In this study LES simulations are initialized with a random solenoidal velocity field and integrated ahead in time with finite viscosity.

Numerous experiments have shown that the boundary layer in a plane turbulent channel flow can be divided into two parts: the inner or near wall region and the outer region. At the near wall region, the dynamics is dominated by the viscous effects. In the outer region, it controlled by the turbulence. Each of these regions is split into several layers, corresponding to different types of dynamics. In the case of canonical boundary layer, the near wall region can be largely subdivided into three layers. These three layers are the viscous sub-layer \((y^+ \leq 5)\), buffer layer \((5 < y^+ \leq 30)\) and logarithmic inertial layer \((y^+ > 30; y/\delta \ll 1)\) [21]. The outer region includes the end of the logarithmic inertial region and wake region.

The profile of the mean velocity non-dimensionalized by the wall-shear velocity corresponding to the lower half of the channel is shown in Figure 2, which is defined as

\[
 u_x^+ = \frac{\langle u_x \rangle}{u_\tau}
\]  

(7.1)

![Figure 2. The mean velocity profile in wall units.](image)

It can be observed that the computed profile cannot trace the data for the whole boundary layer. Here the first computational cell above the wall is located within the buffer layer \((5 < y^+ \leq 30)\), at about \(y^+ = 29.5\). That is, the AWM lead to a significant reduction in the number of computational cells at the near wall region. It has to be noted that our computed profile, LES-AWM under predicts the LES-NWM profile until end of the range. From this figure it is also revealed that the LES-AWM results are almost collapsed with the DNS results for \(y^+ \approx 29.5 - 60\). Here after for \(y^+ \approx 60 - 120\), the LES-AWM profile over predicts the DNS...
profile. After that in rest of the range the LES-AWM profile under predicts the DNS profile. Nonetheless, Figure 3 shows that the agreement of the mean velocity profile for LES-AWM with the DNS data is better than that of the LES-NWM profile.

Errors generated in LES-NWM and LES-AWM approaches are presented in Table 1. From this table it can be observed that initially the percentage of relative error in LES-AWM approach is 1%. After that, upto a certain position the error in this approach increases with the increase of wall units. Then, the error decreases and at \( y^+ \approx 147.50 \) it becomes zero. After this position the error increases gradually with the increase of wall units. It is worth noting here that the errors generated in LES-AWM approach are smaller than that in the LES-NWM approach at maximum positions.

Table 1. Percentage of relative errors in mean velocity.

<table>
<thead>
<tr>
<th>( y^+ )</th>
<th>Error in LES-NWM</th>
<th>Error in LES-AWM</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.50</td>
<td>5.51%</td>
<td>1.00%</td>
</tr>
<tr>
<td>88.50</td>
<td>8.78%</td>
<td>2.80%</td>
</tr>
<tr>
<td>147.50</td>
<td>6.59%</td>
<td>0%</td>
</tr>
<tr>
<td>324.50</td>
<td>2.94%</td>
<td>1.04%</td>
</tr>
<tr>
<td>442.50</td>
<td>1.19%</td>
<td>1.71%</td>
</tr>
</tbody>
</table>

Figure 3(a) shows the DNS and LES profiles of streamwise root mean square(r.m.s.) of velocity components normalized by the wall shear velocity defined as

\[
\begin{align*}
  u_{x \text{ r.m.s.}}^+ &= \sqrt{\langle u_x^2 \rangle - \langle u_x \rangle^2} / u_\tau \quad (7.2) \\
  u_{y \text{ r.m.s.}}^+ &= \sqrt{\langle u_y^2 \rangle - \langle u_y \rangle^2} / u_\tau \quad (7.3) \\
  u_{z \text{ r.m.s.}}^+ &= \sqrt{\langle u_z^2 \rangle - \langle u_z \rangle^2} / u_\tau \quad (7.4)
\end{align*}
\]

Figure 3(a) reveals that above the wall the LES-AWM profile of streamwise root mean square velocity starts with 2.02 at \( y^+ \approx 29.5 \). At this position the value of the DNS and LES-NWM profiles are about 2.48 and 3.54 respectively. After this position the value of LES-AWM profile increases with the increase of wall units and attains the maximum value of about 2.50 at \( y^+ \approx 88.5 \). Where, the value of the DNS and LES-NWM profiles are respectively 1.79 and 2.29. Beyond \( y^+ \approx 88.5 \) the trend of LES-AWM profile is always decreasing like the pattern of DNS and LES-NWM profiles until end of the range. Although there exists a noticeable discrepancy from the DNS and LES-NWM profiles at the near wall region, but away from the wall the LES-AWM profile shows closer agreement with the DNS and LES-NWM profiles.

The profile of wall normal root mean square velocity in wall units is shown in Figure 3(b). From this figure it can be observed that initially from \( y^+ \approx 59 - 120 \), the LES-AWM profile under predicts the DNS profile, but over predicts the LES-NWM profile. Here after, from
There is hardly noticeable difference between the DNS and LES-AWM profiles. After that, in rest of the domain the LES-AWM profile under predicts the DNS profile. It has to be noted that, after a certain position the discrepancy between the LES-NWM and LES-AWM profiles decreases with the increase of wall units, and beyond $y^+ \approx 460$ the LES-NWM and LES-AWM profiles are almost collapsed. However, the profile of wall normal root mean square velocity for LES-AWM shows less discrepancy from the DNS profile than that of the LES-NWM profile.

The spanwise root mean square velocity profiles are displayed in Figure 3(c). This figure reveals that above the wall the LES-AWM profile starts with 1.15 near the peak value of LES-NWM profile at $y^+ \approx 29.5$. From this position the LES-AWM profile under predicts the DNS profile until end of the range and over predicts the LES-NWM profile up to $y^+ \approx 440$. Beyond $y^+ \approx 440$, in some of the regions ($y^+ \approx 120 – 300$) the LES-AWM and LES-NWM profiles are almost collapsed, and in rest of the range the LES-AWM profile under predicts the LES-NWM profile. However, in most of the region the LES-AWM profile shows closer agreement with the DNS profile than that of the LES-NWM profile.

The profile of non-dimensional Reynolds stress, $-\frac{u'_x u'_y}{\nu^2}$ corresponding to the channel half width is shown in Figure 4. In a fully developed channel flow this profile is a straight line when the flow reaches an equilibrium state. Our computed results clearly indicate that this is
the case. This figure also reveals that the discrepancy of the LES-AWM profile with the DNS and LES-NWM profiles decreases with the increase of the value of \( y^+ \). Away from the wall (\( y^+ > 400 \)), there is hardly noticeable difference between the three profiles.

![Reynolds stress profile](image)

**Figure 4.** The Reynolds stress profile in wall coordinates.

7.2. **Flow Structures.** We have calculated streamwise velocity (\( \bar{u}_x \)) distribution at the centerline of the channel and streamwise shear velocity (\( \bar{\omega}_{x+} \)) distribution at the immediate vicinity of the wall at a non-dimensional time, \( t = 202.20 \) when the flow reaches to an equilibrium state. Using these computed data different contour plots of the flow field have been drawn and compared with the contours obtained by Uddin and Mallik [5].

Contour of instantaneous streamwise velocity distribution at the centerline of the channel in \( x - z \) plane is shown in Figure 5(a), (b).

![Streamwise velocity contours](image)

**Figure 5.** Contours of streamwise velocity profiles in \( x - z \) plane for (a) LES using no wall model [5], and (b) LES using AWM.

Figure 5(a) is obtained by Uddin and Mallik [5], where no wall model has been used in LES, and in Figure 5(b), AWM has been used in LES. In these contour plots the value of \( \bar{u}_x \) ranged
in between 19 and 22.6. The highest value of $\bar{u}_x$ is indicated by a red color, and the lowest value by a blue color. It has to be noted that in Figure 5(a) the higher values of $\bar{u}_x$ appear more densely adjacent to the centerline of the channel from both sides. But, in Figure 5(b) the higher values appear more densely in scattered locations of the $x-z$ plane. The distinctive features of the flow patterns in these contour plots are that the existence of high-speed fluid regions are more located in Figure 5(a) than that in Figure 5(b).

Contour of instantaneous streamwise shear velocity distribution at the immediate vicinity of the wall of this channel in $x-z$ plane is shown in Figure 6(a), (b).

![Figure 6. Contours of streamwise shear velocity profiles in x – z plane for (a) LES using no wall model [5], and (b) LES using AWM.](image)

In Figure 6(a) no wall model has been used in LES which is obtained by Uddin and Mallik [5], and in Figure 6(b), AWM has been used in LES. The value of $\bar{u}_x$ ranged from 0.7 to 1.35 in these contour plots. The highest value appears at red regions and lowest value at blue regions. From these contour plots it can be observed that the regions of larger values of $\bar{u}_x$ appear more densely in between the boundary and centerline of the channel. It is also noticeable that the intensity of $\bar{u}_x$ in Figure 6(a) is higher than that located in Figure 6(b).

Figure 7(a), (b) represents the visualization of vortical structures in the turbulent channel flow by iso-surfaces of the second invariant $Q$ of velocity gradient tensor, which is defined as [5, 6]:

$$Q = -\frac{1}{2} \left( S_{ij} S_{ij} - \Omega_{ij} \Omega_{ij} \right)$$  \hspace{1cm} (7.5)

where,

$$S_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \quad \text{and} \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right)$$  \hspace{1cm} (7.6)

are respectively, the strain-rate and rotation tensors, that is, the symmetric and asymmetric part of the velocity gradient tensor:

$$A_{ij} = \frac{\partial \bar{u}_i}{\partial x_j} = S_{ij} + \Omega_{ij}$$  \hspace{1cm} (7.7)

In Figure 7(a), no wall model has been used in LES which is obtained by Uddin and Mallik [5], but in Figure 7(b), AWM has been applied at the near wall region. The visualized region is the whole calculation domain. The level of the iso-surface is selected to be $Q = 5$. For this value of $Q$ the vortical structures are significant and are randomly distributed over the
turbulent flow field. Generally, it can be noted that the vortices are generated in between near the boundary and the centerline of the channel are more intense than the ones are generated around the centerline of the channel. It is also noticeable that the vortices are generated more densely in Figure 7(a) than that of Figure 7(b).

![Figure 7](image.png)

**Figure 7.** Iso-surfaces of the second invariant \( Q = 5 \) in the channel flow for (a) LES using no wall model [5], and (b) LES using AWM.

### 8. Conclusion

A Large eddy simulation of a turbulent channel flow has been successfully carried out using AWM at a Reynolds number, \( Re_x = 590 \) with \( 32 \times 20 \times 32 \) grid points. To reduce the calculation cost, the AWM lead to a significant reduction in the number of computational cells at the near wall region. In spite of resolution limitations, the simulations are able to resolve the essential features of the statistical fields. The statistical results are compared with the DNS and LES data of reference. Maximum discrepancies are found at the near wall region. In comparison with the DNS data the computed results show better agreement than that of LES results using no wall model. Instantaneous streamwise velocity distribution at the centerline of the channel and streamwise shear velocity distribution at the immediate vicinity of the wall have also been measured in the contour plots, and compared these contour plots with those of LES data using no wall model. In our computation the higher values of streamwise velocity appear more densely in scattered locations of the centerline \( x - z \) plane, and the higher values of streamwise shear velocity appear more densely in between the boundary and centerline of the channel. But the existence of high-speed fluid regions are less located in the computed flow field. Visualization of the iso-surfaces of the second invariant \( Q = 5 \) in the turbulent channel flow show that the flow field contains lots of tube-like vortical structures which are randomly distributed over the turbulent flow field. The intensity of the vortical structures is
high in between near the boundary and the centerline of the channel. But the vortices are generated less densely in the computed flow field.

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**REFERENCES**


EXISTENCE AND CONTROLLABILITY OF FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY IN BANACH SPACES

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ABSTRACT. In view of ideas for semigroups, fractional calculus, resolvent operator and Banach contraction principle, this manuscript is generally included with existence and controllability (EaC) results for fractional neutral integro-differential systems (FNIDS) with state-dependent delay (SDD) in Banach spaces. Finally, an examples are also provided to illustrate the theoretical results.

1. INTRODUCTION

The notion of fractional differential equation (FDE) is increasing as a basic scope of research because of the reality it is better in problems in connection with relating hypothesis of conventional differential equations. Fractional order models are often observed to be more adequate than integer order models in some real world problems as fractional derivatives supply an extraordinary application for the depiction of memory and hereditary properties of various materials and procedures. These days, it has been shown that the differential models including derivatives of fractional order occur in lots of engineering and scientific professions as the mathematical modeling of systems and strategies in numerous domains, for case in point, basic sciences, navigation, feed back amplifiers, and neuron modelling so on. For crucial confirmations about fractional frameworks, one can make reference to the treatises [1, 2], and the papers [3–9], and the references cited therein. FDE with delay characteristics happen in a number of domains such as medical and physical with SDD or non-constant delay. Presently, existence
and controllability results of mild solutions for such problems turned quite interesting and numerous researchers working on it. Just lately, number of papers have been released on the fractional order problems with SDD, see for instance [10–19].

On the other side, the thought of controllability has accepted a central part all through the historical backdrop of cutting edge control theory. This is the qualitative property of control frameworks and is of specific significance in control theory. A lot of dynamical systems are such that the control does not affect the complete state of the dynamical system yet just a part of it. Then again, frequently in real industrial procedures it is conceivable to notice just a specific piece of the complete state of the dynamical framework. Along these lines, it is essential to figure out if or not control of the complete state of the dynamical framework is conceivable. In this way, here the thought of complete controllability and approximate controllability exists. Generally discussing, controllability usually indicates that it is conceivable to steer dynamical framework from a arbitrary beginning state to the coveted last state utilizing the set of acceptable controls.

The existence, controllability and other qualitative and quantitative attributes of FDE are the most advancing area of pursuit, in particular, see [20–29]. These days, Santos et al. [16, 23, 24] reviewed the existence of solutions for FIDE with unbounded or SDD delay in Banach spaces. Shu et al. [25] examined the existence outcomes for FDE with nonlocal conditions of order $\alpha \in (1, 2)$. In [26, 27], the writers present sufficient circumstances for the existence and approximate controllability of fractional order neutral differential and stochastic differential system with infinite delay. Kexue et al. [28] analyzed the controllability of nonlocal FDE of order $\alpha \in (1, 2]$. Sakthivel et al. [29] acknowledged the approximate controllability of fractional dynamical system by making use of appropriate fixed point theorem. Lately, in [17–19], the authors outlined the approximate controllability results for FNIDS with SDD by applying the acceptable fixed point theorem. However, EaC results for FNIDS with SDD in $\mathcal{B}_h$ phase space adages have not yet been completely examined.

Inspired by the effort of the previously stated papers [14–16], the principle inspiration driving this manuscript is to research the EaC of mild solutions for FNIDS with SDD of the models

$$D_t^\alpha \left[ x(t) + \mathcal{G} \left( t, x_{\varphi(t,x_t)}, \int_0^t e_1(t, s, x_{\varphi(s,x_s)})ds \right) \right] = \mathcal{A} x(t) + \int_0^t \mathcal{B}(t - s) x(s)ds$$

$$+ \mathcal{F} \left( t, x_{\varphi(t,x_t)}, \int_0^t e_2(t, s, x_{\varphi(s,x_s)})ds \right)$$

$$+ \mathcal{H} \left( t, x_{\varphi(t,x_t)}, \int_0^t e_3(t, s, x_{\varphi(s,x_s)})ds \right), \quad t \in \mathcal{I} = [0, T], \quad (1.1)$$

$$x_0 = \varsigma(t) \in \mathcal{B}_h, \quad x'(0) = 0, \quad t \in (-\infty, 0], \quad (1.2)$$

and the corresponding controllability structure

$$D_t^\alpha \left[ x(t) + \mathcal{G} \left( t, x_{\varphi(t,x_t)}, \int_0^t e_1(t, s, x_{\varphi(s,x_s)})ds \right) \right] = \mathcal{A} x(t) + \int_0^t \mathcal{B}(t - s) x(s)ds$$
deliberate phase spaces

we should talk about the theoretical phase space

the middle at

space characterized in Preliminaries.

\[ \| \cdot \|_{L^1} \]

from

\[ X \]

principal outcomes. Let

\[ SDD, \]

which is communicated in the structures (1.1)–(1.2) and (1.3)–(1.4). To pack this

conceptual idea.

Section 5, as a last point, an appropriate cases are equipped to replicate the efficiency of the

we declare and present the EaC results about by proposes of Banach fixed point theorem. In

tives that will be utilized in this work to accomplish our primary results. In Section 3 and 4,

where the unknown \( x(\cdot) \) needs values in the Banach space \( \mathbb{X} \) having norm \( \| \cdot \| \), \( I = [0, T] \) is an

operational interval, \( D^\alpha_t \) is the Caputo fractional derivative of order \( \alpha \in (1, 2) \), \( I, (B(t))_{t \geq 0} \)

are closed linear operators described on a regular domain which is dense in \( (\mathbb{X}, \| \cdot \|) \) and

\[ D^\alpha_t \sigma(t) := \int_0^t \tilde{\mu}_{\alpha-\alpha}(t-s) \frac{d^n}{ds^n} \sigma(s) ds, \]

where \( n \geq \alpha \) and \( \tilde{\mu}_\beta(t) := \frac{\beta^{-1}}{\Gamma(\beta)} t^\beta \), \( t > 0, \beta \geq 0 \). Further, \( \mathcal{I}, \mathcal{F}, \mathcal{H} : \mathcal{I} \times \mathcal{B}_h \times \mathbb{X} \rightarrow \mathbb{X}, e_i : \mathcal{P} \times \mathcal{B}_h \rightarrow \mathbb{X}, i = 1, 2, 3 ; \mathcal{P} = \{(t, s) \in \mathcal{I} \times \mathcal{I} : 0 \leq s \leq t \leq T \}, g : \mathcal{I} \times \mathcal{B}_h \rightarrow (-\infty, T] \)

are opposite functions, \( C \) is a bounded linear operator from a Banach space \( U \) into \( \mathbb{X} \); the control

function \( u(\cdot) \in L^2(\mathcal{I}, U) \), a Banach space of admissible control functions, and \( \mathcal{B}_h \) is a phase

space characterized in Preliminaries.

For almost any continuous function \( x \) characterized on \( (-\infty, T] \) and any \( t \geq 0 \), we designate

by \( x_t \) the part of \( \mathcal{B}_h \) characterized by \( x_t(\theta) = x(t + \theta) \) for \( \theta \leq 0 \). Now \( x_i(\cdot) \) speaks to the

historical backdrop of the state from every \( \theta \in (-\infty, 0] \) likely the current time \( t \).

We push ahead as takes after. Section 2 is focused on call to mind of some crucial perspectives

that will be utilized in this work to accomplish our primary results. In Section 3 and 4,

we declare and present the EaC results about by proposes of Banach fixed point theorem. In

Section 5, as a last point, an appropriate cases are equipped to replicate the efficiency of the

conceptual idea.

To the best of our insight, there is no work gave an account of the EaC results for FNIDS

with SDD, which is communicated in the structures (1.1)–(1.2) and (1.3)–(1.4). To pack this

gap, in this manuscript, we contemplate this fascinating model.

2. Preliminaries

In this section, we present some primary components which are required to confirm the

principal outcomes. Let \( \mathcal{L}(\mathbb{X}) \) symbolizes the Banach space of all bounded linear operators

from \( \mathbb{X} \) into \( \mathbb{X} \) endowed with the uniform operator topology, having its norm recognized as

\[ \| \|_{\mathcal{L}(\mathbb{X})} \]. Let \( C(\mathcal{I}, \mathbb{X}) \) symbolize the space of all continuous functions from \( \mathcal{I} \) into \( \mathbb{X} \),

having its norm recognized as \( \| \|_{C(\mathcal{I}, \mathbb{X})} \). Moreover, \( B_r(x, \mathbb{X}) \) symbolizes the closed ball in \( \mathbb{X} \) with

the middle at \( x \) and the distance \( r \). It needs to be outlined that, once the delay is infinite, then

we should talk about the theoretical phase space \( \mathcal{B}_h \) in a beneficial way. In this manuscript, we

deliberate phase spaces \( \mathcal{B}_h \) which are same as described in [30]. So, we bypass the details.
We expect that the phase space \((\mathcal{B}_h, \| \cdot \|_{\mathcal{B}_h})\) is a semi-normed linear space of functions mapping \((-\infty, 0]\) into \(\mathbb{X}\), and fulfilling the subsequent elementary adages as a result of Hale and Kato (see case in point in [31, 32]).

If \(x : (-\infty, T] \to \mathbb{X}, T > 0\), is continuous on \(\mathcal{I}\) and \(x_0 \in \mathcal{B}_h\), then for every \(t \in \mathcal{I}\) the accompanying conditions hold:

1. \((P_1)\) \(x_t\) is in \(\mathcal{B}_h\);
2. \((P_2)\) \(\|x(t)\|_{\mathcal{X}} \leq H \|x_t\|_{\mathcal{B}_h}\);
3. \((P_3)\) \(\|x_t\|_{\mathcal{B}_h} \leq \mathcal{P}_1(t) \sup \{\|x(s)\|_{\mathcal{X}} : 0 \leq s \leq t\} + \mathcal{P}_2(t) \|x_0\|_{\mathcal{B}_h}\), where \(H > 0\) is a constant and \(\mathcal{P}_1(\cdot) : [0, +\infty) \to [0, +\infty)\) is continuous, \(\mathcal{P}_2(\cdot) : [0, +\infty) \to [0, +\infty)\) is locally bounded, and \(\mathcal{P}_1, \mathcal{P}_2\) are independent of \(\alpha(\cdot)\).
4. \((P_4)\) The function \(t \to \varsigma_t\) is well described and continuous from the set
   \[\mathcal{R}(q^-) = \{q(s, \varsigma) : (s, \varsigma) \in [0, T] \times \mathcal{B}_h\},\]
   into \(\mathcal{B}_h\) and there is a continuous and bounded functional \(J^t : \mathcal{R}(q^-) \to (0, \infty)\) to ensure that \(\|\varsigma_t\|_{\mathcal{B}_h} \leq J^t(\|\varsigma\|_{\mathcal{B}_h})\) for every \(t \in \mathcal{R}(q^-)\).

Recognize the space
\[\mathcal{B}_T = \{x : (-\infty, T] \to \mathbb{X} : x|_{\mathcal{I}}\text{ is continuous and } x_0 \in \mathcal{B}_h\},\]
where \(x|_{\mathcal{I}}\) is the constraint of \(x\) to the real compact interval on \(\mathcal{I}\). The function \(\| \cdot \|_{\mathcal{B}_T}\) to be a seminorm in \(\mathcal{B}_T\), it is described by
\[\|x\|_{\mathcal{B}_T} = \|s\|_{\mathcal{B}_h} + \sup\{\|x(s)\|_{\mathcal{X}} : s \in [0, T]\}, x \in \mathcal{B}_T.\]

**Lemma 2.1.** [33, Lemma 2.1] Let \(x : (-\infty, T] \to \mathbb{X}\) be a function in a way that \(x_0 = \varsigma\), and if \((P4)\) hold, then
\[\|x_s\|_{\mathcal{B}_h} \leq (\mathcal{P}_2^s + J^s)\|s\|_{\mathcal{B}_h} + \mathcal{P}_1^t \sup\{\|x(\theta)\|_{\mathcal{X}} : \theta \in [0, \max\{0, s\}]\}, s \in \mathcal{R}(q^-) \cup \mathcal{I},\]
where \(J^s = \sup_{t \in \mathcal{R}(q^-)} J^t(s), \mathcal{P}_1^s = \sup_{s \in [0, T]} \mathcal{P}_1(s), \mathcal{P}_2^s = \sup_{s \in [0, T]} \mathcal{P}_2(s)\).

To be able to acquire our outcomes, we believe that the subsequent FIDS
\[D_\alpha^t x(t) = \mathcal{A}x(t) + \int_0^t \mathcal{B}(t - s)x(s)ds,\]
\[x(0) = \varsigma \in \mathbb{X}, \quad x'(0) = 0,\]
has an associated \(\alpha\)-resolvent operator of bounded linear operators \((\mathcal{R}_\alpha(t))_{t \geq 0}\) on \(\mathbb{X}\).

**Definition 2.1.** [23, Definition 2.1] A one parameter family of bounded linear operators \((\mathcal{R}_\alpha(t))_{t \geq 0}\) on \(\mathbb{X}\) is called a \(\alpha\)-resolvent operator of \((2.1)-(2.2)\) if the subsequent conditions are fulfilled.

(a) The function \(\mathcal{R}_\alpha(\cdot) : [0, \infty) \to \mathcal{L}(\mathbb{X})\) is strongly continuous and \(\mathcal{R}_\alpha(0)x = x\) for all \(x \in \mathbb{X}\) and \(\alpha \in (1, 2)\).

(b) For \(x \in D(\mathcal{A}), \mathcal{R}_\alpha(\cdot)x \in C([0, \infty), [D(\mathcal{A})] \cap C^1((0, \infty), \mathbb{X})\), and
   \[D_\alpha^t \mathcal{R}_\alpha(t)x = \mathcal{A}\mathcal{R}_\alpha(t)x + \int_0^t \mathcal{B}(t - s)\mathcal{R}_\alpha(s)xds,\]

(2.3)
Assume that conditions (P1)–(P3) are fulfilled. Then there exists a unique \( \rho_\alpha(G_\alpha) \) the sets
\[
\rho_\alpha(G_\alpha) = \{ \lambda \in \mathbb{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - \mathcal{A} - \mathcal{B}(\lambda))^{-1} \in \mathcal{L}(X) \}.
\]

Presently, we determine the operator family \( (R_\alpha(t))_{t \geq 0} \) by
\[
R_\alpha(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t}G_\alpha(\lambda)d\lambda, & t > 0, \\ I, & t = 0. \end{cases}
\]

Now, we are in a position to present some conventional outcomes from current works.

**Theorem 2.1** ([20, Theorem 2.1]). Assume that conditions (P1)–(P3) are fulfilled. Then there exists a unique \( \alpha \)-resolvent operator for problem (2.1)-(2.2).

**Theorem 2.2** ([20, Lemma 2.5]). The function \( R_\alpha : [0, \infty) \to \mathcal{L}(X) \) is strongly continuous and \( R_\alpha : (0, \infty) \to \mathcal{L}(X) \) is uniformly continuous.

Hereafter, we expect that the conditions (P1) – (P3) are fulfilled. Further, we need to talk about the mild solution for the model (1.1)-(1.2). For this intent, it is necessary to discuss the subsequent non-homogeneous model
\[
D^\alpha_t x(t) = \mathcal{A}x(t) + \int_0^t \mathcal{B}(t-s)x(s)ds + \mathcal{F}(t), \quad t \in \mathcal{I},
\]
\[
x(0) = x_0, \quad x'(0) = 0,
\]
where \( \alpha \in (1, 2) \) and \( \mathcal{F} \in L^1(\mathcal{I}, X) \). In the follow up, \( R_\alpha(\cdot) \) is the operator function characterized by (2.5). Now, we start by presenting the subsequent concept of classical solution.

**Definition 2.2** ([23, Definition 2.5]). A function \( x : \mathcal{I} \to X, 0 < T, \) is called a classical solution of (2.6)-(2.7) on \( \mathcal{I} \) if \( x \in C(\mathcal{I}, [D(\mathcal{A})]) \cap C(\mathcal{I}, X), \bar{\mu}_{n-\alpha}x \in C^1(\mathcal{I}, X), n = 1, 2, \) the condition (2.7) holds and the equation (2.6) is verified on \( \mathcal{I} \).
Definition 2.3 ([23, Definition 2.6]). Let $\alpha \in (1, 2)$, we describe the family $(S_\alpha(t))_{t \geq 0}$ by
\[
S_\alpha(t)x := \int_0^t \tilde{\mu}_{\alpha-1}(t-s)R_\alpha(s)x ds,
\]
for each $t \geq 0$.

Now, just as before, we need to present some additional conventional outcomes from [20].

Lemma 2.2 ([20, Lemma 3.9]). If the function $R_\alpha(\cdot)$ is exponentially bounded in $L(X)$, then $S_\alpha(\cdot)$ is exponentially bounded in $L(X)$.

Lemma 2.3 ([20, Lemma 3.10]). If the function $R_\alpha(\cdot)$ is exponentially bounded in $L([D(\mathcal{A})])$, then $S_\alpha(\cdot)$ is exponentially bounded in $L([D(\mathcal{A})])$.

Theorem 2.3 ([20, Theorem 3.2]). Let $z \in D(\mathcal{A})$. Assume that $F \in C(I, X)$ and $x(\cdot)$ is a classical solution of (2.6)-(2.7) on $I$. Then
\[
x(t) = R_\alpha(t)z + \int_0^t S_\alpha(t-s)F(s) ds, \quad t \in I.
\]

It is obvious from the earlier definition that $R_\alpha(\cdot)z$ is a solution of problem (2.1)-(2.2) on $(0, \infty)$ for $z \in D(\mathcal{A})$.

Definition 2.4 ([23, Definition 2.10]). Let $F \in L^1(I, X)$. A function $x \in C(I, X)$ is called a mild solution of (2.6)-(2.7) if
\[
x(t) = R_\alpha(t)z + \int_0^t S_\alpha(t-s)F(s) ds, \quad t \in I.
\]

Theorem 2.4 ([20, Theorem 3.3]). Let $z \in D(\mathcal{A})$ and $F \in C(I, X)$. If $F \in L^1(I, [D(\mathcal{A})])$ then the mild solution of (2.6)-(2.7) is a classical solution.

Theorem 2.5 ([20, Theorem 3.4]). Let $z \in D(\mathcal{A})$ and $F \in C(I, X)$. If $F \in W^{1,1}(I, X)$, then the mild solution of (2.6)-(2.7) is a classical solution.

In the subsequent result, we signify by $(-\mathcal{A})^\vartheta$ the fractional power of the operator $-\mathcal{A}$, (see [34] for details).

Lemma 2.4 ([23, Lemma 3.1]). Suppose that the conditions $(P1) - (P3)$ are satisfied. Let $\alpha \in (1, 2)$ and $\vartheta \in (0, 1)$ such that $\alpha \vartheta \in (0, 1)$, then there exists positive number $C$ such that
\[
\|(-\mathcal{A})^\vartheta R_\alpha(t)\| \leq Ce^{rt}t^{-\alpha\vartheta}, \quad \|(-\mathcal{A})^\vartheta S_\alpha(t)\| \leq Ce^{rt}t^{(1-\vartheta)-1},
\] 
for all $t > 0$.

Remark 2.1. The verifications of the above results are excessively standard, subsequently we overlook here. For additional data about this idea, we propose the peruser to allude [20, 23].
Let $x_T(\xi; u)$ be the state value of the model (1.3)-(1.4) at terminal time $T$ corresponding to the control $u$ and the initial value $\xi \in \mathcal{B}_h$. Present the set $\mathcal{R}(T, \xi) = \{ x_T(\xi; u)(0) : u(\cdot) \in L^2(\mathcal{I}, U) \}$, which is known as the reachable set of model (1.3)-(1.4) at terminal time $T$.

**Definition 2.6.** The model (1.3)-(1.4) is said to be exactly controllable on $\mathcal{I}$ if $\mathcal{R}(T, \xi) = \mathbb{X}$.

Assume that the fractional differential control model

$$D^a x(t) = A x(t) + (C u)(t), \quad t \in \mathcal{I},$$  \hspace{1cm} (2.11)

$$x_0 = \xi \in \mathcal{B}_h,$$  \hspace{1cm} (2.12)

is exactly controllable. It is practical at this position to present the controllability operator linked with (2.11)-(2.12) as

$$\Gamma^T_0 = \int_0^T S_\alpha(T - s)CC^*S^*_\alpha(T - s)ds,$$

where $C^*$ and $S^*_\alpha(t)$ denotes the adjoints of $C$ and $S_\alpha(t)$, accordingly. It is simple that the operator $\Gamma^T_0$ is a linear bounded operator [35, Theorem 3.2].

**Lemma 2.5.** If the linear fractional model (2.11)-(2.12) is exactly controllable if and only then for some $\gamma > 0$ such that $\langle \Gamma^T_0 x, x \rangle \geq \gamma \| x \|^2$, for all $x \in \mathbb{X}$ and as a result $\| (\Gamma^T_0)^{-1} \| \leq \frac{1}{\gamma}$.

**Remark 2.2.** Further, we assume that the linear fractional control system (2.11)-(2.12) is exactly controllable.

### 3. Existence Results

In this section, we exhibit and demonstrate the existence of solutions for the structure (1.1)-(1.2) under Banach fixed point theorem. In the first place, we present the mild solution for the model (1.1)-(1.2).

**Definition 3.1.** A function $x : (-\infty, T] \rightarrow \mathbb{X}$, is called a mild solution of (1.1)-(1.2) on $[0, T]$, if $x_0 = \xi; x||_{[0,T]} \in C([0, T] : \mathbb{X})$; the function $s \rightarrow A S_\alpha(t-s)\mathcal{G} \left( s, x_{\varrho(s,x_s)}, \int_0^s e_1(s, \tau, x_{\varrho(\tau,x_\tau)})d\tau \right)$ and $s \rightarrow \int_0^s \mathcal{G}(s - \tau)S_\alpha(t-s)\mathcal{G} \left( \tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau, \xi, x_{\varrho(\xi,x_\xi)})d\xi \right)d\tau$ is integrable on $[0, t)$ for all $t \in (0, T]$ and for $t \in [0, T]$,

$$x(t) = \mathcal{R}_\alpha(t) \left[ \xi(0) + \mathcal{G}(0, \xi(0), 0) \right] - \mathcal{G} \left( t, x_{\varrho(t,x_t)}, \int_0^t e_1(t, s, x_{\varrho(s,x_s)})ds \right) - \int_0^t \mathcal{G} S_\alpha(t-s)\mathcal{G} \left( s, x_{\varrho(s,x_s)}, \int_0^s e_1(s, \tau, x_{\varrho(\tau,x_\tau)})d\tau \right)ds$$

$$- \int_0^t \int_0^s \mathcal{G}(s - \tau)S_\alpha(t-s)\mathcal{G} \left( \tau, x_{\varrho(\tau,x_\tau)}, \int_0^\tau e_1(\tau, \xi, x_{\varrho(\xi,x_\xi)})d\xi \right)d\tau ds \hspace{1cm} (3.1)$$
The operator families $R_{\alpha}(t)$ and $S_{\alpha}(t)$ are compact for all $t > 0$, and there exists a constant $M$ in a way that $\|R_{\alpha}(t)\|_{\mathcal{L}(\mathcal{X})} \leq M$ and $\|S_{\alpha}(t)\|_{\mathcal{L}(\mathcal{X})} \leq M$ for every $t \in \mathcal{I}$ and

$$\|(-\mathcal{A})^\alpha S_{\alpha}(t)\|_{\mathcal{X}} \leq Mt^{\alpha(1-\theta)-1}, \quad 0 < t \leq T.$$  

\section*{(H2) The subsequent conditions are fulfilled.}

(a) $\mathcal{B}(\cdot)x \in C(\mathcal{I}, \mathcal{X})$ for every $x \in [D((-\mathcal{A})^{1-\theta})].$

(b) There is a function $\mu(\cdot) \in L^1(\mathcal{I}; \mathbb{R}^+)$, to ensure that

$$\|\mathcal{B}(s)S_{\alpha}(t)\|_{\mathcal{L}(D((-\mathcal{A})^\theta))} \leq M\mu(s) t^{\alpha\theta-1}, \quad 0 \leq s < t \leq T.$$  

\section*{(H3) The function $\mathcal{F} : \mathcal{I} \times \mathcal{B}_h \times \mathbb{X} \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathcal{F}}, L_{\mathcal{F}}^* > 0$ in ways that

$$\|\mathcal{F}(t, \psi_1, x) - \mathcal{F}(t, \psi_2, y)\|_{\mathcal{X}} \leq L_{\mathcal{F}} \|\psi_1 - \psi_2\|_{\mathcal{B}_h} + L_{\mathcal{F}} \|x - y\|_{\mathcal{X}}, \quad t \in \mathcal{I}, \quad x, y \in \mathbb{X},$$

and

$$L_{\mathcal{F}}^* = \max_{t \in \mathcal{I}} \|\mathcal{F}(t, 0, 0)\|_{\mathcal{X}}.$$  

(ii) The function $\mathcal{H} : \mathcal{I} \times \mathcal{B}_h \times \mathbb{X} \to \mathbb{X}$ is continuous and we can find positive constants $L_{\mathcal{H}}, L_{\mathcal{H}}^* > 0$ in ways that

$$\|\mathcal{H}(t, \psi_1, x) - \mathcal{H}(t, \psi_2, y)\|_{\mathcal{X}} \leq L_{\mathcal{H}} \|\psi_1 - \psi_2\|_{\mathcal{B}_h} + L_{\mathcal{H}} \|x - y\|_{\mathcal{X}}, \quad t \in \mathcal{I}, \quad x, y \in \mathbb{X},$$

and

$$L_{\mathcal{H}}^* = \max_{t \in \mathcal{I}} \|\mathcal{H}(t, 0, 0)\|_{\mathcal{X}}.$$  

\section*{(H4) The function $e_i : \mathcal{D} \times \mathcal{B}_h \to \mathbb{X}$ is continuous and we can find constants $L_{e_i}, L_{e_i}^* > 0$ to ensure that

$$\|e_i(t, s, \varsigma) - e_i(t, s, \psi)\|_{\mathcal{X}} \leq L_{e_i} \|\varsigma - \psi\|_{\mathcal{B}_h}, \quad (t, s) \in \mathcal{D}, \quad (\varsigma, \psi) \in \mathcal{B}_h, \quad i = 1, 2, 3,$$

and

$$L_{e_i}^* = \max_{t \in \mathcal{I}} \|e_i(t, s, 0)\|_{\mathcal{X}}, \quad i = 1, 2, 3.$$  

\section*{(H5) The function $g(\cdot) = (-\mathcal{A})^2$-valued, $g : \mathcal{I} \times \mathcal{B}_h \times \mathbb{X} \to [D((-\mathcal{A})^{-\theta})]$ is continuous and there exist positive constants $L_{g}, L_{g}^* > 0$ such that for all $t, \varsigma_j \in \mathcal{I} \times \mathcal{B}_h, j = 1, 2$;

$$\|(-\mathcal{A})^\alpha g(t, \varsigma_1, x) - (-\mathcal{A})^\alpha g(t, \varsigma_2, y)\|_{\mathcal{X}} \leq L_{g} \|\varsigma_1 - \varsigma_2\|_{\mathcal{B}_h} + L_{g} \|x - y\|_{\mathcal{X}}, \quad x, y \in \mathbb{X},$$

and

$$\|(-\mathcal{A})^\alpha g(t, \varsigma, 0)\|_{\mathcal{X}} \leq L_{g} \|\varsigma\|_{\mathcal{B}_h} + L_{g}^*.$$}
where
\[ L^*_g = \max_{t \in \mathcal{F}} \| (-\mathcal{A})^{q/2} \mathcal{G}(t, 0, 0) \|_X. \]

(H6) The following inequalities holds:

(i) Let
\[
\mathcal{M}, \mathcal{M}_0 \left[ L_g \| \varsigma \|_{\mathcal{H}_h} + L^*_g \right] + \left\{ \mathcal{M}_0 + \frac{\mathcal{M} T_{\alpha} \mathcal{G}}{\alpha \theta} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L^*_g + \tilde{L}_g T L e_1 \right)
+ \mathcal{M} \left\{ L^*_g + L^*_{\mathcal{H}} + T (\tilde{L}_g L e_2 + \tilde{L}_g L e_3) \right\}
+ \left( \mathcal{D}^*_r + c_n \right) \left[ \mathcal{M} T \left( (L_g + L_{\mathcal{H}}) + T (\tilde{L}_g L e_2 + \tilde{L}_g L e_3) \right)
+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M} T_{\alpha} \mathcal{G}}{\alpha \theta} \right\} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L_g + \tilde{L}_g T L e_1 \right) \leq r,
\]
for some \( r > 0 \).

(ii) Let

\[ \Lambda = \mathcal{D}^*_r \left[ \mathcal{M} T \left( (L_g + L_{\mathcal{H}}) + T (\tilde{L}_g L e_2 + \tilde{L}_g L e_3) \right)
+ \left\{ \mathcal{M}_0 + \frac{\mathcal{M} T_{\alpha} \mathcal{G}}{\alpha \theta} \right\} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L_g + \tilde{L}_g T L e_1 \right) < 1 \]
be such that \( 0 \leq \Lambda < 1 \).

**Theorem 3.1.** Assume that the conditions (H1)-(H6) hold. Then the structure (1.1)-(1.2) has a unique mild solution on \( \mathcal{F} \).

**Proof.** We will transmute the structure (1.1)-(1.2) into a fixed-point problem. Recognize the operator \( \mathcal{T} : \mathcal{B}_T \to \mathcal{B}_T \) specified by

\[
(\mathcal{T} x)(t) = \begin{cases}
\mathcal{R}_a(t) \left[ \varsigma(0) + \mathcal{G}(0, \varsigma(0), 0) \right] - \mathcal{G} \left( t, x_{\mathcal{G}(t,x_2)}, \int_0^t e_1(t, s, x_{\mathcal{G}(s,x_2)}) ds \right) \\
- \int_0^t \mathcal{B}_{\mathcal{S}_a(t-s)} \mathcal{G} \left( s, x_{\mathcal{G}(s,x_2)}, \int_0^s e_1(t, \tau, x_{\mathcal{G}(\tau,x_2)}) d\tau \right) ds \\
- \int_0^t \int_0^s \mathcal{B}(s-\tau) \mathcal{S}_a(t-s) \mathcal{G} \left( \tau, x_{\mathcal{G}(\tau,x_2)}, \int_0^\tau e_2(t, \xi, x_{\mathcal{G}(\xi,x_2)}) d\xi \right) d\tau ds \\
+ \int_0^t \mathcal{S}_a(t-s) \mathcal{H} \left( s, x_{\mathcal{G}(s,x_2)}, \int_0^s e_2(t, \tau, x_{\mathcal{G}(\tau,x_2)}) d\tau \right) ds \\
+ \int_0^t \mathcal{S}_a(t-s) \mathcal{H} \left( s, x_{\mathcal{G}(s,x_2)}, \int_0^s e_3(t, \tau, x_{\mathcal{G}(\tau,x_2)}) d\tau \right) ds, & t \in \mathcal{F}.
\end{cases}
\]
It is evident that the fixed points of the operator $\overline{T}$ are mild solutions of the model (1.1)-(1.2).

We express the function $y(\cdot) : (-\infty, T] \rightarrow \mathbb{X}$ by

$$
y(t) = \begin{cases} 
\zeta(t), & t \leq 0; \\
R_\alpha(t)z(0), & t \in \mathcal{I},
\end{cases}
$$

then $y_0 = \zeta$. For every function $z \in C(\mathcal{I}, \mathbb{X})$ with $z(0) = 0$, we allocate as $\tilde{z}$ is characterized by

$$
\tilde{z}(t) = \begin{cases} 
0, & t \leq 0; \\
z(t), & t \in \mathcal{I}.
\end{cases}
$$

If $x(\cdot)$ fulfills (3.1), we are able to split it as $x(t) = y(t) + z(t), t \in \mathcal{I}$, which suggests $x_t = y_t + z_t$, for each $t \in \mathcal{I}$ and also the function $y(\cdot)$ fulfills

$$
z(t) = R_\alpha(t)\eta(0, \zeta, 0)
$$

\begin{equation}
+ \int_0^t \mathcal{B}(s - \tau)S_\alpha(t - s)\left(\int_0^s \mathcal{F}(s, \zeta, \zeta + y_s) + \mathcal{F}(\zeta, \zeta + y_s)ds\right) d\tau
\end{equation}

Let $\mathcal{R}_\alpha^0 = \{ z \in \mathcal{R}_\alpha^0 : z_0 = 0 \in \mathcal{R}_h \}$. Let $\| \cdot \|_{\mathcal{R}_\alpha^0}$ be the seminorm in $\mathcal{R}_\alpha^0$ described by

$$
\| z \|_{\mathcal{R}_\alpha^0} = \sup_{t \in \mathcal{I}} \| z(t) \|_{\mathbb{X}} + \| z_0 \|_{\mathcal{R}_h} = \sup_{t \in \mathcal{I}} \| z(t) \|_{\mathbb{X}}, \quad z \in \mathcal{R}_\alpha^0.
$$
as a result \( \mathcal{B}^1 \), \( \| \cdot \|_{\mathcal{B}^1} \) is a Banach space. We delimit the operator \( \overline{\mathbb{T}} : \mathcal{B}^1 \to \mathcal{B}^1 \) by

\[
(\overline{\mathbb{T}} z)(t) = R_\alpha(t)\mathcal{G}(0, \varsigma, 0) - \int_0^t \mathcal{A}_\alpha(t-s) (x) \mathcal{G} \left( s, z_\varrho(s, s, z, y) + y_\varrho(s, s, z, y) \right) ds - \int_0^t \int_0^s \mathcal{B}(s-x) \mathcal{S}_\alpha(t-s) \mathcal{G} \left( x, z_\varrho(s, s, z, y) + y_\varrho(s, s, z, y) \right) ds \] 

It is vindicated that the operator \( \overline{\mathbb{T}} \) has a fixed point if and only if \( \overline{\mathbb{T}} \) has a fixed point.

**Remark 3.1.** Let \( B_r = \{ x \in \mathbb{X} : \| x \| \leq r \} \) for some \( r > 0 \). From the above discussion, we have the subsequent estimates:

(i)

\[
\| z_\varrho(s, s, z, y) + y_\varrho(s, s, z, y) \|_{\mathbb{B}_h} \\
\leq \| z_\varrho(s, s, z, y) \|_{\mathbb{B}_h} + \| y_\varrho(s, s, z, y) \|_{\mathbb{B}_h} \\
\leq D_1^* \sup_{0 \leq \tau \leq s} \| z(\tau) \|_{\mathbb{X}} + (D_2^* + J) \| z_0 \|_{\mathbb{B}_h} + D_1^* \| y(s) \| + (D_2^* + J) \| y_0 \|_{\mathbb{B}_h} \\
\leq D_1^* \sup_{0 \leq \tau \leq s} \| z(\tau) \|_{\mathbb{X}} + D_1^* \| R_\alpha(t) \|_{\mathbb{B}(X)} \| \varsigma(0) \| + (D_2^* + J) \| \varsigma \|_{\mathbb{B}_h} \\
\leq D_1^* \sup_{0 \leq \tau \leq s} \| z(\tau) \|_{\mathbb{X}} + D_1^* M H \| \varsigma \|_{\mathbb{B}_h} + (D_2^* + J) \| \varsigma \|_{\mathbb{B}_h} \\
\leq D_1^* \sup_{0 \leq \tau \leq s} \| z(\tau) \|_{\mathbb{X}} + (D_1^* M H + D_2^* + J) \| \varsigma \|_{\mathbb{B}_h}.
\]
In the event that \( \|z\|_X < r, \ r > 0 \), then
\[
\|z_{\rho(s,z_s+\gamma)} + y_{\rho(s,z_s+y_s)}\|_{B\alpha} \leq \mathcal{D}_1^* r + c_n,
\]
where \( c_n = (\mathcal{D}_1^* \mathcal{M}H + \mathcal{D}_2^* + J^*)\|\mathcal{A}\|_{B\alpha} \).

(ii) From suppositions (H1) and (H5), we sustain
\[
\|\mathcal{R} \alpha(t)\|_{\mathcal{X}(\mathcal{X})} \|\mathcal{A}(0, c, 0)\|_X \leq \mathcal{M}\mathcal{M}_0 \left[ L_{\mathcal{A}} \|\mathcal{A}\|_{B\alpha} + L_{\mathcal{A}}^* \right],
\]
where \( \|(-\mathcal{A})^{-\vartheta}\| = \mathcal{M}_0 \).

(iii)
\[
\left\| \mathcal{G} \left( t, z_{\rho(t,z_t+\gamma)} + y_{\rho(t,z_t+y_t)}, \int_0^t e_1(t, s, z_{\rho(s,z_s+\gamma)} + y_{\rho(s,z_s+y_s)})ds \right) \right\|_X
\leq \left\| (-\mathcal{A})^{-\vartheta} \mathcal{G} \left( t, z_{\rho(t,z_t+\gamma)} + y_{\rho(t,z_t+y_t)}, \int_0^t e_1(t, s, z_{\rho(s,z_s+\gamma)} + y_{\rho(s,z_s+y_s)})ds \right) - (-\mathcal{A})^{-\vartheta} \mathcal{G} (t, 0, 0) \right\|_X + \left\| (-\mathcal{A})^{-\vartheta} \mathcal{G} (t, 0, 0) \right\|_X
\leq \mathcal{M}_0 \left[ L_{\mathcal{G}} \|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{B\alpha} + L_{\mathcal{G}} \int_0^t e_1(t, s, z_{\rho(s,z_s+\gamma)} + y_{\rho(s,z_s+y_s)})ds \right]_X
+ \mathcal{M}_0 L_{\mathcal{G}} \left[ \int_0^t \|e_1(t, s, z_{\rho(s,z_s+y_s)} + y_{\rho(s,z_s+y_s)}) - e_1(t, s, 0)\|_X 
+ \|e_1(t, s, 0)\|_X \right] ds + \mathcal{M}_0 L_{\mathcal{G}}^*
\leq \mathcal{M}_0 L_{\mathcal{G}} (\mathcal{D}_1^* r + c_n) + \mathcal{M}_0 L_{\mathcal{G}} + \mathcal{M}_0 L_{\mathcal{G}} T \left[ L_{e_1} \|z_{\rho(t,z_t+y_t)} + y_{\rho(t,z_t+y_t)}\|_{B\alpha} + L_{e_1}^* \right]
\leq \mathcal{M}_0 L_{\mathcal{G}} (\mathcal{D}_1^* r + c_n) + \mathcal{M}_0 L_{\mathcal{G}}^* + \mathcal{M}_0 L_{\mathcal{G}}^* T L_{e_1} \left( \mathcal{D}_1^* r + c_n \right) + \mathcal{M}_0 L_{\mathcal{G}}^* T L_{e_1}^*,
\]
and
\[
\left\| \mathcal{G} \left( t, z_{\rho(t,z_t+\gamma)} + y_{\rho(t,z_t+y_t)}, \int_0^t e_1(t, s, z_{\rho(s,z_s+\gamma)} + y_{\rho(s,z_s+y_s)})ds \right) \right\|_X
- \mathcal{G} \left( t, z_{\rho(t,z_t+\gamma)} + y_{\rho(t,z_t+y_t)}, \int_0^t e_1(t, s, z_{\rho(s,z_s+\gamma)} + y_{\rho(s,z_s+y_s)})ds \right) \right\|_X
\leq \left\| (-\mathcal{A})^{-\vartheta} \mathcal{G} \left( t, z_{\rho(t,z_t+\gamma)} + y_{\rho(t,z_t+y_t)}\right) \right\|_X
\leq \left\| (-\mathcal{A})^{-\vartheta} \mathcal{G} \left( t, z_{\rho(t,z_t+\gamma)} + y_{\rho(t,z_t+y_t)}\right) \right\|_X.
\[
\int_0^t e_1(t, s, z_{\varphi(s,z_s+y_s)} + y_{\varphi(s,z_s+y_s)})ds
\]
\[- ( -\mathcal{A} )^\beta_{\varphi} \left( t, z_{\varphi(t,z_t+y_t)} + y_{\varphi(t,z_t+y_t)} \right) \int_0^t e_1(t, s, z_{\varphi(s,z_s+y_s)} + y_{\varphi(s,z_s+y_s)})ds \]}
\[
\leq M_0 \left[ \mathcal{L}\| z_{\varphi(t,z_t+y_t)} - z_{\varphi(t,z_t+y_t)} \|_{\mathcal{A}h} + \mathcal{L}\| T L e_1 \| \right] \| z_{\varphi(t,z_t+y_t)} - z_{\varphi(t,z_t+y_t)} \|_{\mathcal{A}h} \]
\[
\leq M_0 \bigg[ \mathcal{L} \bigg[ \| z_{\varphi(t,z_t+y_t)} - z_{\varphi(t,z_t+y_t)} \|_{\mathcal{A}h} \bigg] \bigg] \| z - z \|_{\mathcal{A}^2} .
\]

(iv)
\[
\left\| \int_0^t \mathcal{A} S_{\alpha}(t-s) \mathcal{G} \left( s, z_{\varphi(s,z_s+y_s)} + y_{\varphi(s,z_s+y_s)} \right) \int_0^s e_1(s, \tau, z_{\varphi(\tau,z_{\tau}+y_{\tau})} + y_{\varphi(\tau,z_{\tau}+y_{\tau})})d\tau ds \right\|_{\mathcal{X}}
\]
\[
\leq \int_0^t \left\| ( -\mathcal{A} )^{1-\beta} S_{\alpha}(t-s) \right\|_{\mathcal{X}} \left\| ( -\mathcal{A} )^\beta_{\varphi} \left( s, z_{\varphi(s,z_s+y_s)} + y_{\varphi(s,z_s+y_s)} \right) \right\|_{\mathcal{X}}
\int_0^s e_1(s, \tau, z_{\varphi(\tau,z_{\tau}+y_{\tau})} + y_{\varphi(\tau,z_{\tau}+y_{\tau})})d\tau \right) ds
\]
\[
+ \left\| ( -\mathcal{A} )^\beta_{\varphi} (s, 0, 0) \right\|_{\mathcal{X}} ds
\]
\[
\leq \int_0^t M(t-s)^{\alpha\beta-1} \bigg[ \mathcal{L}\| z_{\varphi(s,z_s+y_s)} + y_{\varphi(s,z_s+y_s)} \|_{\mathcal{A}h} \bigg]
\]
\[
+ \mathcal{L}\int_0^s \left[ \| e_1(s, \tau, z_{\varphi(\tau,z_{\tau}+y_{\tau})} + y_{\varphi(\tau,z_{\tau}+y_{\tau})}) - e_1(s, \tau, 0) \|_{\mathcal{X}} + \| e_1(s, \tau, 0) \|_{\mathcal{X}} \right] d\tau
\]
\[
+ \mathcal{L}^* ds
\]
\[
\leq \frac{MT^{\alpha\beta}}{\alpha\beta} \left[ \mathcal{L}(\mathcal{D}^r_{\varphi} + c_n) + \mathcal{L}\| T L e_1 (\mathcal{D}^r_{\varphi} + c_n) + \mathcal{L}\| T L e_1 + \mathcal{L}^* \right]
\[
\left\| \int_0^t \mathcal{A} S_\alpha(t-s) \mathcal{G} \left( s, z_{\varrho(s,z_s+y_s)} + y_g(s,z_s+y_s) \right) ds \right\| \leq \mathcal{M}^{\alpha\vartheta} \left\| \mathcal{A} S_\alpha(t-s) \mathcal{G} \left( s, z_{\varrho(s,z_s+y_s)} + y_g(s,z_s+y_s) \right) ds \right\| \leq \mathcal{M}^{\alpha\vartheta} \left\| \mathcal{A} S_\alpha(t-s) \mathcal{G} \left( s, z_{\varrho(s,z_s+y_s)} + y_g(s,z_s+y_s) \right) ds \right\|
\]

and

\[
\left\| \int_0^t \mathcal{G} \left( s, z_{\varrho(s,z_s+y_s)} + y_g(s,z_s+y_s) \right) ds \right\| \leq \mathcal{M}^{\alpha\vartheta} \left\| \mathcal{A} S_\alpha(t-s) \mathcal{G} \left( s, z_{\varrho(s,z_s+y_s)} + y_g(s,z_s+y_s) \right) ds \right\| \leq \mathcal{M}^{\alpha\vartheta} \left\| \mathcal{A} S_\alpha(t-s) \mathcal{G} \left( s, z_{\varrho(s,z_s+y_s)} + y_g(s,z_s+y_s) \right) ds \right\|
\]
\[
- \|(-\mathcal{A})^{\alpha} \mathcal{G}(\tau, 0, 0)\|_X + \|(-\mathcal{A})^{\alpha} \mathcal{G}(\tau, 0, 0)\|_X d\tau ds
\leq \left( \frac{MT^{\alpha \vartheta}}{\alpha \vartheta} \right) \int_0^T \mu(\tau) d\tau \left[ L_g \|\mathcal{G}(\tau, z_r + y_r) \| + \| e_1(\tau, y_r) \| \right]^2 + \widetilde{L}_g \int_0^T \left[ \| e_1(\tau, y_r) \| + \| e_1(\tau, 0) \| \right] ds + L_g
\]
and
\[
\left\| \int_0^t \int_0^s \mathcal{B}(s-\tau) \mathcal{S}_\alpha(t-s) \mathcal{G}(\tau, z_\varrho(\tau, z_r + y_r) + y_\varrho(\tau, z_r + y_r)) \right\|_X
\leq \left( \frac{MT^{\alpha \vartheta}}{\alpha \vartheta} \right) \int_0^T \mu(\tau) d\tau \mathcal{M}(t-s)^{\alpha \vartheta-1} \left[ \|(-\mathcal{A})^{\alpha \vartheta} \mathcal{G}(\tau, z_\varrho(\tau, z_r + y_r) + y_\varrho(\tau, z_r + y_r)) \right] + \mathcal{L} \mathcal{G}(\tau, z_\varrho(\tau, z_r + y_r) + y_\varrho(\tau, z_r + y_r)) \|_X d\tau ds
\leq \left( \frac{MT^{\alpha \vartheta}}{\alpha \vartheta} \right) \int_0^T \mu(\tau) d\tau \left[ L_g \| z_\varrho(\tau, z_r + y_r) - z_\varrho(\tau, z_r + y_r) \| + \widetilde{L}_g T L_{\mathcal{E}_1} \| z_\varrho(\tau, z_r + y_r) - z_\varrho(\tau, z_r + y_r) \|_X \right] + \mathcal{L} \mathcal{G}(\tau, z_\varrho(\tau, z_r + y_r) + y_\varrho(\tau, z_r + y_r)) \|_X d\tau ds
\]
\[ \left\| \int_0^t S_\alpha(t-s) \mathcal{F} \left( s, \tilde{z}(s,z_s+y_s) + y\varrho(s,z_s+y_s), \right) \right\| X \]
\[ \leq \int_0^t \left\| S_\alpha(t-s) \right\| X \left[ \left\| \mathcal{F} \left( s, \tilde{z}(s,z_s+y_s) + y\varrho(s,z_s+y_s), \right) \right\| X + \left\| \mathcal{F} (s, 0, 0) \right\| X \right] ds \]
\[ \leq M \int_0^t \left[ L_\mathcal{F} \left\| z(s,z_s+y_s) + y\varrho(s,z_s+y_s) \right\| X \right. \]
\[ + \mathcal{H} \int_0^t \left[ \left\| e_2(s, \tau, \tilde{z}(\tau,z_s+y_s) + y\varrho(\tau,z_s+y_s)) - e_2(s, \tau, 0) \right\| X + \left\| e_2(s, \tau, 0) \right\| X \right] d\tau \]
\[ + L_\mathcal{F} \right] ds \]
\[ \leq MT \left[ (\mathcal{D}_1^* + c) \left\{ L_\mathcal{F} + \mathcal{H} TL e_2 \right\} + \mathcal{H} TL e_2^* + L_\mathcal{F} \right], \]

and
\[ \left\| \int_0^t S_\alpha(t-s) \mathcal{F} \left( s, \tilde{z}(s,z_s+y_s) + y\varrho(s,z_s+y_s), \right) \right\| X \]
\[ - \int_0^t S_\alpha(t-s) \mathcal{F} \left( s, \tilde{z}(s,z_s+y_s) + y\varrho(s,z_s+y_s), \right) \left[ \int_0^s e_2(s, \tau, \tilde{z}(\tau,z_s+y_s) + y\varrho(\tau,z_s+y_s)) d\tau \right] ds \]
\[ \leq \int_0^t \left\| S_\alpha(t-s) \right\| X \left[ L_\mathcal{F} \left\| z(s,z_s+y_s) - \tilde{z}(s,z_s+y_s) \right\| X \right. \]
\[ + \mathcal{H} TL e_2 \left\| z(s,z_s+y_s) - \tilde{z}(s,z_s+y_s) \right\| X \right] ds \]
\[ \leq MT \mathcal{D}_1^* \left\{ L_\mathcal{F} + \mathcal{H} TL e_2 \right\} \left\| z - \tilde{z} \right\| X. \]
Now, we enter the main proof of this theorem. Initially, we demonstrate that $\mathbf{T}$ maps $B_{\tau}(0, \mathcal{B}_T^0)$ into $B_{\tau}(0, \mathcal{B}_T^0)$. For any $z(\cdot) \in \mathcal{B}_T^0$, by employing Remark 3.1, we sustain

$$
\| (\mathbf{T}z)(t) \|_{\mathcal{X}} 
\leq M M_0 \left[ L_{\theta} \| z \|_{\mathcal{X}} + L_{\theta}^* \right] + \left\{ M_0 + \frac{MT_{\alpha}^0}{\alpha \theta} \left( 1 + \int_0^T \mu(t) dt \right) \right\} \left( L_{\theta}^* + L_{\theta} T L_{\epsilon_2}^* \right)
$$

\( \cdot \)
\[ + MT \left\{ (L^* + L) + T(\tilde{L}_{x^2} + \tilde{L}_{x^3}) \right\} \\
+ (D + c_\alpha) \left\{ MT \left( (L^* + L) + T(\tilde{L}_{x^2} + \tilde{L}_{x^3}) \right) \\
+ \left\{ M_0 + \frac{MT_{\alpha\alpha}}{\alpha} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} (L_{x^4} + \tilde{L}_{x^4} T_{x^3}) \right\} \leq r. \]

Therefore, \( \overline{\mathcal{T}} \) maps the ball \( B_r(0, B_{T_0}) \) into itself. Finally, we show that \( \overline{\mathcal{T}} \) is a contraction on \( B_r(0, B_{T_0}) \). For this, let us consider \( z, \overline{z} \in B_r(0, B_{T_0}) \), then from Remark 3.1, we sustain
\[
\|(\mathcal{T}z)(t) - (\mathcal{T}\overline{z})(t)\|_X \\
\leq D_0 \left[ MT \left( (L^* + L) + T(\tilde{L}_{x^2} + \tilde{L}_{x^3}) \right) + \left\{ M_0 + \frac{MT_{\alpha\alpha}}{\alpha} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} (L_{x^4} + \tilde{L}_{x^4} T_{x^3}) \right\] \leq \lambda \|z - \overline{z}\|_{B_{T_0}}.
\]

From the assumption (H6) and in the perspective of the contraction mapping principle, we understand that \( \overline{\mathcal{T}} \) includes a unique fixed point \( z \in B_r(0, B_{T_0}) \) which is a mild solution of the model (1.1)-(1.2) on \( (-\infty, T] \). The proof is now completed. \( \square \)

4. Controllability Results

In this section, we present and prove the controllability of FNIDS with SDD of the structure (1.3)-(1.4) under Banach fixed point theorem. First, we present the mild solution for the model (1.3)-(1.4).

**Definition 4.1.** A function \( x : (-\infty, T] \rightarrow \mathbb{X} \), is called a mild solution of (1.3)-(1.4) on \([0, T]\), if \( x_0 = \xi; x|_{[0,T]} \in C([0, T] : \mathbb{X}); \) the function \( s \rightarrow \mathcal{A}S_\alpha(t-s)G(s, x(t,x_s)), \int_0^s e_1(s, \tau, x(t,x_s))d\tau \)
and \( s \rightarrow \int_0^s \mathcal{B}(s-t)S_\alpha(t-s)G(\tau, x(t,x_s)), \int_0^\tau e_1(\tau, \xi, x(t,x_s))d\tau \) is integrable on \([0, t]\) for all \( t \in (0, T]\) and \( s \in [0, T]\) and \( u \in L^2(\mathcal{A}, U) \),
\[
x(t) = \mathcal{R}_\alpha(t)[\xi(0) + \mathcal{G}(0, \xi(0), 0)] - \mathcal{G}(t, x(t,x_s)), \int_0^t e_1(t, s, x(t,x_s))ds \\
- \int_0^t \mathcal{A}S_\alpha(t-s)G(s, x(t,x_s)), \int_0^s e_1(s, \tau, x(t,x_s))d\tau ds \\
- \int_0^t \int_0^s \mathcal{B}(s-t)S_\alpha(t-s)G(\tau, x(t,x_s)), \int_0^\tau e_1(\tau, \xi, x(t,x_s))d\tau d\tau ds \quad (4.1)
\]
\[
+ \int_0^t S_\alpha(t-s) \mathcal{F} \left( s, x_\varrho(s,x_s), \int_0^s e_2(s,\tau,x_\varrho(\tau,x_\tau)) d\tau \right) ds \\
+ \int_0^t S_\alpha(t-s) \mathcal{H} \left( s, x_\varrho(s,x_s), \int_0^s e_3(s,\tau,x_\varrho(\tau,x_\tau)) d\tau \right) ds + \int_0^t S_\alpha(t-s) \mathcal{C} u(s) ds.
\]

For the study of the structure (1.3)-(1.4), we report the further right after hypothesis:

(H6) The following inequalities holds:

(i) Let

\[
\left( \frac{1}{\gamma} M^2 \mathcal{M}^2 T \right) \| x_T \| + \left( 1 + \frac{1}{\gamma} M^2 \mathcal{M}^2 T \right) \left[ \mathcal{M} \mathcal{M}_0 \left[ L_\varrho \| s \|_{\mathcal{B}_k} + L_\varrho^* \right] \\
+ \left\{ M_0 + \frac{M T \alpha \vartheta}{\alpha \gamma} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L_\varrho^* + L_\varrho T L_e^1 \right) \\
+ M T \left\{ (L_\varrho^* + L_\varrho^* \mathcal{M}) + T(\tilde{L}_\varrho L_e^2 + \tilde{L}_\varrho L_e^3) \right\} \\
+ (\mathcal{D}_1^* r + c_n) \left[ MT \left( (L_\varrho \mathcal{M} + L_\varrho^* \mathcal{M}) + T(\tilde{L}_\varrho L_e^2 + \tilde{L}_\varrho L_e^3) \right) \\
+ \left\{ M_0 + \frac{M T \alpha \vartheta}{\alpha \gamma} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L_\varrho^* + L_\varrho T L_e^1 \right) \right] \leq r,
\]

for some \( r > 0 \).

(ii) Let

\[
\Lambda = \left( 1 + \frac{1}{\gamma} M^2 \mathcal{M}^2 T \right) \mathcal{D}_1^* \left[ MT \left( (L_\varrho \mathcal{M} + L_\varrho^* \mathcal{M}) + T(\tilde{L}_\varrho L_e^2 + \tilde{L}_\varrho L_e^3) \right) \\
+ \left\{ M_0 + \frac{M T \alpha \vartheta}{\alpha \gamma} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L_\varrho^* + L_\varrho T L_e^1 \right) \right] < 1
\]

be such that \( 0 \leq \Lambda < 1 \).

**Theorem 4.1.** Assume that the conditions (H1)-(H5) and (H6) hold. Then the control system (1.3)-(1.4) is exactly controllable on \( \mathcal{F} \).
Proof. Utilizing the hypothesis, for an arbitrary function \( x(\cdot) \), choose the feedback control function as follows:

\[
u_x(t) = \begin{cases} 
C^*S^*_\alpha(T-t)(T^0_T)^{-1} & \left[ x_T - \mathcal{R}_\alpha(T)[\zeta(0) + \mathcal{G}(0,\zeta(0),0)] \right. \\
+ \mathcal{G} \left( T, x_{\phi(T,x_T)} \right), \int_0^T e_1(T,s,x_{\phi(s,x_s)})ds \\
+ \int_0^T \mathcal{A}S_\alpha(T-s)\mathcal{G} \left( s, x_{\phi(s,x_s)}, \int_0^s e_1(s,\tau,x_{\phi(\tau,x_\tau)})d\tau \right) ds \\
+ \int_0^t \int_0^s \mathcal{B}(s-\tau)S_\alpha(T-s) (x_{\phi(s,x_s)}) \int_0^s e_2(s,\tau,x_{\phi(\tau,x_\tau)})d\tau ds \\
- \int_0^T \mathcal{A}S_\alpha(T-s)\mathcal{F} \left( s, x_{\phi(s,x_s)}, \int_0^s e_3(s,\tau,x_{\phi(\tau,x_\tau)})d\tau \right) ds \\
- \int_0^t \int_0^s \mathcal{B}(s-\tau)S_\alpha(T-s) (x_{\phi(s,x_s)}) \int_0^s e_3(s,\tau,x_{\phi(\tau,x_\tau)})d\tau ds \\
- \int_0^t \mathcal{B}(s-\tau) \mathcal{C}u_\alpha(s)ds, & t \in \mathcal{I}.
\end{cases}
\] (4.2)

Presently, we determine the operator \( \Upsilon_1 : \mathcal{B}_T \to \mathcal{B}_T \) by

\[
(\Upsilon_1 x)(t) = \mathcal{R}_\alpha(t)[\zeta(0) + \mathcal{G}(0,\zeta(0),0)] - \mathcal{G} \left( t, x_{\phi(t,x_t)} \right), \int_0^t e_1(t,s,x_{\phi(s,x_s)})ds \\
- \int_0^t \mathcal{A}S_\alpha(t-s)\mathcal{G} \left( s, x_{\phi(s,x_s)}, \int_0^s e_1(s,\tau,x_{\phi(\tau,x_\tau)})d\tau \right) ds \\
- \int_0^t \int_0^s \mathcal{B}(s-\tau)S_\alpha(t-s)\mathcal{G} \left( \tau, x_{\phi(\tau,x_\tau)}, \int_0^\tau e_1(\tau,\xi,x_{\phi(\xi,x_\xi)})d\xi \right) d\tau ds \\
+ \int_0^t \mathcal{A}S_\alpha(t-s)\mathcal{F} \left( s, x_{\phi(s,x_s)}, \int_0^s e_2(s,\tau,x_{\phi(\tau,x_\tau)})d\tau \right) ds \\
+ \int_0^t \mathcal{A}S_\alpha(t-s)\mathcal{F} \left( s, x_{\phi(s,x_s)}, \int_0^s e_3(s,\tau,x_{\phi(\tau,x_\tau)})d\tau \right) ds \\
+ \int_0^t \mathcal{B}(s-\tau) \mathcal{C}u_\alpha(s)ds, & t \in \mathcal{I}.
\]

Observe that the control (4.2) transfers the system (1.3)-(1.4) from the initial state \( \zeta \) to the final state \( x_T \) provided that the operator \( \Upsilon_1 \) has a fixed point. To confirm the exact controllability outcome, it is adequate to demonstrate that the operator \( \Upsilon_1 \) has a fixed point in \( \mathcal{B}_T \).

We express the function \( y(\cdot) : (-\infty, T) \to \mathcal{X} \) by

\[
y(t) = \begin{cases} 
\zeta(t), & t \leq 0; \\
\mathcal{R}_\alpha(t)\zeta(0), & t \in \mathcal{I},
\end{cases}
\]
then $y_0 = z$. For every function $z \in C(\mathcal{I}, \mathbb{R})$ with $z(0) = 0$, we allocate as $\tilde{z}$ is characterized by

$$
\tilde{z}(t) = \begin{cases} 
0, & t \leq 0; \\
z(t), & t \in \mathcal{I}.
\end{cases}
$$

If $x(\cdot)$ fulfills (4.1), we are able to split it as $x(t) = y(t) + z(t)$, $t \in \mathcal{I}$, which suggests $x_t = y_t + z_t$, for each $t \in \mathcal{I}$ and also the function $z(\cdot)$ fulfills

$$
z(t) = \begin{cases} 
\mathcal{R}_\alpha(t)\mathcal{G}(0, \varsigma, 0) - \mathcal{G} \left( t, z_{\phi(t,z_t+y_t)} + y_{\phi(t,z_t+y_t)} \right), \\
\int_0^t e_1(t, s, z_{\phi(s,z_s+y_s)} + y_{\phi(s,z_s+y_s)}) ds \\
- \int_0^s \mathcal{A}\mathcal{S}_\alpha(t-s)\mathcal{G} \left( s, z_{\phi(s,z_s+y_s)} + y_{\phi(s,z_s+y_s)} \right) ds + \int_0^t \int_0^s \mathcal{B}(s-\tau)\mathcal{S}_\alpha(t-s) ds d\tau \\
\int_0^s e_2(s, \tau, z_{\phi(\tau,z_{\tau+s+y_s})} + y_{\phi(\tau,z_{\tau+s+y_s})}) d\tau ds \\
\int_0^s e_3(s, \tau, z_{\phi(\tau,z_{\tau+s+y_s})} + y_{\phi(\tau,z_{\tau+s+y_s})}) d\tau ds + \int_0^t \mathcal{S}_\alpha(t-s)C z_t+y(s) ds,
\end{cases}
$$

where

$$
u_{z+y}(t) = C^*\mathcal{S}^*_\alpha(T-t)(T_0^T)^{-1} \left[ x_T - \mathcal{R}_\alpha(T)\mathcal{G}(0, \varsigma, 0) \\
+ \mathcal{G} \left( T, z_{\phi(T,z_T+y_T)} + y_{\phi(T,z_T+y_T)} \right), \int_0^T e_1(T, s, z_{\phi(s,z_s+y_s)} + y_{\phi(s,z_s+y_s)}) ds \\
+ \int_0^T \mathcal{A}\mathcal{S}_\alpha(T-s)\mathcal{G} \left( s, z_{\phi(s,z_s+y_s)} + y_{\phi(s,z_s+y_s)} \right) ds \\
\int_0^T e_2(s, \tau, z_{\phi(\tau,z_{\tau+s+y_s})} + y_{\phi(\tau,z_{\tau+s+y_s})}) d\tau ds \\
+ \int_0^T \mathcal{B}(s-\tau)\mathcal{S}_\alpha(T-s)\mathcal{G} \left( \tau, z_{\phi(\tau,z_{\tau+s+y_s})} + y_{\phi(\tau,z_{\tau+s+y_s})} \right),
\right]
$$
\[
\int_0^\tau e_1(\tau, \xi, z_\theta(\xi, z_\xi + y_\xi)) + y_\theta(\xi, z_\xi + y_\xi))d\xi d\tau ds
- \int_0^T S_\alpha(T - s)F(s, z_\theta(s, z_s + y_s) + y_\theta(s, z_s + y_s),
\int_0^s e_2(s, \tau, z_\theta(\tau, z_\tau + y_\tau) + y_\theta(\tau, z_\tau + y_\tau))d\tau ds
- \int_0^T S_\alpha(T - s)H(s, z_\theta(s, z_s + y_s) + y_\theta(s, z_s + y_s) ds
\int_0^s e_3(s, \tau, z_\theta(\tau, z_\tau + y_\tau) + y_\theta(\tau, z_\tau + y_\tau))d\tau ds
\]

Let \( S_0 = \{ z \in S_0 : z_0 = 0 \in S_0 \}. \) Let \( \| \cdot \|_{S_0} \) be the seminorm in \( S_0 \), described by
\[
\| z \|_{S_0} = \sup_{t \in \mathcal{J}} \| z(t) \| \chi + \| z_0 \|_{S_0} = \sup_{t \in \mathcal{J}} \| z(t) \| \chi, \quad z \in S_0,
\]
as a result \( (S_0, \| \cdot \|_{S_0} ) \) is a Banach space. We delimit the operator \( \mathcal{Y}_1 : S_0 \rightarrow S_0 \) by
\[
(\mathcal{Y}_1 z)(t) = R_\alpha(t)\mathcal{G}(0, \zeta, 0) - \mathcal{G}(t, z_\theta(t, z_t + y_t) + y_\theta(t, z_t + y_t),
\int_0^t e_1(t, s, z_\theta(s, z_s + y_s) + y_\theta(s, z_s + y_s))ds
- \int_0^T S_\alpha(t - s)\mathcal{G}(s, z_\theta(s, z_s + y_s) + y_\theta(s, z_s + y_s),
\int_0^s e_1(s, \tau, z_\theta(\tau, z_\tau + y_\tau) + y_\theta(\tau, z_\tau + y_\tau))d\tau ds
- \int_0^T S_\alpha(t - s)\mathcal{G}(\tau, z_\theta(\tau, z_\tau + y_\tau) + y_\theta(\tau, z_\tau + y_\tau),
\int_0^\tau e_1(\tau, \xi, z_\theta(\xi, z_\xi + y_\xi) + y_\theta(\xi, z_\xi + y_\xi))d\xi d\tau ds
+ \int_0^t S_\alpha(t - s)\mathcal{F}(s, z_\theta(s, z_s + y_s) + y_\theta(s, z_s + y_s),
\int_0^s e_2(s, \tau, z_\theta(\tau, z_\tau + y_\tau) + y_\theta(\tau, z_\tau + y_\tau))d\tau ds
+ \int_0^T S_\alpha(t - s)\mathcal{H}(s, z_\theta(s, z_s + y_s) + y_\theta(s, z_s + y_s),
\int_0^s e_3(s, \tau, z_\theta(\tau, z_\tau + y_\tau) + y_\theta(\tau, z_\tau + y_\tau))d\tau ds + \int_0^t S_\alpha(t - s)C_{z+\theta}(s)ds.
\]
Remark 4.1. In addition to Remark 3.1, we have the subsequent estimates:

(i) 
\[
\|Cu_{z+y}(s)\| \leq \left( \frac{1}{\gamma} \mathcal{M}^2 \right) \left[ \|x_T\| + \mathcal{M}M_0[L_\theta \|\varsigma\|_{\mathcal{H}} + L_\theta^*] 
+ \left\{ M_0 + \frac{MT^{\alpha\theta}}{\alpha\theta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_\theta^* + L_\theta^* TT^*_e \right) 
+ MT \left\{ (L_\theta^* + L_\theta^*) + T(\tilde{L}_\theta L^*_{e_2} + \tilde{L}_\theta^* L^*_{e_3}) \right\} 
+ (\mathcal{D}_n^* + c_n) \left[ \mathcal{M} \left( (L_\theta + L_{\theta}) + T(\tilde{L}_\theta L_{e_2} + \tilde{L}_\theta^* L_{e_3}) \right) 
+ \left\{ M_0 + \frac{MT^{\alpha\theta}}{\alpha\theta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_\theta + L_\theta^* TT^*_e \right) \right],
\]

and
\[
\|Cu_{z+y}(s) - Cu_{\tilde{z}+y}(s)\| 
\leq \left( \frac{1}{\gamma} \mathcal{M}^2 \right) \mathcal{D}_n^* \left[ \mathcal{M}T \left( (L_\theta + L_{\theta}) + T(\tilde{L}_\theta L_{e_2} + \tilde{L}_\theta^* L_{e_3}) \right) 
+ \left\{ M_0 + \frac{MT^{\alpha\theta}}{\alpha\theta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_\theta + L_\theta^* TT^*_e \right) \right] \|z - \tilde{z}\|_{\mathcal{H}_T}.
\]

Therefore, we have
\[
\left\| \int_0^t S_\alpha(t-s)Cu_{z+y}(s)ds \right\|_{\mathcal{V}} \leq \tilde{C}_1 + \tilde{C}_2,
\]

where
\[
\tilde{C}_1 = \left( \frac{1}{\gamma} \mathcal{M}^2 \mathcal{D}_n^* T \right) \|x_T\|,
\]
\[
\tilde{C}_2 = \left( \frac{1}{\gamma} \mathcal{M}^2 \mathcal{D}_n^* T \right) \left[ M_0 \mathcal{M}M_0[L_\theta \|\varsigma\|_{\mathcal{H}} + L_\theta^*] 
+ \left\{ M_0 + \frac{MT^{\alpha\theta}}{\alpha\theta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_\theta^* + L_\theta^* TT^*_e \right) 
+ MT \left\{ (L_\theta^* + L_\theta^*) + T(\tilde{L}_\theta L^*_{e_2} + \tilde{L}_\theta^* L^*_{e_3}) \right\} 
+ (\mathcal{D}_n^* + c_n) \left[ \mathcal{M}T \left( (L_\theta + L_{\theta}) + T(\tilde{L}_\theta L_{e_2} + \tilde{L}_\theta^* L_{e_3}) \right) 
+ \left\{ M_0 + \frac{MT^{\alpha\theta}}{\alpha\theta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_\theta + L_\theta^* TT^*_e \right) \right].
\]
\[
+ \left\{ M_0 + \frac{M T^{\alpha \vartheta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_{\varphi} + \tilde{L}_{\varphi} T L e_1 \right) \right] ,
\]

and
\[
\left\| \int_0^t S_\alpha(t-s) \left[ C u_{z+y}(s) - C u_{z+y}(s) \right] ds \right\|_X
\leq \left( \frac{1}{\gamma} M^2 M^2 T \right) \mathcal{D}_t^\alpha \left[ MT \left( (L_{\varphi} + L_{\varphi}') + T(\tilde{L}_{\varphi} L e_2 + \tilde{L}_{\varphi} L e_3) \right) 
+ \left\{ M_0 + \frac{M T^{\alpha \vartheta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_{\varphi} + \tilde{L}_{\varphi} T L e_1 \right) \right] \| z - \bar{z} \|_{\mathcal{B}_F}.
\]

Now, we enter the main proof of this theorem. Initially, we demonstrate that \( \Upsilon_1 \) maps \( B_r(0, \mathcal{B}_F^0) \) into \( B_r(0, \mathcal{B}_F^0) \). For any \( z(\cdot) \in \mathcal{B}_F^0 \), by employing Remark 3.1 and Remark 4.1, we sustain
\[
\left\| (\Upsilon_1 z)(t) \right\|_X \leq \bar{C}_3 \leq r,
\]

where
\[
\bar{C}_3 = \left( 1 + \frac{1}{\gamma} M^2 M^2 T \right) \mathcal{M} M_0 [L_{\varphi} \| s \|_{\mathcal{B}_F} + L_{\varphi}^*] 
+ \left\{ M_0 + \frac{M T^{\alpha \vartheta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_{\varphi}^* + \tilde{L}_{\varphi} T L e_1 \right) 
+ MT \left\{ (L_{\varphi}^* + L_{\varphi}^*) + T(\tilde{L}_{\varphi} L e_2 + \tilde{L}_{\varphi} L e_3) \right\} 
+ (\mathcal{D}_t^\alpha + c_n) \left[ MT \left( (L_{\varphi} + L_{\varphi}') + T(\tilde{L}_{\varphi} L e_2 + \tilde{L}_{\varphi} L e_3) \right) 
+ \left\{ M_0 + \frac{M T^{\alpha \vartheta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_{\varphi} + \tilde{L}_{\varphi} T L e_1 \right) \right] \right].
\]

Therefore, \( \Upsilon_1 \) maps the ball \( B_r(0, \mathcal{B}_F^0) \) into itself. Finally, we show that \( \Upsilon_1 \) is a contraction on \( B_r(0, \mathcal{B}_F^0) \). For this, let us consider \( z, \bar{z} \in B_r(0, \mathcal{B}_F^0) \), then from Remark 3.1 and Remark 4.1, we sustain
\[
\left\| (\Upsilon_1 z)(t) - (\Upsilon_1 \bar{z})(t) \right\|_X
\leq \left( 1 + \frac{1}{\gamma} M^2 M^2 T \right) \mathcal{D}_t^\alpha \left[ MT \left( (L_{\varphi} + L_{\varphi}') + T(\tilde{L}_{\varphi} L e_2 + \tilde{L}_{\varphi} L e_3) \right) 
+ \left\{ M_0 + \frac{M T^{\alpha \vartheta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau)d\tau \right) \right\} \left( L_{\varphi} + \tilde{L}_{\varphi} T L e_1 \right) \right].
\]
+ \left\{M_{0} + \frac{M_{0}}{\alpha^{\alpha}} \left(1 + \int_{0}^{T} \mu(\tau) d\tau\right) \right\} \left(\mathcal{L}_{\mathcal{G}} + \tilde{\mathcal{L}}_{\mathcal{G}} T L_{\varepsilon_{1}}\right)
\leq \Lambda \|z - \pi\|_{\mathcal{G}}^\alpha.

From the assumption (H6) and in the perspective of the contraction mapping principle, we understand that \( T_{1} \) includes a unique fixed point \( z \in \mathcal{B}_{f} \). Thus, the model (1.3)-(1.4) is exactly controllable on \( \mathcal{S} \). The proof is now completed. \( \square \)

5. Applications

Example 5.1:
To exemplify our theoretical outcomes, first we treat the FNIDS with SDD of the model
\[
\begin{align*}
D_{t}^{\alpha} u(t, x) + & \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \psi_{1}(s) \psi_{2}(\|u(s)\|), x)}{49} ds \\
+ & \int_{0}^{t} \sin(t-s) \int_{-\infty}^{s} e^{2(\tau-s)} \frac{u(\tau - \psi_{1}(\tau) \psi_{2}(\|u(\tau)\|), x)}{36} d\tau ds \\
+ & \int_{0}^{t} (t-s) \delta e^{-\gamma(t-s)} \frac{\partial^{2}}{\partial x^{2}} u(s, x) ds + \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \psi_{1}(s) \psi_{2}(\|u(s)\|), x)}{9} ds \\
+ & \int_{-\infty}^{t} \sin(t-s) \int_{-\infty}^{s} e^{2(\tau-s)} \frac{u(\tau - \psi_{1}(\tau) \psi_{2}(\|u(\tau)\|), x)}{25} d\tau ds \\
+ & \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \psi_{1}(s) \psi_{2}(\|u(s)\|), x)}{64} ds \\
+ & \int_{0}^{t} \sin(t-s) \int_{-\infty}^{s} e^{2(\tau-s)} \frac{u(\tau - \psi_{1}(\tau) \psi_{2}(\|u(\tau)\|), x)}{16} d\tau ds,
\end{align*}
\tag{5.1}
\]
\[
\begin{align*}
\text{where } D_{t}^{\alpha} \text{ is Caputo’s fractional derivative of order } \alpha \in (1, 2), \delta \text{ and } \gamma \text{ are positive numbers and } \varsigma \in \mathcal{B}_{h}. \text{ We consider } X = L^{2}[0, \pi] \text{ having the norm } \| \cdot \|_{L^{2}} \text{ and determine the operator } \mathcal{A} : D(\mathcal{A}) \subset X \to X \text{ by } \mathcal{A} u = w'' \text{ with the domain }
\]
\[
D(\mathcal{A}) = \{ w \in X : w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0 \}.
\]
Then
\[
\mathcal{A} w = \sum_{n=1}^{\infty} n^{2} \langle w, w_{n} \rangle w_{n}, \quad w \in D(\mathcal{A}),
\]
in which \( w_{n}(s) = \sqrt{\frac{2}{\pi}} \sin(ns), n = 1, 2, \ldots \) is the orthogonal set of eigenvectors of \( \mathcal{A} \). It is long familiar that \( \mathcal{A} \) is the infinitesimal generator of an analytic semigroup \((T(t))_{t \geq 0}\) in \( X \).
and is provided by

\[ T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, w_n \rangle w_n, \quad \text{for all } w \in X, \quad \text{and every } t > 0. \]

Hence (H1) is fulfilled. If we fix \( \vartheta = \frac{1}{2} \), then the operator \((-\mathscr{A})^{\frac{1}{2}}\) is given by

\[ (-\mathscr{A})^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n \langle w, w_n \rangle w_n, \quad w \in (D(-\mathscr{A})^{\frac{1}{2}}), \]

in which \((D(-\mathscr{A})^{\frac{1}{2}}) = \left\{ \omega(\cdot) \in X : \sum_{n=1}^{\infty} n \langle \omega, w_n \rangle w_n \in X \right\}\) and \((-\mathscr{A})^{-\frac{1}{2}} = 1\). Therefore, \(\mathscr{A}\) is sectorial of type and the properties (P1) hold. We also take into account the operator \(\mathcal{B}(t) : D(\mathscr{A}) \subseteq X \to X, t \geq 0, \mathcal{B}(t)x = t^\delta e^{-\gamma t} A x \) for \(x \in D(\mathscr{A})\). In addition, it is simple to see that conditions (P2)-(P3)\(\) are fulfilled with \(b(t) = t^\delta e^{-\gamma t}\) and \(D = C_0^\omega([0, \pi])\), where \(C_0^\omega([0, \pi])\) is the space of infinitely differentiable functions that vanish at \(x = 0\) and \(x = \pi\). From the Lemma 2.4, it is simple to see that condition (H2) is fulfills. For the phase space, we choose \(h = e^{2s}, \ s < 0\), then \(l = \int_{-\infty}^{0} h(s) ds = \frac{1}{2} < \infty, \) for \(t \leq 0\) and determine

\[ ||\varsigma||_{B_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s, 0]} ||\varsigma(\theta)||_{L_2} ds. \]

Hence, for \((t, \varsigma) \in [0, T] \times \mathcal{B}_h\), where \(\varsigma(\theta)(x) = \varsigma(\theta, x), \ (\theta, x) \in (-\infty, 0) \times [0, \pi]\). Set

\[ u(t)(x) = u(t, x), \quad \varrho(t, \varsigma) = \varrho_1(t) \varrho_2(||\varsigma(0)||), \]

we have

\[ \mathcal{G}(t, \varsigma, \mathcal{H}\varsigma)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{S}{49} ds + (\mathcal{H}\varsigma)(x), \]

\[ \mathcal{F}(t, \varsigma, \mathcal{H}\varsigma)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{S}{9} ds + (\mathcal{H}\varsigma)(x), \]

\[ \mathcal{H}(t, \varsigma, \mathcal{H}\varsigma)(x) = \int_{-\infty}^{0} e^{2(s)} \frac{S}{64} ds + (\mathcal{H}\varsigma)(x), \]

where

\[ (\mathcal{H}\varsigma)(x) = \int_{0}^{t} \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{S}{36} d\tau ds, \]

\[ (\mathcal{F}\varsigma)(x) = \int_{0}^{t} \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{S}{25} d\tau ds, \]

\[ (\mathcal{H}\varsigma)(x) = \int_{0}^{t} \sin(t-s) \int_{-\infty}^{0} e^{2(\tau)} \frac{S}{16} d\tau ds, \]
EXISTENCE AND CONTROLLABILITY RESULTS

then using these configurations, the system (5.1)-(5.3) is usually written in the theoretical form of design (1.1)-(1.2).

To treat this system we assume that 
\( \varrho_i: [0, \infty) \to [0, \infty) \), \( i = 1, 2 \) are continuous. Now, we can see that for \( t \in [0, T] \), \( \varsigma, \tilde{\varsigma} \in \mathcal{B}_h \), we have

\[
\|(-\mathcal{A})^{\frac{1}{2}} \mathcal{F}(t, \varsigma, \tilde{\mathcal{F}}\tilde{\varsigma})\|_X \\
\leq \left( \int_0^t \left( \int_{-\infty}^0 e^{2(s)} \left\| \frac{\varsigma - \tilde{\varsigma}}{49} \right\| ds + \int_0^t \| \sin(t-s) \| \int_{-\infty}^0 e^{2(\tau)} \left\| \frac{\varsigma - \tilde{\varsigma}}{36} \right\| d\tau ds \right)^2 dx \right)^{\frac{1}{2}}
\leq \left( \int_0^t \left( \frac{1}{49} \int_{-\infty}^0 e^{2(s)} \sup \| \varsigma \| ds + \frac{1}{36} \int_{-\infty}^0 e^{2(s)} \sup \| \varsigma \| ds \right)^2 dx \right)^{\frac{1}{2}}
\leq \frac{\sqrt{\pi}}{49} \| \varsigma \| \mathcal{B}_h + \frac{\sqrt{\pi}}{36} \| \varsigma \| \mathcal{B}_h
\leq L_{\mathcal{A}} \| \varsigma \| \mathcal{B}_h + \hat{L}_{\mathcal{A}} \| \varsigma \| \mathcal{B}_h,
\]

where \( L_{\mathcal{A}} + \hat{L}_{\mathcal{A}} = \frac{85 \sqrt{\pi}}{1764} \), and

\[
\|(-\mathcal{A})^{\frac{1}{2}} \mathcal{F}(t, \varsigma, \tilde{\mathcal{F}}\tilde{\varsigma})\|_X \\
\leq \left( \int_0^t \left( \int_{-\infty}^0 e^{2(s)} \left\| \frac{\varsigma - \tilde{\varsigma}}{49} \right\| ds + \int_0^t \| \sin(t-s) \| \int_{-\infty}^0 e^{2(\tau)} \left\| \frac{\varsigma - \tilde{\varsigma}}{36} \right\| d\tau ds \right)^2 dx \right)^{\frac{1}{2}}
\leq \left( \int_0^t \left( \frac{1}{49} \int_{-\infty}^0 e^{2(s)} \sup \| \varsigma - \tilde{\varsigma} \| ds + \frac{1}{36} \int_{-\infty}^0 e^{2(s)} \sup \| \varsigma - \tilde{\varsigma} \| ds \right)^2 dx \right)^{\frac{1}{2}}
\leq \frac{\sqrt{\pi}}{49} \| \varsigma - \tilde{\varsigma} \| \mathcal{B}_h + \frac{\sqrt{\pi}}{36} \| \varsigma - \tilde{\varsigma} \| \mathcal{B}_h
\leq L_{\mathcal{A}} \| \varsigma - \tilde{\varsigma} \| \mathcal{B}_h + \hat{L}_{\mathcal{A}} \| \varsigma - \tilde{\varsigma} \| \mathcal{B}_h.
\]

Similarly, we conclude

\[
\| \tilde{\mathcal{F}}(t, \varsigma, \tilde{\mathcal{F}}\tilde{\varsigma})\|_{L^2} \\
\leq \left( \int_0^t \left( \int_{-\infty}^0 e^{2(s)} \left\| \frac{\varsigma}{9} \right\| ds + \int_0^t \| \sin(t-s) \| \int_{-\infty}^0 e^{2(\tau)} \left\| \frac{\varsigma}{25} \right\| d\tau ds \right)^2 dx \right)^{\frac{1}{2}}
\leq \left( \int_0^t \left( \frac{1}{9} \int_{-\infty}^0 e^{2(s)} \sup \| \varsigma \| ds + \frac{1}{25} \int_{-\infty}^0 e^{2(s)} \sup \| \varsigma \| ds \right)^2 dx \right)^{\frac{1}{2}}
\leq \frac{\sqrt{\pi}}{9} \| \varsigma \| \mathcal{B}_h + \frac{\sqrt{\pi}}{25} \| \varsigma \| \mathcal{B}_h
\leq L_{\tilde{\mathcal{F}}} \| \varsigma \| \mathcal{B}_h + \hat{L}_{\tilde{\mathcal{F}}} \| \varsigma \| \mathcal{B}_h.
\]
where $L_F + \tilde{L}_F = \frac{34\sqrt{\pi}}{225}$, and

\[
\|\mathcal{F}(t, \varsigma, \mathcal{H}\varsigma) - \mathcal{F}(t, \zeta, \mathcal{H}\zeta)\|_{L^2} \\
\leq \left( \int_0^\pi \left( \int_{-\infty}^0 e^{2(s)}\left| \frac{\varsigma}{9} - \frac{\zeta}{9} \right| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left| \frac{\varsigma}{25} - \frac{\zeta}{25} \right| d\tau ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \left( \int_0^\pi \left( \frac{1}{9} \int_{-\infty}^0 e^{2(s)} \sup_{\varsigma - \zeta} \|s - \tau\| ds + \frac{1}{25} \int_{-\infty}^0 e^{2(s)} \sup_{\varsigma - \zeta} \|s - \tau\| ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \sqrt{\frac{\pi}{9}} \|s - \zeta\|_{\mathcal{B}h} + \sqrt{\frac{\pi}{25}} \|s - \zeta\|_{\mathcal{B}h} \\
\leq L_F \|s - \zeta\|_{\mathcal{B}h} + \tilde{L}_F \|s - \zeta\|_{\mathcal{B}h}.
\]

Correspondingly, we have

\[
\|\mathcal{H}(t, \varsigma, \mathcal{H}\varsigma)\|_{L^2} \\
\leq \left( \int_0^\pi \left( \int_{-\infty}^0 e^{2(s)}\left| \frac{\varsigma}{64} - \frac{\zeta}{64} \right| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left| \frac{\varsigma}{16} - \frac{\zeta}{16} \right| d\tau ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \left( \int_0^\pi \left( \frac{1}{64} \int_{-\infty}^0 e^{2(s)} \sup_{\varsigma - \zeta} \|s - \tau\| ds + \frac{1}{16} \int_{-\infty}^0 e^{2(s)} \sup_{\varsigma - \zeta} \|s - \tau\| ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \sqrt{\frac{\pi}{64}} \|s - \zeta\|_{\mathcal{B}h} + \sqrt{\frac{\pi}{16}} \|s - \zeta\|_{\mathcal{B}h} \\
\leq L_H \|s - \zeta\|_{\mathcal{B}h} + \tilde{L}_H \|s - \zeta\|_{\mathcal{B}h},
\]

where $L_H + \tilde{L}_H = \frac{80\sqrt{\pi}}{1024}$, and

\[
\|\mathcal{H}(t, \varsigma, \mathcal{H}\varsigma) - \mathcal{H}(t, \zeta, \mathcal{H}\zeta)\|_{L^2} \\
\leq \left( \int_0^\pi \left( \int_{-\infty}^0 e^{2(s)}\left| \frac{\varsigma}{64} - \frac{\zeta}{64} \right| ds + \int_0^t \|\sin(t-s)\| \int_{-\infty}^0 e^{2(\tau)}\left| \frac{\varsigma}{16} - \frac{\zeta}{16} \right| d\tau ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \left( \int_0^\pi \left( \frac{1}{64} \int_{-\infty}^0 e^{2(s)} \sup_{\varsigma - \zeta} \|s - \tau\| ds + \frac{1}{16} \int_{-\infty}^0 e^{2(s)} \sup_{\varsigma - \zeta} \|s - \tau\| ds \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \sqrt{\frac{\pi}{64}} \|s - \zeta\|_{\mathcal{B}h} + \sqrt{\frac{\pi}{16}} \|s - \zeta\|_{\mathcal{B}h} \\
\leq L_H \|s - \zeta\|_{\mathcal{B}h} + \tilde{L}_H \|s - \zeta\|_{\mathcal{B}h}.
\]
Therefore the conditions (H3) and (H5) are all fulfilled. Furthermore, we assume that \( D_1^\alpha = 1, M_0 = 1, M = 1, T = 1, \alpha = \frac{3}{2}, L_{e_1} = 1, L_{e_2} = 1, L_{e_3} = 1 \) and \( \int_0^1 \mu(\tau) d\tau = 1 \). Then
\[
D_1^\alpha \left[ MT \left( (L_\mathcal{F} + L_\mathcal{F}^\ast) + T(\tilde{L}_\mathcal{F} L_{e_2} + \tilde{L}_\mathcal{F} L_{e_3}) \right) \right. \\
+ \left. \{ M_0 + \frac{MT^{\alpha \delta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \} \right] \approx 0.71941 < 1.
\]
Thus the condition (H6) holds. Hence by Theorem 3.1, we realize that the system (5.1)–(5.3) has a unique mild solution on \([0, 1]\).

**Example 5.2:**

In this section, as an application of Theorem 4.1, we treat the FNIDS with SDD of the model
\[
D_1^\alpha \left[ u(t, x) + \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \varrho_1(s) \varrho_2(\|u(s)\|, x))}{49} ds \right.
\]
\[
+ \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau-s)} \frac{u(\tau - \varrho_1(\tau) \varrho_2(\|u(\tau)\|, x))}{36} d\tau ds \right] = \frac{\partial^2}{\partial x^2} u(t, x)
\]
\[
+ \int_0^t (t-s) \delta e^{-\pi(t-s)} \frac{\partial^2 u(s, x)}{\partial x^2} ds + \nu(t, x) + \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \varrho_1(s) \varrho_2(\|u(s)\|, x))}{9} ds 
\]
\[
+ \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau-s)} \frac{u(\tau - \varrho_1(\tau) \varrho_2(\|u(\tau)\|, x))}{25} d\tau ds 
\]
\[
+ \int_{-\infty}^t e^{2(s-t)} \frac{u(s - \varrho_1(s) \varrho_2(\|u(s)\|, x))}{64} ds 
\]
\[
+ \int_0^t \sin(t-s) \int_{-\infty}^s e^{2(\tau-s)} \frac{u(\tau - \varrho_1(\tau) \varrho_2(\|u(\tau)\|, x))}{16} d\tau ds,
\]
(5.4)

with the conditions (5.2)-(5.3). \( D_1^\alpha, \alpha, \delta \) and \( \pi \) are same as defined in Example 5.1. Further, we define the operator \( C : U \to \mathbb{X} \) by \( C u(t, x) = \nu(t, x), 0 < x < \pi, u \in U \), where \( \nu : [0, 1] \times [0, \pi] \to [0, \pi] \). In perspective of Example 5.1 and using these configurations, the system (5.4) with the conditions (5.2)-(5.3) is usually written in the theoretical form of design (1.3)-(1.4).

Furthermore, we assume that \( D_1^\alpha = \frac{1}{2}, M_0 = 1, M = 1, M_C = 1, \gamma = 1, T = 1, \alpha = \frac{3}{2}, L_{e_1} = 1, L_{e_2} = 1, L_{e_3} = 1 \) and \( \int_0^1 \mu(\tau) d\tau = 1 \). Then
\[
\left( 1 + \frac{1}{\gamma} M^2 M_C^2 T \right) D_1^\alpha \left[ MT \left( (L_\mathcal{F} + L_\mathcal{F}^\ast) + T(\tilde{L}_\mathcal{F} L_{e_2} + \tilde{L}_\mathcal{F} L_{e_3}) \right) \right.
\]
\[
+ \left. \{ M_0 + \frac{MT^{\alpha \delta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \} \right] \approx 0.71941 < 1.
\]
\[
+ \left\{ M_0 + \frac{MT^{\alpha \vartheta}}{\alpha \vartheta} \left( 1 + \int_0^T \mu(\tau) d\tau \right) \right\} \left( L_0 + \tilde{L}_0 TL_1 \right) \approx 0.71941 < 1.
\]

Thus the condition \((H6)^*\) holds. Hence by Theorem 4.1, we realize that the system (5.4) with the conditions (5.2)–(5.3) has a unique mild solution on \([0, 1]\).

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REFERENCES


COMPARISON OF NUMERICAL METHODS FOR TERNARY FLUID FLOWS: IMMERSED BOUNDARY, LEVEL-SET, AND PHASE-FIELD METHODS

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ABSTRACT. This paper reviews and compares three different methods for modeling incompressible and immiscible ternary fluid flows: the immersed boundary, level set, and phase-field methods. The immersed boundary method represents the moving interface by tracking the Lagrangian particles. In the level set method, an interface is defined implicitly by using the signed distance function, and its evolution is governed by a transport equation. In the phase-field method, the advective Cahn–Hilliard equation is used as the evolution equation, and its order parameter also implicitly defines an interface. Each method has its merits and demerits. We perform the several simulations under different conditions to examine the merits and demerits of each method. Based on the results, we determine the most suitable method depending on the specific modeling needs of different situations.

1. INTRODUCTION

A double emulsion, or compound droplet, is a specific case of a ternary fluid mixture. It has a smaller drop or drops inside a larger drop and has a high level of potential for many applications that use liquid membranes for selective mass transport, such as drug delivery and controlled drug release, because of its three phases: a small inner drop, the surrounding medium, and a third fluid [1]. However, modeling or simulating the interfaces of ternary fluid flows is a challenging problem since the diffusion phenomenon is more complex than two-component mixtures [2]. Figure 1 presents a schematic of the coaxial microcapillary fluidic device and the geometry of the double emulsion that it generates [3].

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There are two major approaches to simulating multi-phase or multi-component flows to characterize moving interfaces: the interface tracking method and the interface capturing method. The interface tracking method uses computational mesh to track interfaces and a velocity field is generated by adjusting the position of nodes. Examples of the interface tracking method include the volume of fluid, front tracking, and immersed boundary method (IBM). In contrast, interface capturing methods implicitly define an interface by using the contours of particular scalar functions. Examples include the level set method (LSM) and phase-field method (PFM).

In this paper, we review three methods, IBM, LSM, and PFM, to simulate ternary fluid flows with a double emulsion case and describe the basic techniques of each method in detail.

IBM was originally developed to model the blood flow in the heart by Peskin [4]; this method has been applied to various biological or industrial modeling problems. IBM has been used to research the hydrodynamics of a compound drop for application to leukocyte modeling [5, 6]. IBM has been applied not only to biological modeling problems as originally developed but also to fluid dynamics modeling problems [7, 8, 9]. The dynamics of a compound droplet in shear flow was researched in [10]. See the articles [11, 12, 13] to refer to the treatment of the multiple junction case with a foam model.

LSM uses a level set function to capture moving interfaces and has become popular in many disciplines since its development by Osher and Sethian in the 1980s [14]. More detailed reviews of classical LSM are given by [15, 17]. There have been developments to capture the fluid flow interfaces of ternary or even more phases by using LSM. Merrian et al. [18] represented each phase by using an individual level set function. The projection method, which uses only \((n - 1)\)-level set functions to represent the interfaces of \(n\)-phases, was developed to resolve the triple junction problem by Smith et al. [19].

PFM is a popular method for modeling the dynamics of multi-phase fluids coupled with the Navier–Stokes equation [20]. It has a diffused interface with a finite but small width between distinct phases and characterizes physical quantities such as the density and viscosity by using an order parameter governed by the modified Cahn–Hilliard (CH) equation. The CH equation was first introduced by Cahn and Hilliard [21] to describe the initial stage of spinodal decomposition. It is often used to model interface dynamics, including surface minimization and sharp topological changes like pinch-off and phase separation. See [22] and the references
therein for detailed review of the method. The multi-component system was first generalized in the literature by de Fontaini [23] and Eyre studied its differences with the binary case and its dynamics [24]. Numerical studies of ternary CH systems have been vigorously pursued [25, 26, 27, 28, 29, 30].

The main goal of this paper is to review and compare three different methods. By comparing fundamental weakness and strength, it is expected to give an advice about a choice of the methods for the beginners in ternary fluid flows problem. Moreover, this paper could be helpful also to experts of this field developing a hybrid method as in the research of Hou et al.[31] by understanding unfamiliar methods to them.

This paper is organized as follows. In Section 2, we present the governing equations for IBM, LSM, and PFM. We summarize the formulas for the surface tension force in Section 3. In Section 4, we present the numerical method to solve the discrete Navier–Stokes equations and the respective equations for the interface. The numerical results are presented in Section 5. Finally, the conclusions are drawn in Section 6.

2. Governing equations and interface representation

We consider incompressible and immiscible ternary fluids in a two-dimensional domain $\Omega$ for simplicity. Its extension to a three-dimensional problem is straightforward. For more details, refer to [10] for IBM, [32, 33, 34] for on LSM, and [28, 35] for on PFM. The motion of fluid flows is generally described by the modified Navier–Stokes (NS) equations with the surface tension force:

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \left[ \eta (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \right] + \mathbf{SF} \text{ in } \Omega, \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \quad (2.2)$$

where $\rho(x, t)$ is the density, $\mathbf{u}(x, t) = (u(x, t), v(x, t))$ is the velocity, $p(x, t)$ is the pressure, $\eta(x, t)$ is the viscosity, $x = (x, y)$ is the Cartesian coordinate, $t$ is the time variable, and $\mathbf{SF}$ is the surface tension force density. We assumed that $\rho$ and $\eta$ are constant for simplicity. A schematic of the three-phase domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ is shown in Fig. 2. $\Gamma_k$ represents the interface between the fluids $k$ and $k + 1$.

We can rewrite (2.1) and (2.2) by using dimensionless parameters as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{SF} \text{ in } \Omega, \quad (2.3)$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega. \quad (2.4)$$

Here, $Re = \rho U^* L^*/\eta$ is the Reynolds number, $U^*$ is the characteristic velocity, and $L^*$ is the characteristic length.

Now, we briefly describe how IBM, LSM, and PFM represent the interfaces of multi-phase fluids by using a Lagrangian variable, level set function, and phase-field function, respectively. The governing equations of the interface evolution are also introduced in each section.
2.1. IBM. In IBM, the interfaces $\Gamma_1$ and $\Gamma_2$ are described by the Lagrangian variables $\mathbf{X}_1(s_1, t)$ and $\mathbf{X}_2(s_2, t)$, respectively. Here, $0 \leq s_k \leq L_k(t)$ and $L_k(t)$ are the lengths of interfaces at time $t$ for $k = 1, 2$. The evolution of the interface is governed by

$$\frac{\partial \mathbf{X}_k(s_k, t)}{\partial t} = \mathbf{U}_k(s_k, t), \quad (2.5)$$

$$\mathbf{U}_k(s_k, t) = \int_{\Omega} \mathbf{u}(\mathbf{x}, t) \delta^2(\mathbf{x} - \mathbf{X}_k(s_k, t)) d\mathbf{x}, \quad \text{for } k = 1, 2, \quad (2.6)$$

where $\mathbf{U}_k(s_k, t)$ is the velocity of the Lagrangian variable $\mathbf{X}_k(s_k, t)$, $\mathbf{u}(\mathbf{x}, t)$ is the velocity field on a Cartesian grid, $\delta^2(\mathbf{x})$ is the two-dimensional Dirac-delta function defined by the product of the one-dimensional Dirac-delta functions, $\delta^2(\mathbf{x}) = \delta(x)\delta(y)$. Figure 3 shows the Lagrangian variables representing interfaces on the domain $\Omega$. 

**Figure 2.** Schematic of a three-phase domain.

**Figure 3.** Lagrangian variables representing interfaces in the domain.
The principal advantage of IBM compared to the other two methods is that it can use a large number of interfacial marker points to handle the interface geometry for high accuracy. The major drawback is the difficulty of representing topological changes without additional work. Moreover, area conservation does not hold in general because the interfaces between each fluids move discretely [10].

2.2. LSM. In LSM, the interfaces of each two phases are defined implicitly with the level set functions $\phi_k(x, t)$, $k = 1, 2$. Here, $\phi_1$ and $\phi_2$ are the signed distances that satisfy $|\nabla \phi_k| = 1$ from the interfaces $\Gamma_1$ and $\Gamma_2$, respectively. Note that the values of $\phi_k$ become zero at the interfaces, i.e., the zero contours of $\phi_k$ represent the interfaces. Figure 4 shows the zero contour of the signed distance function and surface plots with zero contours. In addition, note that $\phi_k$ has the opposite sign in each phase (see Fig. 4).

\[
\Gamma_1 : \{ \phi_1 = 0 \} \\
\Gamma_2 : \{ \phi_2 = 0 \} \\
\phi_1 > 0, \quad \phi_2 > 0 \\
\phi_1 < 0, \quad \phi_2 < 0 \\
\phi_1, \phi_2 < 0
\]

The evolution equation of $\phi_k$ is governed by the transport equation:

\[
(\phi_k)_t + u \cdot \nabla \phi_k = 0.
\] (2.7)

During the process of interface evolution, $\phi_k$ tends to deviate from the signed distance function. However, we maintained $\phi_k$ as the signed distance function because the density and surface tension depend on $\phi_k$ [36]. The reinitialization step makes $\phi_k$ recover to the signed distance function without changing its zero contour and is given as follows:

\[
\frac{\partial d_k(x, t)}{\partial \tau} = S(\phi_k(x, t))(1 - |\nabla d_k(x, t)|),
\] (2.8)

\[
d_k(x, 0) = \phi_k(x, t),
\] (2.9)

where $\tau$ is the pseudo-time and $S(\phi_k)$ is the sign function. In numerical implementations, we can use the smoothed sign function $S_\beta(\phi_k) = \phi_k/\sqrt{\phi_k^2 + \beta^2}$ where $\beta$ is one or two grid lengths. $d_k(x, \tau_s)$ replaces $\phi_k(x, t)$ after the function is solved up to the steady-state where $\tau_s$ is the steady-state pseudo-time. A more detailed description is given in [15].
The advantages of LSM include a simple implementation, ability to automatically capture the merging and break-up of interfaces, and flexibility to describe the complex interface geometry. Whereas, the major disadvantage is the lack of mass (area) conservation and the hybrid methods have been proposed until nowadays to overcome this [16].

2.3. PFM. In PFM, the order parameters $c_k(x, t)$ are used, where $k = 1, 2, 3$ which are measures of the relative composition or the volume fraction of the three components. The functions $c_k$ are distributed continuously on thin interfacial layers and uniformly in the bulk phases. Here, the order parameter is defined by $c_k \approx 1$ in one fluid and $c_k \approx 0$ in the other fluid, while the interfaces $\Gamma_k$ are defined by $c_k = 0.5$. The sharp fluid interfaces are replaced by thin (but nonzero) thickness transition regions. Figure 5 shows the numerical interfaces of the order parameters and surface plots with interfaces.

![Figure 5](image-url)

**Figure 5.** (a) Numerical interfaces of the order parameters $c_1$ and $c_2$, and surface plots with the interfaces of (b) $c_1$, (c) $c_2$, and (d) $c_3 = 1 - c_1 - c_2$.

The evolution equations of the phase-field function $\mathbf{c} = (c_1, c_2, c_3)$ are governed by the advective multi-component CH as follows:

$$\frac{\partial c_k}{\partial t} + \mathbf{u} \cdot \nabla c_k = \frac{1}{Pe} \Delta \mu_k, \quad k = 1, 2, 3$$

$$\mu_k = \frac{\partial F(c_1, c_2, c_3)}{\partial c_k} - C \Delta c_k + \gamma(c_1, c_2, c_3),$$
where $Pe$ is the Peclet number, defined by $L^* U^*/(M \mu^*)$, $M$ is the constant mobility, $\mu^*$ is the characteristic value of the chemical potentials, $\mu_k$ is the chemical potential, $F(c_1, c_2, c_3) = 0.25 \sum_{k=1}^{3} c_k^2 (1 - c_k)^2$ is the bulk energy density, $C$ is the Cahn number, defined by $\epsilon^2 / \mu^*$, $\epsilon$ is the measure of the interface thickness and $\gamma(c_1, c_2, c_3) = -\sum_{k=1}^{3} \partial F / (3 \partial c_k)$ is the Lagrangian multiplier which makes the sum of chemical potential $\mu_k$ zero. See [37] for a detailed derivation of $\gamma$. Here, we only need to solve $c_1$ and $c_2$ because the sum of the mole fractions is unity ($c_1 + c_2 + c_3 = 1$) from the definition of the order parameter. We can use the zero Neumann boundary condition for the CH systems:

$$\nabla c_k \cdot \mathbf{n} = \nabla \mu_k \cdot \mathbf{n} = 0 \text{ on } \partial \Omega,$$

(2.12)

where $\mathbf{n}$ is the unit normal vector to $\partial \Omega$. The boundary condition (2.12) is natural and conserves the total mass in $\Omega$.

As shown in Fig. 6, the concentration field varies from 0.05 to 0.95 over a distance about $\xi = 2\sqrt{2} \tan^{-1}(0.9)$ from the equilibrium profile $c(x) = 0.5 + 0.5 \tanh(x/(\sqrt{2}\epsilon))$ in the infinite domain [38].

![Figure 6](image)

**Figure 6.** Phase transition of the equilibrium profile $c(x) = 0.5 + 0.5 \tanh(x/(\sqrt{2}\epsilon))$.

The advantages of LSM given above also apply PFM. Moreover, physical meanings of the order parameters can be applied to many physical phase states such as miscible, immiscible, and partially miscible. However, PFM needs a relatively large number of grid points near the interface because the phase-field function changes quickly near the interface. Moreover, it is important to choose appropriate $\epsilon$ values for accurate calculations. An excessively large $\epsilon$ can produce nonphysical solutions, whereas an excessively small $\epsilon$ can cause numerical difficulties [39].
3. Surface Tension Force

The singular surface tension force $\mathbf{SF}$ is represented by the continuum surface force (CSF) model [40]:

$$\mathbf{SF} = -\frac{\sigma \kappa \delta \Gamma \mathbf{n}}{We},$$

(3.1)

where $\kappa$ is the mean curvature of the interface and $\delta \Gamma$ is the surface delta function, $We = \rho(U^s)^2 L^s / \sigma$ is the Weber number, and $\sigma$ is the surface tension coefficient. Instead of $\delta \Gamma$, a smoothed delta function is usually used to adapt the CSF framework to spread the interfacial force to the nearby grid points in numerical implementations.

We describe how to define the surface tension force for each method in the remainder of this section.

3.1. IBM. The surface tension force in IBM is given by

$$\mathbf{SF}(x, t) = \sum_{k=1}^{2} \int_{\Gamma_k} \frac{1}{We_k} \mathbf{F}_k(s_k, t) \delta^2(x - \mathbf{x}_k(s_k, t)) \, ds,$$

(3.2)

$$\mathbf{F}_k(s_k, t) = \sigma_k \frac{\partial^2 \mathbf{x}_k(s_k, t)}{\partial s_k^2},$$

(3.3)

where $\mathbf{F}_k(s, t)$ is the boundary force defined for each particle of the $k$-th interface and $We_k$ is the Weber number with the $k$-th interface’s surface tension coefficient $\sigma_k$. The smoothed delta function $\delta(x)$ is defined as [41]:

$$\delta(x) = \begin{cases} 
0.125 \left( 3 - 2|x| + \sqrt{1 + 4|x|^2 - 4x^2} \right), & \text{if } |x| \leq 1, \\
0.125 \left( 5 - 2|x| - \sqrt{-7 + 12|x|^2 - 4x^2} \right), & \text{if } 1 < |x| \leq 2, \\
0, & \text{otherwise.} 
\end{cases}$$

(3.4)

We usually call (3.4) a four-point delta function. The schematics of the smoothed delta function $\delta(x)$ and its two-dimensional version $\delta^2(x) = \delta(x)\delta(y)$ are shown in Figs. 7 (a) and (b), respectively.

Note that $\partial^2 \mathbf{x}_k(s_k, t) / \partial s_k^2$ accounts for the interface curvature $\kappa$. Because the marker points of the moving interfaces and the grid points of the velocity field do not coincide directly, the interpolation is performed by using (3.4) to spread the surface tension force into the underlying grid points.

3.2. LSM. The surface tension force using the level set functions $\phi_k(x, t)$ is given by

$$\mathbf{SF}(x, t) = -\sum_{k=1}^{2} \frac{1}{We_{k,k+1}} \nabla \cdot \left( \frac{\nabla \phi_k}{|\nabla \phi_k|} \right) \delta_\alpha(\phi_k) \frac{\nabla \phi_k}{|\nabla \phi_k|},$$

(3.5)

where $We_{k,k+1}$ is the Weber number with the physical surface tension coefficient $\sigma_{k,k+1}$ between the fluids $k$ and $k+1$, which satisfies $\sigma_{k,k+1} = \sigma_k + \sigma_{k+1}$ for the phase specific surface...
tension coefficient $\sigma_k$ (see [19]). Recall that $\sigma_{3,1}$ would not be defined in our compound droplet case. In addition, the smoothed delta function $\delta_\alpha$ as follows [36]:

$$\delta_\alpha(\phi) = \begin{cases} 
\frac{1}{2\alpha} + \frac{1}{2\alpha} \cos \left( \frac{\pi x}{\alpha} \right), & \text{if } |\phi| \leq \alpha, \\
0, & \text{otherwise.} 
\end{cases}$$

Note that the interface curvature $\kappa$ is calculated by $\nabla \cdot \left( \nabla \phi_k / |\nabla \phi_k| \right)$ and that the unit normal vector $n$ is represented by $-\nabla \phi_k / |\nabla \phi_k|$. Here, we only consider the physical surface tension coefficient for $k = 1, 2$ because our focus is on the compound droplet case. Meanwhile, the phase specific surface tension coefficient $\sigma_k$ is uniquely defined as $\sigma_1 = (\sigma_{12} - \sigma_{23} + \sigma_{13})/2$, $\sigma_2 = (\sigma_{12} + \sigma_{23} - \sigma_{13})/2$, and $\sigma_3 = (-\sigma_{12} + \sigma_{23} + \sigma_{13})/2$ by the relation between physical surface tension coefficients.

3.3. PFM. The surface tension force using the phase-field functions $c_k(x, t)$ is written in the form

$$\mathbf{SF}(x, t) = -\frac{\alpha \epsilon}{W c_k} \nabla \cdot \left( \frac{\nabla c_k}{|\nabla c_k|} \right) |\nabla c_k| |\nabla k_2|,$$

where $\alpha$ is the variable to match the surface tension of the sharpened interface model and satisfies

$$\int_{-\infty}^{\infty} \alpha \epsilon \left| \nabla c_k^0(x,y) \right|^2 \, dx = 1.$$  

(3.8)

Here, $c_k^0(x,y) = 0.5[1 + \tanh(x/(2\sqrt{2}\epsilon))]$ is an equilibrium profile in the infinite domain $(-\infty, \infty) \times (-\infty, \infty)$ with $c_3 \equiv 0$ [42]. Therefore, we get $\alpha = 6\sqrt{2}$ from Eq. (3.8). Here, we only consider the case of $k = 2$ because our focus is on the compound droplet case.

4. Numerical Solution

In this section, we briefly describe the numerical solutions for the dimensionless NS equations (2.3) and (2.4), evolution equations (2.5) and (2.6), (2.7), (2.10) and (2.11) for IBM, LSM, and PFM in the two-dimensional domain.
4.1. Discretization. We first discretize the computational domain $\Omega = (a, b) \times (c, d)$ before solving the governing equations numerically. In Cartesian geometry, we consider $h = (b - a)/N_x$ and $h = (d - c)/N_y$ to be a uniform spatial step size where $N_x$ and $N_y$ are the numbers of cells in the $x$- and $y$-directions, respectively. This implies that $x_{ij} = (x_i, y_j)$ is located at the cell center where $x_i = a + (i - 0.5)h$ and $y_j = c + (j - 0.5)h$ for $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$. We denote $u(x_i, y_j, n\Delta t)$ as $u_{ij}^n$ where $\Delta t$ is a temporal step size in the discretized domain. The discrete gradient operator $\nabla_h$ is defined with the forward difference as

$$\nabla_h \phi_{ij}^n = \left( \frac{\phi_{i+1,j}^n - \phi_{ij}^n}{h}, \frac{\phi_{ij+1}^n - \phi_{ij}^n}{h} \right).$$

The discrete Laplacian operator $\Delta_h$ is defined with the central difference as

$$\Delta_h \phi_{ij}^n = \frac{\phi_{i+1,j}^n + \phi_{i-1,j}^n + \phi_{i,j+1}^n + \phi_{i,j-1}^n - 4\phi_{ij}^n}{h^2}$$

in Cartesian coordinates. We can use a staggered marker-and-cell (MAC) mesh that stores the pressure value $p_{ij}$ at a cell center and the velocity values $u_{i+1/2,j}$ and $v_{i,j+1/2}$ at the cell edges in the $x$- and $y$-directions, respectively (see Fig. 8). The level set function, phase-field function, and surface tension values $\phi_{k,ij}^n$, $c_{k,ij}^n$, and $SF_{ij}^n$ are also stored in the cell centers as pressure values.

![Figure 8. MAC mesh that stores pressure value $p_{ij}$ at a cell center and the velocity values $u_{i+1/2,j}$ and $v_{i,j+1/2}$ at the cell edges in the $x$- and $y$-directions, respectively.](https://example.com/figure8)

In contrast, IBM uses a set of Lagrangian points, whose coordinates do not depend on the MAC mesh grid, to discretize the immersed boundary. There are $M_1$ Lagrangian points $X_{1,l}^n = (X_{1,l}^n, Y_{1,l}^n)$ for $l = 1, \ldots, M_1$ to represent the inner droplet boundary and $M_2$ Lagrangian points $X_{2,l}^n = (X_{2,l}^n, Y_{2,l}^n)$ for $l = 1, \ldots, M_2$ to represent the outer droplet boundary.

4.2. Fluid solution. The temporal discretization of (2.3) and (2.4) is as follows:

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = -\nabla_h p_{ij}^{n+1} + \frac{1}{Re} \Delta_h u_{ij}^n + SF_{ij}^n - (u \cdot \nabla_h u)_{ij}^n,$$

$$\nabla_h \cdot u_{ij}^{n+1} = 0.$$

(4.1)

(4.2)
Here, the discrete surface tension force $\mathbf{SF}^n$ is calculated from the variable $\mathbf{X}^n$, $\phi^n_k$, or $c^n_k$. These represent the interfaces for each method, as discussed in the previous section 4.3. At each time step, (4.1) and (4.2) are solved to find $\mathbf{u}^{n+1}$ and $p^{n+1}$ from the given $\mathbf{u}^n$. We can apply the projection method, which was developed by Chorin [43]. Here, we present the outline of the main procedures of the method.

First, we consider the intermediate velocity $\tilde{\mathbf{u}}$ and split the discrete equation (4.1) as follows:

$$\frac{\tilde{\mathbf{u}} - \mathbf{u}^n}{\Delta t} = \frac{1}{Re} \Delta_h \mathbf{u}^n + \mathbf{SF}^n - (\nabla_h \cdot \mathbf{u})^n, \quad (4.3)$$

$$\frac{\mathbf{u}^{n+1} - \tilde{\mathbf{u}}}{\Delta t} = -\nabla_h p^{n+1}. \quad (4.4)$$

By applying $\nabla_h$ to both sides of (4.4) and the divergence free condition (4.2), we get the discrete Poisson equation for the pressure field:

$$\Delta_h p^{n+1} = \frac{1}{\Delta t} \left( \nabla_h \cdot \tilde{\mathbf{u}} \right). \quad (4.5)$$

We can solve (4.5) by using a multigrid method—specifically, V-cycles with a Gauss–Seidel relaxation.

In summary, we first update the intermediate velocity $\tilde{\mathbf{u}}$ from (4.3). Next, we update the pressure field by solving (4.5). Finally, the velocity $\mathbf{u}^{n+1}$ is calculated from (4.4).

4.3. Surface tension force. In this section, we present how to derive the discrete surface tension forces by using interface variables for each method. We store values of the force at cell-centers as the pressure values, i.e., $\mathbf{SF}^n_{ij}$ is defined in this section. However, the interpolated values $(SF^x_{i+\frac{1}{2},j}, SF^y_{i,j+\frac{1}{2}})$ at the cell-edges are used in the fluid equations (4.3) and (4.4) to match the stencils of the velocities.

4.3.1. IBM. By discretizing (3.2) and (3.3), we get the discrete surface tension force for IBM as below:

$$\mathbf{SF}^n_{ij} = \sum_{k=1}^{M_k} \sum_{l=1}^{M_k} \frac{1}{W e_k} \mathbf{F}^n_{k,l} \delta^2(x_{ij} - X^n_{k,l}) \Delta s_{k,l}, \quad (4.6)$$

$$\mathbf{F}^n_{k,l} = \sigma_k \left( \frac{X^n_{k,l+1} - X^n_{k,l}}{\Delta s_{k,l}} - \frac{X^n_{k,l} - X^n_{k,l-1}}{\Delta s_{k,l-1}} \right) / \Delta s_{k,l} + \Delta s_{k,l-1} \frac{1}{2}, \quad (4.7)$$

where $\Delta s_{k,l} = s_{k,l+1} - s_{k,l}$ is a line segment of each interface. Note that (4.7) is a multiple of the mean curvature and the normal vector at $X^n_{k,l}$. Refer to [44] for a more detailed description and calculation of (4.7).

4.3.2. LSM. In LSM, the discrete surface tension force is derived from (3.5), and the force is given as

$$\mathbf{SF}^n_{ij} = -\sum_{k=1}^{M_k} \frac{1}{W e_{k,k+1}} \nabla_h \cdot \left( \nabla_h (\phi^n_k) \nabla_h (\phi^n_k) \right) \delta_k (\phi^n_{k,ij}) \nabla_h (\phi^n_{k,ij}). \quad (4.8)$$
Note that $\alpha$ is usually taken as $h$ or $2h$. Here, we select $2h$.

4.3.3. PFM. With PFM, the discrete surface tension force can be derived similarly to the LSM case. The force is formulated as follows:

$$
\mathbf{SF}_{ij}^n = -\frac{6\sqrt{2}\epsilon}{Wc_2} \nabla h \cdot \left( \frac{\nabla h c_{2,ij}^n}{|\nabla h c_{2,ij}^n|} \right) |\nabla h c_{2,ij}^n| \nabla h c_{2,ij}^n.
$$

(4.9)

Because the interfaces are already diffused when using PFM, a delta function is not required to represent the surface tension force in (4.9).

4.4. Governing equations of interfaces. In this section, we discretize the governing equations of interfaces for each method and present their numerical solutions.

4.4.1. IBM. By using the updated fluid velocity $u_i^{n+1}$ in (4.3) and (4.4), we can evaluate the immersed boundary velocity $U_{i,l}^{n+1}$ and new boundary position $X_{i,l}^{n+1}$ according to the following equations:

$$
U_{i,l}^{n+1} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} u_{ij}^{n+1} \delta_2(x_{ijk} - X_{i,l}^{n+1} h^2),
$$

(4.10)

$$
X_{i,l}^{n+1} = X_{i,l}^{n} + \Delta t U_{i,l}^{n+1},
$$

(4.11)

where $k = 1, 2$ and $l = 1, \cdots, M_k$. We can also apply the algorithms introduced by [44] and [45] for the high-quality distribution of the interface points and the area conservation property, respectively. See each reference for detailed descriptions of the properties.

4.4.2. LSM. The numerical solution of the evolution equation (2.7) is derived from the following discrete transport equation:

$$
\frac{\phi_{k,ij}^{n+1} - \phi_{k,ij}^n}{\Delta t} = -\frac{u_{i+1/2,j}^n (\phi_{k,i+1,j}^n - \phi_{k,ij}^n) + u_{i-1/2,j}^n (\phi_{k,i-1,j}^n - \phi_{k,ij}^n)}{2h} - \frac{v_{i,j+1/2}^n (\phi_{k,i,j+1}^n - \phi_{k,ij}^n) + v_{i,j-1/2}^n (\phi_{k,i,j-1}^n - \phi_{k,ij}^n)}{2h}.
$$

(4.12)

It is a very basic numerical solver and more accurate, stable, and conservative numerical methods such as a WENO-type difference and Godunov’s scheme can be founded in [15]. Next, the discrete equations of the reinitialized steps (2.8) and (2.9) are given as

$$
\tilde{d}_{k,ij} = d_{k,ij} + \Delta \tau \frac{\phi_{k,ij}^n}{\sqrt{(\phi_{k,ij}^n)^2 + \beta^2}} \left( 1 - \sqrt{(D_x d_{k,ij})^2 + (D_y d_{k,ij})^2} \right)
$$

(4.13)

where the initial condition of $d_{k,ij}$ is given as $\phi_{k,ij}^n$. The operators $D_x$ and $D_y$ are discrete differentiations in the WENO sense [36, 46] with respect to $x$ and $y$, respectively. After a few iterations, we can update $\phi_{k,ij}$ as the result of the final $\tilde{d}_{k,ij}$. 
4.4.3. PFM. To discretize the CH equation, we consider the nonlinear splitting scheme over time. If the variable Lagrangian multiplier \( \gamma(c_1, c_2, c_3) \) is determined by \( c_1^n, c_2^n, \) and \( c_3^n \), i.e., \( \gamma \) is treated explicitly, the solutions at the time level \( n + 1 \) have no relation to each other. This implies that the ternary component CH system can be solved in a decoupled manner. Therefore, we can discretize (2.10) and (2.11) for \( k = 1, 2 \) as follows:

\[
\frac{c_{k,ij}^{n+1} - c_{k,ij}^n}{\Delta t} = \frac{1}{P_e} \Delta h \mu_{k,ij}^{n+1} + \Delta h \left( \gamma(c_{1,ij}^n, c_{2,ij}^n, c_{3,ij}^n) - 0.25 c_{k,ij}^n \right) - \frac{u_{i+\frac{1}{2},j}^n (c_{k+1,ij}^n - c_{k,ij}^n) + u_{i-\frac{1}{2},j}^n (c_{k,i+1,ij}^n - c_{k,i,j-1}^n)}{2h} - \frac{v_{i,j+\frac{1}{2}}^n (c_{k,i,j+1}^n - c_{k,i,j}^n) + v_{i,j-\frac{1}{2}}^n (c_{k,i,j+1}^n - c_{k,i,j}^n)}{2h}.
\]

(4.14)

\[
\mu_{k,ij}^{n+1} = f(c_{1,ij}^{n+1}, c_{2,ij}^{n+1}, c_{3,ij}^{n+1}) + 0.25 c_{k,ij}^{n+1} - C \Delta h c_{k,ij}^{n+1}.
\]

(4.15)

This means that we can solve the ternary CH system by solving the binary CH equation twice. A nonlinear multigrid method can be used to solve (4.14) and (4.15). A detailed description is given by[47].

5. Numerical experiments

Before performing numerical experiments, we note that a relation between the \( \epsilon \) value and the width of the transition layer for PFM. As mentioned in section 2.3, the equilibrium state of the concentration has a tangent hyperbolic profile. If we want to set \( \epsilon \) value to be about \( m \) grid points, the value is set as \( \epsilon = h m / 4 \sqrt{2} \tanh^{-1}(0.9) \) [48, 49]. Unless otherwise specified, we use \( \epsilon = \epsilon_4 \).

5.1. Pressure difference. The pressure gradient and surface tension force are balanced in the absence of viscous, gravitational, and other external forces. The pressure difference can be expressed by \( [p]_\Gamma = \sigma / R \) with Laplace’s formula for a spherical liquid surrounded by an ambient fluid in a two-dimensional space, where \( R \) is the radius of the droplet. Therefore, the pressure jump of the compound droplet is defined by

\[
[p]_\Gamma = [p]_{\Gamma_1} + [p]_{\Gamma_2} = \frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2}.
\]

(5.1)

Here, circular droplets with \( R_1 = 0.5 \) and \( R_2 = 1 \) were taken as the initial conditions, and \( \sigma_1 = \sigma_2 = 1 \) was used. Our numerical simulation was in the domain \((0, 3) \times (0, 3)\) with the uniform grids \( h = 1/2^n \) for \( n = 5, 6, 7, 8 \) and 9 for one time step. Table 1 lists the convergence of the pressure jump for each method between the ambient fluids and inner droplets as we refined the mesh size. Figures 9(b) and (c) show the pressure field on the xy-plane and along the line \( y = 1.5 \), respectively, for the compound droplets.
TABLE 1. Numerical pressure jump between the ambient fluids and inner droplet as the mesh size was refined for each method. The theoretical pressure jump was 3.

<table>
<thead>
<tr>
<th>Mesh size (h)</th>
<th>Method</th>
<th>1/32</th>
<th>1/64</th>
<th>1/128</th>
<th>1/256</th>
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</thead>
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<tr>
<td></td>
<td>LSM</td>
<td>3.0350</td>
<td>3.0216</td>
<td>3.0097</td>
<td>3.0066</td>
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<tr>
<td></td>
<td>PFM</td>
<td>2.5435</td>
<td>2.8875</td>
<td>2.9728</td>
<td>2.9854</td>
</tr>
<tr>
<td></td>
<td>IBM</td>
<td>3.0027</td>
<td>3.0015</td>
<td>3.0017</td>
<td>3.0018</td>
</tr>
</tbody>
</table>

Figure 9. (a) Schematic illustration of a drop-in-drop surrounded by ambient fluid. (b) Pressure field for compound drop. (c) Slice of the pressure field at \( y = 1.5 \) (dotted line in (a)) for each method.

Figure 10. Schematic of a compound drop in the ambient fluid under a simple shear flow.

5.2. Deformation of compound droplet under shear flow. The imposed flow was a simple shear flow given by \( U = \dot{\gamma} y \) and \( v = 0 \), where \( \dot{\gamma} \) is the shear rate. Figure 10 presents a schematic of a compound drop in the ambient fluid under a simple shear flow.
We first introduce the Taylor deformation number $D$, defined as $D = (L - B)/(L + B)$, where $L$ and $B$ are the major and minor semiaxes of the droplet (See Fig. 10). $D$ is usually used to measure the magnitude of the droplet deformation.

We confirm the effect of parameters such as the numbers of Lagrangian particles $M_1$ and $M_2$ in IBM, the number of repetitions of the reinitialization process in LSM, and $Pe$ in PFM. The simulations are performed on a squared domain $\Omega_h = (-2, 2) \times (-2, 2)$ with a $128 \times 128$ meshgrid for 12500 iterations unless otherwise stated. The radii of the inner and outer droplets are $R_1 = 0.5$ and $R_2 = 1$, respectively. We used the parameters $\Delta t = 0.1h^2Re$, $Pe = 1$, $We = 0.2$, and $\dot{\gamma} = 0.5$, i.e., the velocity on the top of the domain is $1$.

Figure 11 represents the shapes of the compound droplet with different numbers of immersed boundary points with $64 \times 64$ meshgrid. The values in the legend mean initial distances of each Lagrangian particle, for example, about 4 particles are in the one mesh grid of the $4h$ case. For each case, $(M_1, M_2)$ are (101, 51), (51, 26), and (26, 13) from top to bottom. It is convergent enough when there are at least one particle in one mesh grid as shown in the Fig. 11 and we will set the distance between each particle as about $h/4$.

![Figure 11](image-url)

**FIGURE 11.** Shapes of the compound droplet with different number of immersed boundary points. The values in the legend mean initial distances of each Lagrangian particle.

We also compare the effect of $n_r$ which is the number of repetition of reinitialization process. Figure 12 shows deformed shape of the compound droplet using the contour line at $-2h$, 0, and $2h$ level, respectively. As shown in Fig. 12, the results with $n_r = 0$ which mean that the reinitialization process has not taken place, have the difference with the numerical results when $n_r = 1$ or 5. The result is compatible with the suggestion, in the reference [36].

Before checking the effect of $Pe$, we first confirm that the choice of $\epsilon_4$ is suitable enough. In Fig. 13, changes of deformation number $D$ is presented varying time until $T = 0.2941$. The result shows that $\epsilon_4$ and $\epsilon_6$ cases have a better consistency with LSM and IBM than $\epsilon_2$. 
Next, the simulations are performed to compare the effect of $Pe$ in PFM. Figure 14 represents the deformed shapes of compound droplet with different $Pe$ values. The top and bottom rows represent contour lines at 0.1, 0.5, and 0.9 levels of $\phi_1$ and $\phi_2$, respectively. The $Pe$ values are $0.01/\epsilon$, $1/\epsilon$, and $100/\epsilon$ in Fig. 14(a), (b), and (c), respectively. The result shows that degree of deformation could be too tenuous in the smallest $Pe$ case (Fig. 14(a)) and thickness of the contour lines is not uniform in the biggest $Pe$ case (Fig. 14(c)). Moreover, we compare shapes of compound droplets using LSM and PFM with different $Pe$ values in Fig. 15. The results shows that the case using $Pe = 1/\epsilon$ is the most consistent with LSM case. Therefore, We use $Pe = 1/\epsilon$ in the later simulations unless otherwise stated.
Figure 14. Shapes of compound droplet with different Pe values. The top and bottom rows represent contour lines at 0.1, 0.5, and 0.9 levels of \( \phi_1 \) and \( \phi_2 \), respectively. The Pe values are (a) 0.01/\( \epsilon \), (b) 1/\( \epsilon \), and (c) 100/\( \epsilon \), respectively.

Figure 15. Shapes of compound droplets using LSM and PFM with different Pe values.
5.3. Comparison with each method in specific cases.

5.3.1. Multiple compound droplets case. We perform the simulations of specific cases which are suitable to distinguish advantages and disadvantages for each method. At first, the multiple compound droplets case like in the right side of Fig. 1 is chosen. We only consider two couples of emulsion for simplicity. The initial radii are $R_1 = 0.5$ and $R_2 = 1$ for each emulsion whose centers are located at $(-0.74, -0.74)$ and $(0.74, 0.74)$ on $(-2, 2) \times (-2, 2)$. The other parameters, except $\dot{\gamma} = 0$, have same values used in the simulations of section 5.2. If there is no flow outside of the droplets, emulsions do not collide or merge each other even though their distance is comparatively near in vivo.

Figure 16 represents the shapes of droplets at initial condition and at $T = 0.1221$ solved by IBM, LSM, and PFM. The result shows that only IBM maintains the topological phase. It implies that IBM is the best in three methods when the densely distributed compound droplets are stabilized, or prevented the coalescence between droplets by employing surfactants. On the other hand, LSM or PFM is recommended to model the merged droplets without surfactants.

5.3.2. Different radius of smaller droplet. The coalescence can be happened between not only different emulsions, but also between an inner droplet and an outer droplet in one emulsion. Here, we consider another specific cases with different radii of smaller droplets to check whether each method can be treated such conditions well or not. We fix $R_1 = 0.5$, $Re = 1$, $We = 0.1$, $\dot{\gamma} = 0.5$, $\Omega = (-2, 2) \times (-2, 2)$, $\Delta t = 0.1h^2 Re$ and $T = 22500\Delta t$.

Figure 17 represents the shapes of droplets at $T$ using each method with different $R_2$. The emulsions have the same shape using any methods in a shear flow in Fig. 17 (a) $R_2 = 0.7$; however, the inner droplet whose initial radius $R_2$ is 0.8 is broken when only PFM is implemented as shown in Fig. 17 (b). The brokenness stems from a numerical error when interfaces of outer and inner droplets are too close since PFM uses a diffused-interface. From the result, we suggest that applying IBM or LSM might draw a better result rather than using PFM when sizes of outer and inner droplets in an emulsion are too similar to avoid merging each other.

5.3.3. Mass conservation. To compare mass conservation property in each method, we perform the simulation under the condition with a strong surface tension. The initial compound droplets, whose radii are $R_1 = 0.5$ and $R_2 = 1$, are located at the center of the domain $\Omega = (-2, 2) \times (-2, 2)$. We choose $We = 0.001$ and other parameters are same as the simulation in section 5.3.1.

Figure 18 represents the shapes of the droplets (a) at $T = 0.4900$ using IBM, (b) at $T = 2.9591$ using LSM, and (c) at $T = 2.9591$ using PFM (dotted line) with the initial condition (solid line). Except PFM, the mass of droplets does not conserve and their area shrinks after several iterations even though there are no external force without surface tension force. Otherwise, the droplets solving PFM conserves its shape during longer or same temporal evolution than other methods. Therefore, using PFM is recommended when the simulation is performed with a strong surface tension from our result.
5.3.4. Different shear rate. We checked the suitable or recommended cases using IBM or PFM by previous simulations. In this section, we examine the merits of using LSM rather than other two methods. Clearly, LSM has a better performance than IBM when there is any topological changes for droplets. Furthermore, it could be confirmed that LSM has an advantage over PFM of choosing compatible parameter values.

The initial radii, computational domain, and other parameters except the Peclet number $Pe$, the final time $T$ and shear rate $\dot{\gamma}$ are same as the simulation in section 5.3.1. The simulations with different $\dot{\gamma} = 0.5$ and 5 are performed with $Pe = 0.1/\epsilon$ until $T = 0.2941$. In Fig. 19, the changes of the deformation number varying time for LSM, PFM, and IBM (upper row) and the shapes of droplets at time $T$ (lower row) are shown. As shown in Fig. 19, the evolution of droplet shapes and deformation numbers are distinct with different shear rate. To resolve this phenomenon, $Pe$ should be differently chosen for different conditions, i.e., choosing suitable $Pe$ is quite important in PFM. However, LSM does not have this restriction and is independent
on this choice problem. We suggest that LSM is better method than other two methods when comparison of different shear rate conditions is required.

6. Conclusion

The main goal of this paper was to review and compare three different methods such as immersed boundary, level set, and phase-field methods for incompressible, immiscible ternary fluid flows. We performed the simulations to investigate advantages and disadvantages of each method. Immersed boundary method was good for defining multiple droplets closely located each other and prevented from the coalescence. However, it was difficult to model topological transition phenomena. Level set method can deal with merging and pinch-off of interfaces, however it suffered from mass conservation. Phase-field method had a good property of mass
conservation, however it had to choose an appropriate relaxation parameter such as mobility. Therefore, depending on one’s need for modeling, we chose the most suitable method.

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<table>
<thead>
<tr>
<th>Title</th>
<th>Authors</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIVIDED DIFFERENCES AND POLYNOMIAL CONVERGENCES</td>
<td>Suk Bong Park, Gang Joon Yoon, and Seok-Min Lee</td>
<td>1</td>
</tr>
<tr>
<td>DYNAMIC CHARACTERISTICS OF A ROTATING TIMOSHENKO BEAM</td>
<td>Babatope Omolofe and Segun Nathaniel Ogunyebi</td>
<td>17</td>
</tr>
<tr>
<td>LARGE EDDY SIMULATION OF TURBULENT CHANNEL FLOW USING ALGEBRAIC WALL MODEL</td>
<td>Muhammad Saiful Islam Mallik and Md. Ashraf Uddin</td>
<td>37</td>
</tr>
<tr>
<td>EXISTENCE AND CONTROLLABILITY OF FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL SYSTEMS WITH STATE-DEPENDENT DELAY IN BANACH SPACES</td>
<td>Subramanian Kailasavalli, Selvaraj Suganya, and Mani Mallika Arjunan</td>
<td>51</td>
</tr>
<tr>
<td>COMPARISON OF NUMERICAL METHODS FOR TERNARY FLUID FLOWS: IMMERSED BOUNDARY, LEVEL-SET, AND PHASE-FIELD METHODS</td>
<td>Seunggyu Lee, Darae Jeong, Yongho Choi, and Junseok Kim</td>
<td>83</td>
</tr>
</tbody>
</table>