Journal of the KSIAM

Editor-in-Chief
Kwak, Minkyu (Chonnam National University, Korea)

Associate Editor-In-Chief
Chen, Zhiming (Chinese Academy of Sciences, China)
Lee, June-Yub (Ewha w. University, Korea)
Tang, Tao (Hong Kong Baptist University, Hong Kong)

Managing Editors
Cho, Jin-Yeon (Inha University, Korea)
Kim, Junseok (Korea University, Korea)

Editorial Board
Ahn, Hyung Taek (University of Ulsan, Korea)
Ahn, Jaemyung (KAIST, Korea)
Ha, Tae Young (NIMS, Korea)
Han, Pigong (Chinese Academy of Sciences, P.R.China)
Hodges, Dewey (Georgia Tech., USA)
Hwang, Gang Uk (KAIST, Korea)
Jung, Il Hyo (Pusan Natl University, Korea)
Kim, Hyea Hyun (Kyung Hee University, Korea)
Kim, Hyoun Jin (Seoul Natl University, Korea)
Kim, Jeong-ho (Inha University, Korea)
Kim, Kyu Hong (Seoul Natl University, Korea)
Kim, Yunho (UNIST, Korea)
Lee, Eunjung (Yonsei University, Korea)
Lee, Jihoon (Chung-Ang University, Korea)
Li, Zhilin (North Carolina State University, USA)
Liu, Hongyu (Hong Kong Baptist University, Hong Kong)
Min, Chohong (Ewha w. University, Korea)
Park, Soo Hyung (Konkuk University, Korea)
Shim, Gyoocheol (Ajou University, Korea)
Shin, Eui Seop (Chonbuk Natl University, Korea)
Shin, Sang Joon (Seoul Natl University, Korea)
Simone Garatti (Politecnico di Milano, Italy)
Sohn, Sung-Ik (Gangneung-Wonju Natl University, Korea)
Wang, Zhi Jian (North Carolina State University, USA)

Aims and Scope

Journal of the KSIAM is devoted to theory, experimentation, algorithms, numerical simulation, or applications in the fields of applied mathematics, engineering, economics, computer science, and physics as long as the work is creative and sound.

Copyright

It is a fundamental condition that submitted manuscripts have not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their article is transferred to the publisher if and when the article is accepted for publication. The copyright covers the exclusive rights to reproduce and distribute the article, including reprints, photographic reproductions, microform or any other reproductions of similar nature, and translation. Photographic reproduction, microform, or any other reproduction of text, figures or tables from this journal is prohibited without permission obtained from the publisher.

Subscription

Subscription rate for individuals is $50/year; Domestic university library rate is $150/year; Foreign or industrial institute rate is $300/year.

Contact info.

THE KOREAN SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS
Room 405, Industry-University Research Center, Yonsei Univ.
50, Yonsei-ro, Seodaemun-gu, Seoul 03722, Republic of Korea
Jung, Eunok (e-mail : jksiam@ksiam.org) Tel: +82-2-2123-8078

Journal of KSIAM was launched in 1997, is published four times a year by the Korean Society for Industrial and Applied Mathematics. Total or part of the articles in this journal are abstracted in KSCI, Mathematical Reviews, CrossRef, Korean Science, and NDSL. Full text is available at http://www.ksiam.org/archive.

Copyright © 2018, the Korean Society for Industrial and Applied Mathematics
The Korean Society for Industrial and Applied Mathematics

Volume 22, Number 1, March 2018

Contents

AN OPTIMAL BOOSTING ALGORITHM BASED ON NONLINEAR CONJUGATE GRADIENT METHOD
JOOYEON CHOI, BORA JEONG, YESOM PARK, JIWON SEO, CHOHOONG MIN ........................................... 1

ACCELERATION OF MACHINE LEARNING ALGORITHMS BY TCHEBYCHEV ITERATION TECHNIQUE
MIKHAIL P. LEVIN ........................................................................................................... 15

STABILITY OF DELAY-DISTRIBUTED HIV INFECTION MODELS WITH MULTIPLE VIRAL PRODUCER CELLS
A. M. ELAIW, E. KH. ELNAHARY, A. M. SHEHATA, M. ABUL-EZ .................. 29

EFFECT OF PERTURBATION IN THE SOLUTION OF FRACTIONAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS
MOHAMMED. S. ABDO, SATISH. K. PANCHAL .......................................................... 63

COMPARATIVE STUDY OF NUMERICAL ALGORITHMS FOR THE ARITHMETIC ASIAN OPTION
JIAN WANG, JUNGYUP BAN, SEONGJIN LEE, CHANGWOO YOO .......... 75
AN OPTIMAL BOOSTING ALGORITHM BASED ON NONLINEAR CONJUGATE
GRADIENT METHOD

JOOYEON CHOI, BORA JEONG, YESOM PARK, JIWON SEO, CHOHONG MIN†

DEPARTMENT OF MATHEMATICS, EWha WOMANS UNIVERSITY, KOREA
E-mail address: chohong@ewha.ac.kr

ABSTRACT. Boosting, one of the most successful algorithms for supervised learning, searches
the most accurate weighted sum of weak classifiers. The search corresponds to a convex pro-
gramming with non-negativity and affine constraint. In this article, we propose a novel Con-
jugate Gradient algorithm with the Modified Polak-Ribiera-Polyak conjugate direction. The
convergence of the algorithm is proved and we report its successful applications to boosting.

1. INTRODUCTION

Boosting refers to constructing a strong classifier based on the given training set and weak
classifiers, and has been one of the most successful algorithms for supervised learning [1, 8, 9].
A first and seminal boosting algorithm, named AdaBoost, was introduced by [3]. AdaBoost can
be understood as a gradient descent algorithm to minimize the margin, a measure of confidence
of the strong classifier [3, 7, 10].

Though simple and explicit, AdaBoost is still one of the most popular boosting algorithms
for classification and supervised learning. According to the analysis by [10], AdaBoost tries to
minimize a smooth margin. The hard margin refers to a direct sum of the confidence of each
data, and the soft margin takes the log-sum-exponential function. LPBoost invented by [2, 4]
minimizes the hard margin, resulting in a linear programming. It is observed that LPBoost does
not perform well in most cases compared to AdaBoost [11].

The strong classifier is a weighted sum of the weak classifiers. AdaBoost determines the
weight by the stagewise and unconstrained gradient descent. AdaBoost increases the support of
the weight one-by-one for each iteration. Due to the stagewise search and the stop of its search
when the support is enough, AdaBoost is not the optimal search.

The optimal solution needs to be sought among all the linear combinations of weak classi-
cifiers. The optimization becomes valid with a constraint that sum of the weights is bounded,
and the bound was observed to be proportional to the support size of the weight [11].
In this article, we propose a new and efficient algorithm that solves the constrained optimization problem. Our algorithm is based on the Conjugate-Gradient method with non-negativity constraint by [5]. They showed the convergence of CG with the modified Polak-Ribiera-Polyak (MPRP) conjugate direction.

The optimization that arise in Boosting has the non-negativity constraint and an affine constraint. Our novel algorithm extends the CG with non-negativity to hold the affine constraint.

We present a mathematical setting of boosting in section 2, introduce the novel CG and prove its convergence in section 3, and report its applications to benchmark problems of boosting in section 4.

2. MATHEMATICAL FORMULATION OF BOOSTING

In boosting, one is given with a set of training examples \( \{ x_1, \cdots, x_M \} \) with binary labels \( \{ y_1, \cdots, y_M \} \subset \{ \pm 1 \} \), and weak classifiers \( \{ h_1, h_2, \cdots, h_N \} \). Each weak classifier \( h_j \) gives a label to each example, and hence it is a function \( h_j : \{ x_1, \cdots, x_M \} \to \{ \pm 1 \} \).

A strong classifier \( F \) is made up of a weighted sum of the weak classifiers, so that
\[
F(x) := \sum_{j=1}^{N} w_j h_j(x)
\]
for some \( w \in \mathbb{R}^N \) with \( w \geq 0 \).

For each example \( x_i \), a label +1 is put when \( F(x_i) > 0 \), and −1 otherwise. Hence the strong classifier is successful on \( x_i \) if the sign of \( F(x_i) \cdot y_i \) matches the given label \( y_i \), or \( \text{sign}(F(x_i)) \cdot y_i = +1 \) and unsuccessful on \( x_i \) if \( \text{sign}(F(x_i)) \cdot y_i = -1 \).

The hard margin, which is a measure of the fidelity of the strong classifier, is thus given as
\[
\text{(Hard margin)} : -\sum_{i=1}^{M} \text{sign}(F(x_i)) \cdot y_i
\]

When the margin is smaller, more of \( \text{sign}(F(x_i)) \cdot y_i \) are +1, and \( F \) can be said to be more reliable. Due to the discontinuity present in the hard margin, the soft margin of AdaBoost takes the form, via the monotonicity of log and exponential,
\[
\text{(Soft margin)} : \log \left( \sum_{i=1}^{M} e^{-F(x_i) \cdot y_i} \right)
\]

The composition of log-sum-exponential functions is referred to lse. Let us denote by \( A \in \{ \pm 1 \}^{M \times N} \), the matrix whose entry is \( a_{ij} = h_j(x_i) \cdot y_i \). Then the soft margin can be simply put to \( \text{lse}(-Aw) \), where \( w = [w_1, \cdots, w_N]^T \).

The main goal of this work is to find out a weight that minimizes the soft margin, which is to solve the following optimization problem.
\[
\text{minimize } \text{lse}(-Aw) \text{ subject to } w \geq 0 \text{ and } w \cdot 1 = \frac{1}{T} \quad (2.1)
\]

Here, \( A \in \{ \pm 1 \}^{M \times N} \) is a given matrix from the training data and weak classifiers, and \( T \) is a parameter to control the support size of \( w \). We finish this section with the lemma that shows that the optimization is a convex programming, and we will introduce a novel algorithm to solve the optimization.

**Lemma 1.** lse \((-Aw)\) is a convex function with respect to \( w \).
Proof. Given any \( w, \tilde{w} \in \mathbb{R}^N \) and any \( \theta \in (0, 1) \), let \( z = -Aw \) and \( \tilde{z} = -A\tilde{w} \).

\[
\begin{align*}
&= (1 - \theta) \text{lse} (z) + \theta \text{lse} (\tilde{z}) \\
&= \log \left( \left( \sum_{i=1}^{M} e^{z_i} \right)^{1-\theta} \cdot \left( \sum_{i=1}^{M} e^{\tilde{z}_i} \right)^{\theta} \right) \\
&= \log \left( \left( \sum_{i=1}^{M} e^{z_i(1-\theta)} \right)^{1-\theta} \cdot \left( \sum_{i=1}^{M} e^{\tilde{z}_i\theta} \right)^{\theta} \right) \\
&\leq \log \left( \sum_{i=1}^{M} e^{z_i(1-\theta)} \cdot e^{\tilde{z}_i\theta} \right) \text{ by the Hlder's inequality.} \\
&= \text{lse} \left( (1 - \theta) z + \theta \tilde{z} \right) \\
&= \text{lse} \left( -A \left( (1 - \theta) w + \theta \tilde{w} \right) \right).
\end{align*}
\]

\[\square\]

3. Conjugate Gradient Method

In this section, we introduce a conjugate gradient method for solving the convex programming (2.1).

\[
\min f(w) \text{ subject to } w \geq 0 \text{ and } w \cdot 1 = \frac{1}{T}
\]

Throughout this section, \( f(w) \) denotes the convex function \( \text{lse}(-Aw) \), and \( g(w) \) denotes its gradient \( \nabla f(w) \). Let \( d \) be the direction at a position \( w \) to seek the next position. When \( w \) is located on the boundary of the constraint, \( w \) cannot be moved to a certain direction \( d \) due to the constraints \( \{ w \in \mathbb{R}^N \mid w \geq 0 \text{ and } w \cdot 1 = 1/T \} \).

We refer \( d \) to be feasible at \( w \), if \( w + \alpha d \) stays in the constraint set for sufficiently small \( \alpha > 0 \).

**Definition 1. (Feasible direction) Given a direction \( d \in \mathbb{R}^N \) at a position \( w \in \mathbb{R}^N \) with \( w \geq 0 \) and \( w \cdot 1 = \frac{1}{T} \), the feasible direction \( d^f = d^f(d, w) \) associated with \( d \) is defined as the nearest vector to \( d \) among the feasible directions at the position. Precisely, it is defined by the minimization

\[
d^f = \arg\min_{y \in \mathcal{I}(w) \geq 0 \text{ and } y \cdot 1 = 0} \| d - y \|
\]

where \( \mathcal{I}(w) = \{ i \mid w_i = 0 \} \). The domain of the minimization is convex, and the functional is strictly convex and coercive, so that \( d^f \) is determined uniquely.

Define the index set \( J(w) = \{ j \mid w_j > 0 \} \).

**Lemma 2.** \( \forall \omega \text{ with } w \cdot 1 = \frac{1}{T}, \forall d, \text{ let } d^f = d^f(d, w), \text{ then } w + \alpha d^f \geq 0 \text{ and } (w + \alpha d^f) \cdot 1 = \frac{1}{T} \) for sufficiently small \( \alpha \geq 0 \).
Figure 1. For a given direction $d$ at a position $w$, a colored region is a feasible region of $d$. Since $d^f$ is the nearest vector to $d$ among the feasible directions at $w$, it is the orthogonal projection of $d$ onto the colored region (a). $d^f$ is decomposed into two orthogonal components, $d^f = d^t + d^w$, where $d^t$ is the orthogonal projection of $d^f$ onto the tangent space (b).

Proof. Clearly, $\forall \alpha, (w + \alpha d^f) \cdot 1 = w \cdot 1 + 0 = 1 / T$. 

$$\forall \alpha \geq 0, \quad \text{if } i \in I(w), \quad w_i + \alpha d^f_i = 0 + \alpha d^f_i \geq 0, \text{ and}$$
$$\quad \text{if } j \in J(w), \quad w_j + \alpha d^f_j \geq w_j - \alpha \left(\left|d^f_j\right| + 1\right).$$

Thus for any $\alpha \geq 0$ with $\alpha \leq \min_{j \in J(w)} \frac{w_j}{\left|d^f_j\right| + 1}$, $w + \alpha d^f \geq 0$. □

Proposition 1. (Calculation of the feasible direction) For a given direction $d$ at a position $w$, $d^f$ is calculated as

$$\begin{cases} d^f_i = (d_i - r)^+, & i \in I \\ d^f_j = d_j - r, & j \in J \end{cases}$$

where $r$ is a zero of $(d_J - r \cdot 1_J) \cdot 1_J + (d_{I_1} - r)^+ + \cdots + (d_{I_k} - r)^+, k = |I|.$

Proof. Since $d^f$ is the KKT point of (Def.2), there exist $\lambda_I$ and $\mu$ such that

$$d^f - d = \begin{bmatrix} \lambda_I \\ 0 \end{bmatrix} - r \cdot 1, \text{ with } d^f_I \geq 0, \lambda_I \cdot d^f_I = 0, d^f \cdot 1 = 0.$$  

From these conditions, we get $d^f_j = d_j - r \cdot 1_J$ and $d_i - r = d^f_i - \lambda_i$, for $i \in I$. 

If $d_i - r > 0$, then $d^f_i > 0$ and $\lambda_i = 0$. Thus, $d^f_i = d_i - r$. 

If $d_i - r \leq 0$, then $d^f_i = 0$ and $\lambda_i \geq 0$. 

4 J. CHOI ET AL.
Algorithm 1 Computing the feasible direction, $d^f$.

Input: $w, d$

Output: $d^f$

Procedure:
1: Make index sets $I(w) := \{ i | w(i) = 0 \}$ and $J(w) := \{ j | w(j) > 0 \}$
2: Define a function $p(r) = \sum_{i \in I} (d_i - r) + \sum_{j \in J} (d_j - r)$. And find $\alpha = \arg \max_{i \in I, p(d_i) > 0} i$
3: $r \leftarrow \text{zero of } \sum_{j \in J} (d_j - r) + \sum_{i \in I, i \leq \alpha} (d_i - r) - \sum_{i \in I, i > \alpha} (d_i - r)$
4: Compute $d^f$ as following:
   $d^f_i = \begin{cases} d_i - \max\{0, -d_i + r\} + r, & i \in I \\ d_i - r, & i \in J \end{cases}$

By combining these two, we have $d^f_i = (d_i - r)^+$, for $i \in I$. Since $d^f \cdot 1 = 0$, $d^f \cdot 1 = d^f_1 \cdot 1 + d^f_i \cdot 1_i$

$$= (d_1 - r) \cdot 1 + (d_{i_1} - r)^+ + (d_{i_2} - r)^+ + \cdots + (d_{i_k} - r)^+ = 0.$$ $r$ is the root of the monotonically decreasing function. The monotone function is piecewisely linear, so that the root can be easily obtained by probing intervals between $\{d_1, \cdots, d_k\}$ where the monotone function changes the sign. After $r$ is obtained, $d^f$ is defined as stated. □

**Definition 2.** (Tangent Space) The domain for $w$ is the simplex $\{ w | w \geq 0 \text{ and } w \cdot 1 = 0 \}$. When $w > 0$, $w$ is inside and the tangent space $T = 1^\perp$. When $w_i = 0$ and $w_j > 0 (\forall j \neq i)$, $w$ is on the boundary, and the tangent space becomes smaller $T_w = \{ 1, e_i | i \in I \}^\perp$. In general, we define the tangent space of $w$ as $T_w := [1 \cup \{ e_i | w_i = 0 \}]^\perp \subset \mathbb{R}^N$.

**Definition 3.** (Orthogonal decomposition of direction) Given a direction $d \in \mathbb{R}^N$ on a position $w \in \mathbb{R}^N$ with $w \geq 0$ and $1 \cdot w = \frac{1}{w}$, the direction is decomposed into three mutually orthogonal vectors; tangential, wall, and non-feasible components.

$$d = d^f + (d - d^f)$$

$$= d^f + d^w + (d - d^f).$$

Here, $d^f = d^f(d, w)$ is the feasible direction. $d^f$ is its orthogonal projection onto the tangent space $T_w$, and $d^w = d^f - d^f \in T_w^\perp$. Their mutual orthogonality is proved below.
Lemma 3. The above vectors $d^t$, $d^w$, and $(d - d^f)$ are orthogonal to each other. Furthermore, $d - d^f \in T_w^\perp$.

Proof. By the definition of the orthogonal projection, $d^t \perp d^w$. The KKT condition of the minimization (3.1) is

$$
(d^f - d^t) = \begin{bmatrix}
\lambda_I \\
0
\end{bmatrix} + rI \text{ for some } \lambda_I \geq 0 \text{ with } \lambda_I \cdot d^f = 0 \text{ and some } r \text{ with } (d \cdot 1) = 0,
$$

where $I = I(w)$. Since $d^t \in T_w = \{1, e_I\}^\perp$, $d^t \cdot \begin{bmatrix}
\lambda_I \\
0
\end{bmatrix} = 0$ and $d^t \cdot 1 = 0$, thus $d^t \perp d^f - d^f$.

From $d^t \cdot (d - d^f) = d^f \cdot \lambda_I + r(1 \cdot d) = 0$, we have $d^t \perp d - d^f$ and $d^w = d^f - d^t \perp d - d^f$ which completes the proof of their mutual orthogonalities.

Since $T_w = \{1, e_I\}^\perp$ and $d - d^f \in \text{span}\{1, e_I\}$, $d - d^f$ is orthogonal to the tangent space. \qed

Definition 4. (MPRP direction) Let $w$ be a point with $w \geq 0$ and $w \cdot 1 = \frac{1}{T}$, and let $g = \nabla f(w)$. Putting tilde for the variable in the previous step: let $\tilde{g}$ be the gradient and $\tilde{d}$ be the search direction in the previous step, then the modified Polak-Ribiera-Polyak direction $d_{MPRP} = d_{MPRP} (w, \tilde{g}, \tilde{d})$ is defined as

$$
d_{MPRP} = -(g)^t - \frac{(g - \tilde{g})^t}{\tilde{g} \cdot \tilde{g}} \tilde{d}^t + \frac{(g - \tilde{g})^t}{\tilde{g} \cdot \tilde{g}} (g - \tilde{g})^t
$$

Theorem 1. (KKT condition) \forall w \geq 0 with $w \cdot 1 = \frac{1}{T}, \forall \tilde{g}, \forall \tilde{d}$, let $g = \nabla f(w)$ and $d = d_{MPRP} (w, \tilde{g}, \tilde{d})$, then $(-g)^t \cdot d \geq 0$. Moreover $(-g)^t \cdot d = 0$ if and only if $w$ is a KKT point of the minimization problem (2.1).

Proof.

$$
(-g)^t \cdot d = (-g)^t \cdot \left[\frac{(-g)^t - (g - \tilde{g})^t}{\tilde{g} \cdot \tilde{g}} \tilde{d}^t + \frac{(g - \tilde{g})^t}{\tilde{g} \cdot \tilde{g}} (g - \tilde{g})^t\right] = ||(-g)^t||^2 + \frac{1}{\tilde{g} \cdot \tilde{g}} \left[-\left[(-g)^t \cdot (g - \tilde{g})^t\right] \left[(-g)^t \cdot d^t\right] + \left[(-g)^t \cdot (g - \tilde{g})^t\right] \left[(-g)^t \cdot \tilde{d}^t\right]\right]
$$

Since $(-g)^w \perp T_w$, $(-g)^w \cdot (g - \tilde{g})^t = 0$ and $(-g)^f \cdot (g - \tilde{g})^t = (-g)^t \cdot (g - \tilde{g})^t$.

Similarly, $(-g)^f \cdot \tilde{d}^t = (-g)^t \cdot \tilde{d}^t$, and we have $(-g)^f \cdot d = ||(-g)^f||^2 \geq 0$.

The KKT condition for 2.1 is that

$$
g = \lambda + r \cdot 1 \text{ for some } \lambda \geq 0 \text{ with } \lambda \cdot w = 0$$

and some $r$ with $r \left(w \cdot 1 - \frac{1}{T}\right) = 0$. 
Algorithm 2 Algorithm based on nonlinear conjugate gradient

Input: Given constants \( \rho \in (0, 1), \delta > 0, \epsilon > 0 \). Initial point \( w_0 \succeq 0 \). Let \( k = 0 \), and \( g = \nabla f (w_0) \) where \( f = lse (-Aw) \).

Output: \( w \)

Procedure:
1: Compute \( d = (d_I, d_J) \) by Algorithm 1.
   If \( \left| (g)^f \cdot d \right| \leq \epsilon \), then stop.
   Otherwise, go to the next step.
2: Determine \( \alpha = \max \left\{ \frac{-d_k \cdot \nabla f (w_0)}{\nabla^2 f (w_0) d_k \rho j, j = 0, 1, 2, \cdots} \right\} \) satisfying \( w + \alpha d \succeq 0 \) and \( f (w + \alpha d) \leq f (w) - \delta \alpha \| d \|^2 \)
3: \( w \leftarrow w + \alpha d \)
4: \( k \leftarrow k + 1 \), and go to step 2.

Since \( w_J > 0 \) and \( \lambda \geq 0 \), \( \lambda_J = 0 \). Since \( w \cdot \mathbf{1} = \frac{1}{T} \), and \( w_I = 0 \), the conditions \( r \left( w \cdot \mathbf{1} - \frac{1}{T} \right) = 0 \) and \( \lambda \cdot w = \lambda_I \cdot w_I + \lambda_J \cdot w_J = 0 \) are unnecessary. Therefore, the KKT condition is simplified as

\[
g = \begin{bmatrix} \lambda_I \\ 0 \end{bmatrix} + r \cdot \mathbf{1} \text{ for some } \lambda_I \geq 0 \text{ and some } r.
\]

On the other hand, \( (g)^f \cdot d = \| (g)^f \|^2 = 0 \) if and only if \( (g)^f = \arg \min_{y_I \geq 0 \text{ and } y \cdot \mathbf{1} = 0} \| (g) - y \| \), whose KKT condition is that

\[
g = \begin{bmatrix} \lambda_I \\ 0 \end{bmatrix} + r \cdot \mathbf{1} \text{ for some } \lambda_I \geq 0 \text{ and some } r.
\]

Each of the two minimization problems has a unique minimum point, accordingly a unique KKT condition. Since their KKT conditions are same, we have

\[
(g)^f \cdot d = 0 \iff w \text{ is the KKT point of the minimization problem 2.1.}
\]

Next, we introduce some properties of \( f(w) \) and Algorithm 2 to prove the global convergence of Algorithm 2.

Properties
Let \( V = \{ w \in \mathbb{R}^N \mid w \succeq 0 \text{ and } w \cdot \mathbf{1} = \frac{1}{T} \} \).

1) Since the feasible set \( V \) is bounded, the level set \( \{ w \in \mathbb{R}^N \mid f(w) \leq f(w_0) \} \) is bounded. Thus, \( f \) is bounded from below.
(2) The sequence \( \{w_k\} \) generated by Algorithm 2 is a feasible point sequence and the function value sequence \( \{f(w_k)\} \) is decreasing. In addition, since \( f(w) \) is bounded below,

\[
\sum_{k=0}^{\infty} \alpha_k^2 \|d_k\|^2 < \infty.
\]

Thus we have

\[
\lim_{k \to \infty} \alpha_k \|d_k\| = 0.
\]

(3) \( f \) is continuously differentiable, and its gradient is the Lipschitz continuous; there exists a constant \( L > 0 \) such that

\[
\|\nabla f(w) - \nabla f(y)\| \leq \|x - y\|, \forall x, y \in \mathbf{V}
\]

These imply that there exists a constant \( \gamma_1 \) such that

\[
\|\nabla f(w)\| \leq \gamma_1, \forall x \in \mathbf{V}.
\]

**Lemma 4.** If there exists a constant \( \epsilon \geq 0 \) such that

\[
\|g(x_k)\| \geq \epsilon, \forall k,
\]

then there exists a constant \( M > 0 \) such that

\[
\|d_k\| \leq M, \forall k.
\]

**Proof.**

\[
\|d_k^{MPRP}\| \leq \|(-g)^f\| + 2\|(-g)^f \cdot (g - \tilde{g})^f \| \cdot \|\tilde{d}_k\| \leq \gamma_1 + \frac{2\gamma_1 L \alpha_k \|\tilde{d}_k\|}{\epsilon^2} \|\tilde{d}_k\|
\]

Since \( \lim_{k \to \infty} \alpha_k \|d_k\| = 0, \exists \) a constant \( \gamma \in (0, 1) \) and \( k_0 \in \mathbb{Z} \) such that

\[
\frac{2L\gamma_1}{\epsilon^2}\alpha_{k-1} \|\tilde{d}_{k-1}\| \leq \gamma \text{ for all } k \geq k_0.
\]

Hence, for any \( k \geq k_0 \),

\[
\|d_k^{MPRP}\| \leq 2\gamma_1 + \gamma \|d_{k-1}\| \leq 2\gamma_1 \left(1 + \gamma + \cdots + \gamma^{k-k_0-1}\right) + \gamma^{k-k_0} \|d_{k_0}\| \leq \frac{2\gamma_1}{1 - \gamma} + \|d_{k_0}\|
\]

Let \( M = \max \left\{\|d_1\|, \|d_2\|, \cdots, \|d_{k_0}\|, \frac{2\gamma_1}{1 - \gamma} + \|d_{k_0}\|\right\} \). Then \( \|d_k^{MPRP}\| \leq M, \forall k. \) \( \square \)
Lemma 5. (Success of Line search) In Algorithm 2, the line search step is guaranteed to succeed for each $k$. Precisely speaking,
\[ f(w_k + \alpha_k d_k) \leq f(w_k) - \delta \alpha_k^2 \| d_k \|^2 \]
for all sufficiently small $\alpha_k$.

Proof. By the Mean Value Theorem,
\[ f(w_k + \alpha_k d_k) - f(w_k) = \alpha_k g(w_k + \alpha_k \theta_k d_k) \cdot d_k, \]
for some $\theta_k \in (0, 1)$. The line search is performed only if $(-g(w_k))^T d_k > \epsilon$. In Lemma 7, we showed that $(-g(w_k)) - (-g(w_k))^T T w$ and $(-g(w_k))^T (1 - (-g(w_k))) = 0$ and we have
\[ -g(w_k) \cdot d_k = (-g(w_k))^T d_k > \epsilon. \]

From the continuity of $g(w)$,
\[ -g(w_k + \alpha_k \theta_k d_k) \cdot d_k > \frac{\epsilon}{2} \]
for sufficiently small $\alpha_k$. Choosing $\alpha_k \in \left(0, \frac{\epsilon}{2\|d_k\|^2}\right)$, we get
\[ f(w_k + \alpha_k d_k) = f(w_k) + \alpha_k g(w_k + \alpha_k \theta_k d_k) \cdot d_k \]
\[ < f(w_k) - \frac{\epsilon}{2} \alpha_k \]
\[ \leq f(w_k) - \delta \alpha_k^2 \| d_k \|^2. \]

\[ \square \]

Theorem 2. Let $\{w_k\}$ and $\{d_k\}$ be the sequence generated by Algorithm 2, then
\[ \liminf_{k \to \infty} (-g_k)^T d_k = 0. \]
Thus the minimum point $w^*$ of our main problem (2.1) is a limit point of the set $\{w_k\}$ and Algorithm 2 is convergent.

Proof. We first note that $(-g_k)^T d_k = -g_k \cdot d_k$ that appeared in the proof of Lemma 11. We prove the theorem by contradiction. Assume that the theorem is not true, then there exists an $\epsilon > 0$ such that
\[ \| (-g_k)^T d_k \|^2 = (-g_k)^T d_k > \epsilon, \text{ for all } k. \]

By Lemma 10, there exists a constant $M$ such that
\[ \| d_k \| \leq M, \text{ for all } k. \]
If $\liminf_{k \to \infty} \alpha_k > 0$, then $\lim_{k \to \infty} \| d_k \| = 0$. Since $\| g \|_{\infty} < -r$, $\lim_{k \to \infty} (-g_k)^T d_k = 0$. This contradicts assumption.
If $\liminf_{k \to \infty} \alpha_k = 0$, then there is an infinite index set $K$ such that
\[ \lim_{k \in K, k \to \infty} \alpha_k = 0. \]
It follows from the step 2 of Algorithm 2, that when \( k \in K \) is sufficiently large, \( \rho^{-1} \alpha_k \) does not satisfy \( f(w_k + \alpha_k d_k) \leq f(w_k) - \delta \alpha_k^2 \| d_k \|^2 \), that is
\[
\begin{align*}
f(w_k + \rho^{-1} \alpha_k d_k) - f(w_k) &> -\delta \rho^{-2} \alpha_k^2 \| d_k \|^2 \\
\end{align*}
\]
(3.2)

By the Mean Value Theorem and Lemma 10, there is \( h_k \in (0, 1) \) such that
\[
\begin{align*}
f(w_k) - f(w_k + \rho^{-1} \alpha_k d_k) &\leq \rho^{-1} \alpha_k g(w_k + h_k \rho^{-1} \alpha_k d_k) \cdot d_k \\
&\leq \rho^{-1} \alpha_k g(w_k) \cdot d_k + \rho^{-1} \alpha_k (g(w_k + h_k \rho^{-1} \alpha_k d_k) - g(w_k)) \cdot d_k \\
&\leq \rho^{-1} \alpha_k g(w_k) \cdot d_k + L \rho^{-2} \alpha_k^2 \| d_k \|^2
\end{align*}
\]

Substitute the last inequality into (3.2) and applying \(-g(w_k) \cdot d_k = (-g)^T (w_k) \cdot d_k \), we get for all \( k \in K \) sufficiently large,
\[
0 \leq (-g)^T (w_k) \cdot d_k \leq \rho^{-1} (L + \delta) \alpha_k \| d_k \|^2.
\]

Taking the limit on both sides of the equation, then by combining \| d_k \| \leq M and recalling \( \lim_{k \in K, k \to \infty} \alpha_k = 0 \), we obtain the \( \lim_{k \in K, k \to \infty} |(-g)^T (x_k) \cdot d_k| = 0 \).

This also yields a contradiction. \( \square \)

**Remark 1.** To say the existence of \( k \) which satisfies (3.2), we should verify that \( w_k + \rho^{-1} \alpha_k d_k \) is feasible. Since \( d_k \cdot 1 = 0 \), \( (w_k + \rho^{-1} \alpha_k d_k) \cdot 1 = w_k \cdot 1 = \frac{1}{2} \). So, we should check \( w_k + \rho^{-1} \alpha_k d_k \geq 0 \). Since \( \lim_{k \in K, k \to \infty} \alpha_k = 0 \), \( \alpha_k \) is near to zero for sufficiently large \( k \). Thus, \( w_k + \rho^{-1} \alpha_k d_k \geq 0 \) except very special cases.

4. **Numerical results**

In this section, we test our proposed CG algorithm on two boosting examples of non-negligible size. Through the tests, we check if their numerical results match the analyses presented in section 3.

Our algorithm is supposed to generate a sequence \( \{w_k\} \) on which the soft margin monotonically decreases, which is the first check point. According to Theorem (2), the stopping criteria \( (-g_k)^T d_k < \epsilon \) should be satisfied after a finite number of iterations for any given threshold \( \epsilon > 0 \), which is the second check point. According to Theorem(1), the solution \( w_k \) with the stopping criteria satisfied is the KKT point, which is the third one. The KKT point is the global minimizer of the soft margin, the optimal strong classifier, which is the final one.

4.1. **Low dimensional example.** We solve a boosting problem that minimizes \( lse(-Aw) \) with \( w \geq 0 \) and \( w \cdot 1 = \frac{1}{2} \), where \( A \) is a \( 4 \times 3 \) matrix given below.

\[
A = \begin{bmatrix}
-1 & 1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & -1 \\
1 & -1 & -1
\end{bmatrix}
\]
As shown in Figure 2, the soft margin $\text{lse}(-Aw)$ monotonically decreases and the stopping criteria $(-g_k)^T \cdot d_k$ drops to a very small number in finite iterations, which is equivalent to the statement of Theorem 2, $\liminf_{k \to \infty} (-g_k)^T \cdot d_k = 0$.

4.2. **Classifying win/loss of sports games.** One of the primal applications of boosting is to classify win/loss of sports games [6]. As an example, we take the vast amount of statistics from the basketball league of a certain country*(for a patent issue, we do not disclose the details).

The statistics of each game is represented by the following 36 numbers.
In a whole year, there were 538 number of games with the win/loss results, from which we take a training data \( \{ x_1, \cdots, x_M = 269 \} \) with the win/loss of the home team \( \{ y_1, \cdots, y_M \} \subset \{ \pm 1 \} \). Each \( x_i \) represents the statistics of a game, and \( x_i \in \mathbb{R}^{269 \times 36} \).

Similarly to the previous example, Figure 3 shows that the soft margin monotonically decreases and the stopping criteria drops to a very small number in finite iterations, matching the analyses in Section 3.

5. Conclusion

We proposed a new Conjugate Gradient method for solving convex programings with the non-negative constraints and a linear constraint, and successfully applied the method to the boosting problems. We also presented a convergence analysis for the method. Our analysis shows that the method is convergent in a finite iteration for any small stopping threshold. The solution with the stopping criteria satisfied is shown to be the KKT point of the convex programing and hence the global minimizer of the programing. We solved two benchmark boosting problems by the CG method, and obtained numerical results that completely cope with the analysis. Our algorithm with the guaranteed convergence can be successful in other boosting problems as well as other convex programings.

References

Figure 3. the convergences of the CG method for example 4.2


ACCELERATION OF MACHINE LEARNING ALGORITHMS BY TCHEBYCHEV ITERATION TECHNIQUE

MIKHAIL P. LEVIN

INSTITUTE OF SYSTEM PROGRAMMING OF RUSSIAN ACADEMY OF SCIENCES, MOSCOW, RUSSIA
E-mail address: Mikhail.levin@hotmail.com; mlevin@ispras.ru

ABSTRACT. Recently Machine Learning algorithms are widely used to process Big Data in various applications and a lot of these applications are executed in run time. Therefore the speed of Machine Learning algorithms is a critical issue in these applications. However the most of modern iteration Machine Learning algorithms use a successive iteration technique well-known in Numerical Linear Algebra. But this technique has a very low convergence, needs a lot of iterations to get solution of considering problems and therefore a lot of time for processing even on modern multi-core computers and clusters. Tchebychev iteration technique is well-known in Numerical Linear Algebra as an attractive candidate to decrease the number of iterations in Machine Learning iteration algorithms and also to decrease the running time of these algorithms those is very important especially in run time applications. In this paper we consider the usage of Tchebychev iterations for acceleration of well-known K-Means and SVM (Support Vector Machine) clustering algorithms in Machine Learning. Some examples of usage of our approach on modern multi-core computers under Apache Spark framework will be considered and discussed.

1. INTRODUCTION

Now a day Machine Learning algorithms are widely uses in Data Mining for solution of various application problems related with Big Data processing. In this paper we restrict our consideration by two very popular Machine Learning (ML) algorithms, namely K-Means and SVM (Support Vector Machine). These algorithms are among of the 10 the most popular algorithms [1] in ML and they are widely used in various applications.

K-Means and SVM algorithms are widely used in Machine Vision, Drag Design, Genomic and Bio-informatics, in Medical Cybernetics, in Finance and some other applications. For example, K-Means is used

- for direct clustering of Big Data;
- in Computer Graphics for color quantization (reducing color palette of images to a fixed number of colors);
- for preliminary acceleration of Community Detection clustering methods;
- as a preliminary procedure for parallelization of SVM clustering method.

Received by the editors May 11 2017; Accepted December 18 2017; Published online February 15 2018.
In Computer Vision K-Means clustering technology is one of the basic technologies, because it is used in 6 of 8 branches of Computer Vision, namely in: Signal Processing and Compression, Data Mining, Machine Learning and Artificial Intelligence, Computer Graphics, Automatic Control and Robotics, Applied Mathematics. Other two branches of Computer Vision, namely Physics Imaging and Neurobiology also use this technology but not so often as previous six.

One of attractive properties of K-Means clustering algorithm is its simplicity. But the payment for simplicity is weak properties of this algorithm, those are

- a lot of iterations especially in case of very big data and correspondingly a very big CPU time for this data processing;
- convergence to a local minimum.

The first of above mentioned weak point is a principle obstacle in K-Means clustering, because a lot of applications especially in Computer Vision are running in run time. This means that the time of reaction in these applications should be very small. Therefore decreasing of this time is one of the critical issue in these applications and it is very important and urgent problem in the usage of K-Means clustering.

2. BACKGROUND OF K-MEANS CLUSTERING

K-Means clustering algorithm was proposed by Steinhaus in 1956 and developed by Lloyd in 1957. It takes about $K N$ operations for clustering, where $K$ is a number of clusters and $N$ is a number of points in data. In Lloyd version K-Means comprises of the following steps:

- It starts by setting initial centers of clusters (so called seeding);
- Then assigns each data point to the closest cluster by evaluating the distance between each data point to each cluster centers and allocate each point to the nearest cluster;
- Re-evaluates each cluster center for each cluster group;
- Evaluates maximal absolute value of difference between centers on current and previous iterations;
- Repeats last three steps 2,3, and 4 until each cluster has stable center and members in appropriate cluster;
- Evaluates a sum of square error to estimate quality of clustering.

Evaluation of centers for each cluster in Lloyd’s version of algorithm is provided as follows. Let any cluster with index $k$ on $i$-th iteration consists of $K_i$ points with coordinate vectors $p_j = \{p_{j1}, p_{j2}, ..., p_{jn}\}, j = 1, 2, 3, ..., K_i$.

Then the center of $k$ cluster on $i$ iteration is evaluated as follows

$$x_i = \frac{1}{K_i} \sum_{j=1}^{K_i} p_j$$

(2.1)

Iterations are doing until

$$\max_k ||x_i - x_{i-1}|| < \varepsilon$$

(2.2)
Here $\varepsilon$ is an exactness (usually about $10^{-6}$ and $||.||$ is Euclidean norm in space $\mathbb{R}_n$. The scheme of classic Lloyd’s version of K-Means algorithm is shown on Fig. 1.

**Figure 1.** The scheme of Lloyd’s version of K-Means clustering.

3. **Profile-Guided Tchebychev Algorithm**

To date it were a lot of attempts to speed-up K-Means clustering technology. Mainly these attempts were connected with changing of data structure, usage of another type of norm in distance evaluation instead of classical Euclidean norm. Here we should mention Dan Pelleg and Andrew Moore paper [2] in which kd-trees were used to accelerate nearest-neighbour search queries. This technique modifies steps 4 and 5 in Lloyd’s algorithm by decreasing the number of operations on steps 4 and 5.
Another attempts were done in papers [3-7] and they are related with effect of seeding. In [3] a preclustering technique instead of random seeding was used to decrease the number of iterations and running time. Other algorithms of seeding were considered in [4-7]. All these modifications concern step 3 in Lloyd’s algorithm.

Jenks in [8] suggested to use modified function in minimization problem and also suggested another seeding. He obtained about 90 percent speed-up of Lloyd’s method, but only in some cases.

In 2004 Ya Guan, Ali Ghobani, and Nabil Belacel proposed K-Means+ modification [9]. It selects the number of clustering automatically basing on initial data analysis.

Unfortunately in all above cited approaches there were not any attempts to change iteration process and decrease by this manner the number of iterations. In all these approaches the successive iteration technique is used. Now we suggest to use the Tchebychev iteration technique instead successive iteration technique. In this case at first we represent the formula of clusters computation in evaluation form as follows

\[ x_i - x_{i-1} = \frac{1}{K_i} \sum_{j=1}^{K_i} p_j - \frac{1}{K_{i-1}} \sum_{j=1}^{K_{i-1}} p_j \]  

(3.1)

or as follows

\[ x_i = x_{i-1} + \tau A(x_{i-1}, x_{i-2}) \]

(3.2)

Here

\[ A(x_{i-1}, x_i) = \frac{1}{K_i} \sum_{j=1}^{K_i} p_j - \frac{1}{K_{i-1}} \sum_{j=1}^{K_{i-1}} p_j, \quad \tau = 1 \]  

(3.3)

Now let us consider four layers Tchebychev iteration method [10] for centers of clusters evaluation. We denote \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) minimal and maximal eigenvalues of operator \( A \).

Also we denote so called “optimal” time step in the successive iteration method as follows

\[ \tau_0 = \frac{2}{\lambda_{\text{min}} + \lambda_{\text{max}}} \]

and value

\[ \varrho_0 = \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\lambda_{\text{min}} + \lambda_{\text{max}}} \]

We suggest that the time step \( \tau \) in iteration process is changing with respect to the number of iteration \((i + m)\) and evaluate it on each sequential 4 iterations by the following formulas

Let \( i = 4q + m \), then

\[ \tau_{(4q+m)} = \tau_0 \frac{\tau_0}{1 + \varrho_0 \cos[(2m - 1)\pi/8]} , \quad (m = 1, 2, 3, 4) \]

Here \( \tau_{(4q+m)} \), \((m = 1, 2, 3, 4)\) are roots of Tchebychev polynomial of the 4th order. Thus we evaluate the centers of clusters by the following iteration formulas

\[ x_i = x_{i-1} + \tau_{(4q+m)} A(x_{i-1}, x_{i-2}) , \quad i = (4q + m) , \quad (m = 1, 2, 3, 4) \]  

(3.4)
Unfortunately in the formulas above we can not expressed operator \( A \) in the explicit analytical form and therefore we can not evaluate or estimate its maximal and minimal eigenvalues analytically or numerically. At the first glance this problem is seemed unresolvable in mathematical sense. To solve it we go out of mathematics and go into informatics. To solve our problem we formulate two important properties basing on which we try to solve our problem.

**Property 1:** Operator \( A \) should be compressible to provide the convergence of iteration process.

**Property 2:** The number of iterations in the considering iteration process should be at least minimal as possible.

Basing **Property 1** and **Property 2** we suggest the following algorithm to evaluate the maximal and minimal eigenvalues of operator \( A \).

**Profile-Guided Tchebychev Algorithm:**

- **STEP 1.** Let us choose any small data subset \( d \) of the small volume from the considering data set \( D = p_1, p_2, p_3, ..., p_K \). Here \( K \) is a number of points in the considering data set \( D \).
- **STEP 2.** The volume of subset \( d \) should be small to provide evaluations by Lloyd’s algorithm in a short time. In practice it is possible to use a random choice selection of subset \( d \) from \( D \) data set.
- **STEP 3.** Providing evaluations clusters centers with iteration formulas (3.4) on subset \( d \) for any fixed \( \lambda_{\max} \) and \( \lambda_{\min} \), we consider the following unrestricted minimization problem for maximal and minimal eigenvalues choosing as follows:

\[
\text{Find } N_{\text{iter}}(\lambda_{\max}, \lambda_{\min}) \to \min
\]

The solution of above mentioned problem should be obtained by well-known alternative descent method [11]. The example of appropriate MatLab code is available in [12].

The scheme of profile-guided Tchebychev version of K-Means clustering algorithm is shown on Fig. 2. On this scheme the new blocks in comparison with Lloyd’s version of K-Means algorithm are shown in oval frames. The overhead of the proposed modification is very small in comparison with evaluation of centers of clusters computation. It should be omit, because the time steps can be tabulated in advance and it needs only to provide evaluations by formula (3.2). These additional evaluations contain only 1 add and 1 multiply operations.

4. **Performance of Profile-Guided Tchebychev Algorithm**

Now let us consider an example of usage 4 Layers Profile-Guided Tchebychev Algorithm (4LPGTch) for K-Means clustering on the real well-known data file kmeans_train_3089 consisting of 3089 lines with 5 columns. An exactness of convergence in this example was taken
equal to $10^{-7}$. For comparison with our approach Successive Iteration Method (Lloyd's algorithm), 3 Layers B.T.Polyak’s Iteration Method (3LPIM) [13], and Successive Over Relaxation Method (SORM) with fitting of changing relaxation coefficients were taken. The appropriate results are presented in the following Table 1.

Thus the usage of 4 Layers Profile-Guided Tchebychev Algorithm essentially decreases the number of iterations (up to $13/4 = 3.25$) times in comparison with successive iteration algorithm uses in Lloyd’s version of K-Means clustering. In comparison with other iteration algorithms such as 3 Layers B.T.Polyak’s Iteration Method and Successive Over Relaxation Method with fitting of changing relaxation coefficients, it has $11/4 = 2.75$ and $7/4 = 1.75$ advantage.
Now let us consider one additional example of big volume data file whose was generated by repeating of 2000 times of kmeans_train_3089 data. We provided computations by Kmeans∥ algorithm from Apache Spark ML Library. This algorithm is written on Scala programming language. Also we modified this software with 4, 6, and 8 Layers Profile-Guided Tchebychev ideas. We denote this version as MLPGTch,Kmeans∥. Since Kmeans∥ and MLPGTch,Kmeans∥ use random initial data in seeding of cluster centers, their run times are changing also randomly. Therefore for estimation of results we need to use some statistic estimations. We used the sets of 50 runs. Of course, this statistic is not perfect, but it gives us some raw statistic estimations of performance of considering algorithms. The appropriate results are presented in Table 2 below. The number of clusters in these examples was taken equal to 5 and the number of layers M was taken qual to 4, 6 and 8. The following notations are used avNI is an average number of iterations, avIT is an average time of iteration block run in second, avTT is an average total time run in seconds, and speedups $su$ were evaluated as follows

\[
\begin{align*}
\text{su}(\text{avNI}) &= \frac{\text{avNI}(\text{KMeans}∥)}{\text{avNI}(\text{MLPGTch,KMeans}∥)}, \\
\text{su}(\text{avIT}) &= \frac{\text{avIT}(\text{KMeans}∥)}{\text{avIT}(\text{MLPGTch,KMeans}∥)}, \\
\text{su}(\text{avTT}) &= \frac{\text{avTT}(\text{KMeans}∥)}{\text{avTT}(\text{MLPGTch,KMeans}∥)}.
\end{align*}
\]

### Table 1. Comparison of various algorithms for K-Means clustering

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Number of iterations</th>
<th>Speed up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lloyd’s algorithm</td>
<td>13</td>
<td>0% (basement)</td>
</tr>
<tr>
<td>3LPIM</td>
<td>11</td>
<td>+15%</td>
</tr>
<tr>
<td>SORM</td>
<td>7</td>
<td>+46%</td>
</tr>
<tr>
<td>4LPGTch</td>
<td>4</td>
<td>+69%</td>
</tr>
</tbody>
</table>

### Table 2. Comparison of KMeans∥ and MLPGTch,KMeans∥ with M 4, 6, 8

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>avNI</th>
<th>avIT</th>
<th>avTT</th>
<th>su(avNI)</th>
<th>su(avIT)</th>
<th>su(avTT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>KMeans∥_ML_Lib</td>
<td>31</td>
<td>173''</td>
<td>197''</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>4PGTch,KMeans∥</td>
<td>23</td>
<td>130''</td>
<td>155''</td>
<td>1.36</td>
<td>1.34</td>
<td>1.27</td>
</tr>
<tr>
<td>6PGTch,KMeans∥</td>
<td>22</td>
<td>119''</td>
<td>141''</td>
<td>1.40</td>
<td>1.46</td>
<td>1.40</td>
</tr>
<tr>
<td>8PGTch,KMeans∥</td>
<td>22</td>
<td>134''</td>
<td>160''</td>
<td>1.40</td>
<td>1.29</td>
<td>1.23</td>
</tr>
</tbody>
</table>

This table shows that the best results are obtained with 6PGTch,KMeans∥ algorithm.
5. BACKGROUND OF SVM CLUSTERING

SVM (Support Vector Machine) algorithm was proposed by Vladimir Vapnik and Alexey Chervonenkis [1,14-15] in 1963. It widely uses in Machine Vision, Drag Design, Genomic and Bioinformatics, in Medical Cybernetic, in Finance for a direct clustering of big data.

In recent years SVM is very popular approach non-hierarchical clustering and binary classification. It can use a lot of different kernels and different type of measures to provide a best fitting of clustering data.

SVM has three important advantages:

- Firstly, it has a regularization parameter, which makes the user thinking about avoiding over-fitting;
- Secondly, it uses a kernel trick, so can build in expert knowledge about the problem via engineering the kernel;
- Thirdly, SVM is solved by usage a convex optimization problem, namely Quadratic Programming Problem (QPP) which avoids local minimization solutions and for which developed various efficient methods.

The weak points of SVM are

- a lot of iterations for big data analysis and therefore a big CPU time for clustering;
- SVM theory only really covers the determination of the parameters for a given value of regularization and kernel parameters;
- SVM moves the problem of over-fitting from optimizing the parameters to the model selection. Sadly the kernel models can be quite sensitive to over-fitting the model selection criterion.

To evaluate a separate hyper-plane parameters, SVM uses Quadratic Programming Problem (QPP). To date there are a few different methods for solution of QPP in SVM. One of the most promising approach was proposed by John C. Platt [16] from Microsoft Research. In this approach instead the full multidimensional QPP the sequence of 2 dimensional QPP is solved step-by-step in successive iteration process.

Thus one of the challenge problem in SVM clustering is an acceleration of convergence and decreasing CPU time, especially in run-time applications, there the response time is a critical value, and in big data analysis.

In a linear case SVM problem consists in evaluation of a separate plane

$$u \equiv \overrightarrow{w} \cdot \overrightarrow{x} - b = 0$$  \hspace{1cm} (5.1)

subject to the following inequality

$$y_i (\overrightarrow{w} \cdot \overrightarrow{x_i}) - b \geq 0, \quad 1 \leq i \leq n$$  \hspace{1cm} (5.2)

Here $u$ is output of SVM, $b$ is a threshold, $\overrightarrow{x_i}$ is a training example to the input $\overrightarrow{x}$, $y_i \in \{-1, +1\}$ is a desired output and $n$ is a dimension of problem.

The problem of searching the separate plane (5.1) subject to conditions (5.2) is reduced to minimization of $\|\overrightarrow{w}\|$ subject to conditions (5.2). According to Kurosh-Kuhn-Tucker theorem
(KKT) the above mentioned problem of Quadratic Optimization (QOP) is equivalent to the dual problem of the saddle point to Lagrange function evaluation as follows

$$L(\overrightarrow{w}, b, \overrightarrow{\alpha}) \equiv \frac{1}{2}||\overrightarrow{w}||^2 - \sum_{i=1}^{n} \overrightarrow{\alpha}_i y_i ((\overrightarrow{w} \cdot \overrightarrow{x}_i) - b) \rightarrow \min_{(\overrightarrow{w}, b)} \max_{\overrightarrow{\alpha}}.$$  \hspace{1cm} (5.3)

Here $\overrightarrow{\alpha} = \{\alpha_i\}_{i=1}^{n}$ are Lagrange multipliers.

QOP (5.3) can be rewritten with respect to Lagrange multipliers as the following Quadratic Programming Problem (QPP)

$$-L(\overrightarrow{\alpha}) \equiv -\sum_{i=1}^{n} \overrightarrow{\alpha}_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j y_i y_j (\overrightarrow{x}_i \cdot \overrightarrow{x}_j) \rightarrow \max_{\overrightarrow{\alpha}},$$  \hspace{1cm} (5.4)

$$0 \leq \overrightarrow{\alpha}_i \leq C, \quad 1 \leq i \leq n,$$  \hspace{1cm} (5.5)

$$\sum_{i=1}^{n} \alpha_i y_i = 0.$$  \hspace{1cm} (5.6)

Training of SVM consists in evaluation of $\overrightarrow{\alpha}$ from solution of QPP (5.4-5.6) and evaluation of $\overrightarrow{w}$ and $b$ as follows

$$\overrightarrow{w} = \sum_{i=1}^{n} \alpha_i y_i \overrightarrow{x}_i,$$  \hspace{1cm} (5.7)

$$b = \overrightarrow{w} \cdot \overrightarrow{x}_i - y_i, \quad \alpha_i > 0.$$  \hspace{1cm} (5.8)

SMO algorithm consists in iterative solution step-by-step with respect to any pair of Lagrange multipliers $\alpha_l, \alpha_m$. Each QP sub-problem with respect to $\alpha_l, \alpha_m$ is solved analytically. Iterations are terminated when all Karush-Kuhn-Tucker (KKT) optimality conditions has been valid

if $\alpha_i = 0$, then $y_i u_i \geq 1,$ \hspace{1cm} (5.9)

if $0 \leq \alpha_i < C$, then $y_i u_i = 1,$ \hspace{1cm} (5.10)

if $\alpha_i = C$, then $y_i u_i \leq 1.$ \hspace{1cm} (5.11)

The SVM-SMO scheme of principle is shown on Fig. 3.

6. PROFILE-GUIDED TCHEBYCHEV MODIFICATION OF SVM-SMO CLUSTERING

Now let us consider a modification of SVM-SMO algorithm using ideas of Profile-Guided Tchebychev approach applied before to K-Means algorithm. At first we present formulas for parameters of separation plane computation in evaluation form

$$\overrightarrow{w}^{(k)} - \overrightarrow{w}^{(k-1)} = \sum_{i=1}^{n} \alpha^{(k)}_i y_i \overrightarrow{x}_i - \overrightarrow{w}^{(k-1)},$$  \hspace{1cm} (6.1)

$$b^{(k)} - b^{(k-1)} = \overrightarrow{w}^{(k)} \cdot \overrightarrow{x}_i - y_i - b^{(k-1)}, \quad \alpha_i > 0.$$  \hspace{1cm} (6.2)

Formulas (6.1-6.2) can be presented in the following vector form
where $\vec{Z} = \{\vec{w}, b\}$ is a searching vector.

Formula (6.3) represents a successful iteration technique for evaluation of parameters of separation plane. Now we will consider application of Multi-Layers Tchebychev iteration technique for evaluation of parameters of separation plane in SVM-SMO algorithm.
Let \( \tau \) is not constant and is changing in iteration process with respect to the number of iterations and is evaluated on each \( M \) iterations as follows

\[
\tau_{(k+m)} = \frac{\tau_0}{1 + q_0 \cos \left( \frac{(2m - 1)\pi}{2M} \right)}, \quad \left( m = 1, 2, 3, \ldots, M \right),
\]

where

\[
\tau_0 = \frac{2}{\lambda_{\text{min}} + \lambda_{\text{max}}}, \quad q_0 = \frac{\lambda_{\text{max}} - \lambda_{\text{min}}}{\lambda_{\text{min}} + \lambda_{\text{max}}},
\]

\( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) are minimal and maximal eigenvalues of operator \( \vec{A} \).

Now let us consider Multi-Layers Profile-Guided Tchebychev (MLPGTch) algorithm for SVM-SMO clustering. This algorithm is quite similar to 4LPGTch algorithm considered before for K-Means clustering, but it has additional parameter the number of layers \( M \).

**Multi-Layers Profile-Guided Tchebychev Algorithm:**

1. Let us choose any small data subset \( d \) of the small volume from the considering data set \( D = p_1, p_2, p_3, \ldots, p_K \). Here \( K \) is a number of points in the considering data set \( D \).
2. The volume of subset \( d \) should be small to provide fast evaluations by SVM-SMO algorithm in a short time. In practice it is possible to use a random choice selection of subset \( d \) from \( D \) data set.
3. Providing evaluations of separation plane by iteration formulas (6.3) on subset \( d \) for any fixed number of layers \( M \), eigenvalues \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \), we consider the following unrestricted minimization problem for maximal and minimal eigenvalues, and the number of layers choosing as follows:

\[
\text{Find } N_{\text{iter}}(M, \lambda_{\text{max}}, \lambda_{\text{min}}) \rightarrow \text{min}
\]

The solution of above mentioned problem also as for K-Means algorithm can be obtained by well-known alternative descent method.

The scheme of multi-layers profile-guided Tchebychev version of SVM-SMO clustering algorithm is shown on Fig. 4. On this figure the new blocks in comparison with SVM-SMO algorithm are shown by rectangles with rounded corners.

In our numerical profiling we restricted consideration by case \( M = \{4, 6, 8\} \). Solving minimization problem of test data set of small volume we obtained the following solution: \( M = 6, \lambda_{\text{max}} = 1.0010, \lambda_{\text{min}} = 0.7955 \). As our experience shows it is possible to use this solution in MLPGTch clustering for other data sets including data sets of huge volume. Perhaps this is related with weak dependence of operator \( \vec{A} \) with data sets. But this is only our hypothesis confirmed by our numerical experiments. Below we consider some numerical results obtained with MLPGTch algorithm.
Now let us consider some examples of usage Multi-Layers Profile-Guided Tchebychev SVM-SMO Algorithm for clustering on some examples. At first we consider two simple 2D examples for which it is possible to construct the separation line by using the symmetry properties of data. The appropriate data are accumulated in the following Table 3. The exactness of convergence in all examples was taken equal to $10^{-4}$. For comparison with our approach original SVM-SMO algorithm was taken. The appropriate results are presented in the following Table 4.
### Table 3. Data for Example 1 and Example 2

<table>
<thead>
<tr>
<th>Data</th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
</table>
| $x$ matrix | \[
\begin{pmatrix}
-3.0 & 2.0 \\
-2.0 & 2.0 \\
-2.0 & 3.0 \\
2.0 & -3.0 \\
3.0 & -2.0 \\
2.0 & -2.0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
2.0 & 2.0 \\
2.0 & 1.0 \\
1.0 & 2.0 \\
-2.0 & -1.0 \\
-2.0 & -2.0 \\
-1.0 & -2.0 \\
\end{pmatrix}
\] |
| $y$ vector | \[
\begin{pmatrix}
1 \\
1 \\
1 \\
-1 \\
-1 \\
-1 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 \\
1 \\
1 \\
-1 \\
-1 \\
-1 \\
\end{pmatrix}
\] |

Here $N_{iter(Algorithm)}$ is a number of iterations with usage of the appropriate algorithm and $\text{Speed}_{up}=\frac{N_{iter(SVM-SMO)}}{N_{iter(MLPGTch)}}$.

### Table 4. Comparison of algorithms for SVM-SMO clustering

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Number of iterations</th>
<th>Speed_up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1: Original SVM-SMO</td>
<td>162</td>
<td>1.00</td>
</tr>
<tr>
<td>Example 1: MLPGTch M=6</td>
<td>99</td>
<td>1.64</td>
</tr>
<tr>
<td>Example 2: Original SVM-SMO</td>
<td>709</td>
<td>1.00</td>
</tr>
<tr>
<td>Example 2: MLPGTch M=6</td>
<td>54</td>
<td>13.13</td>
</tr>
</tbody>
</table>

Thus we can see that usage of Multi-Layers Profile-Guided Tchebychev Algorithm essentially decreases the number of iterations (up to 13.13) times in comparison with successive iteration algorithm used in original SVM-SMO.

### 8. Conclusions

The presented results show that Multi-Layers Profiled-Guided Tchebychev technique can be effective in acceleration of iteration algorithms in Machine Learning.

### References


STABILITY OF DELAY-DISTRIBUTED HIV INFECTION MODELS WITH MULTIPLE VIRAL PRODUCER CELLS

A. M. ELAIW,† E. KH. Elnahary, A. M. SHEHATA, AND M. ABUL-EZ

1DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KING ABDULAZIZ UNIVERSITY, JEDDAH 21589
SAUDI ARABIA
E-mail address: a.m.elaiw@yahoo.com

2DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE SOHAG UNIVERSITY, SOHAG EGYPT

3DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AL-AZHAR UNIVERSITY, ASSIUT EGYPT

ABSTRACT. We investigate a class of HIV infection models with two kinds of target cells: CD4+ T cells and macrophages. We incorporate three distributed time delays into the models. Moreover, we consider the effect of humoral immunity on the dynamical behavior of the HIV. The viruses are produced from four types of infected cells: short-lived infected CD4+ T cells, long-lived chronically infected CD4+ T cells, short-lived infected macrophages and long-lived chronically infected macrophages. The drug efficacy is assumed to be different for the two types of target cells. The HIV-target incidence rate is given by bilinear and saturation functional response while, for the third model, both HIV-target incidence rate and neutralization rate of viruses are given by nonlinear general functions. We show that the solutions of the proposed models are nonnegative and ultimately bounded. We derive two threshold parameters which fully determine the positivity and stability of the three steady states of the models. Using Lyapunov functionals, we established the global stability of the steady states of the models. The theoretical results are confirmed by numerical simulations.

1. INTRODUCTION

Mathematical modeling and analysis of within-host human immunodeficiency virus (HIV) dynamics have become one of the hot topics during the last decades [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. These works can help researchers for better understanding the HIV dynamical behavior and providing new suggestions for clinical treatment. Most of the mathematical models presented in the literature suppose that HIV infects just the CD4+ T cells [7, 8, 9, 19, 20, 21, 22], while others suppose that there exist another target cells are called macrophages that HIV infects it in addition to CD4+ T cells [12, 13, 14, 15, 18]. For more accurate mathematical models for the HIV dynamics, the model should included both

Received by the editors December 16 2017; Accepted February 26 2018; Published online March 12 2018.
2000 Mathematics Subject Classification. 34D20; 34D23; 37N25; 92B05.
Key words and phrases. HIV infection models, humoral immunity, global stability, time delay, Lyapunov method.
† Corresponding author.
CD4$^+$ T cells and macrophages. In [3], an HIV mathematical model has been presented by considering two types of infected cells, short-lived infected cells $y_i$ and long-lived chronically infected cells $u_i$ as:

$$
\dot{s}_1 = \rho_1 - \beta_1 s_1 - (1 - \varepsilon)\lambda_1 s_1 p, \quad (1.1)
$$

$$
\dot{s}_2 = \rho_2 - \beta_2 s_2 - (1 - f\varepsilon)\lambda_2 s_2 p, \quad (1.2)
$$

$$
\dot{y}_1 = (1 - q)(1 - \varepsilon)\lambda_1 s_1 p - \pi y_1, \quad (1.3)
$$

$$
\dot{y}_2 = (1 - q)(1 - f\varepsilon)\lambda_2 s_2 p - \pi y_2, \quad (1.4)
$$

$$
\dot{u}_1 = q(1 - \varepsilon)\lambda_1 s_1 p - au_1, \quad (1.5)
$$

$$
\dot{u}_2 = q(1 - f\varepsilon)\lambda_2 s_2 p - au_2, \quad (1.6)
$$

$$
\dot{p} = N\pi(y_1 + y_2) + Ma(u_1 + u_2) - cp, \quad (1.7)
$$

where $i = 1, 2$, are denote, respectively, CD4$^+$T cells and the macrophages. The variables $s_i$ and $p$ represent the concentrations of uninfected cells and free HIV particles, respectively. $\rho_i, \beta_i$ and $\lambda_i$ represent the creation rate, the death rate and the infection rate of the uninfected cells, respectively. Parameters $\pi$ and $a$ are the death rate constants of the two types of infected cells, and $c$ is the death rate of HIV. The model incorporates reverse transcriptase inhibitor (RTI) with efficacy $\varepsilon$ for the CD4$^+$ T cells and $f\varepsilon$ for the macrophages where $\varepsilon \in [0, 1]$ and $f \in (0, 1)$. The uninfected target cells become short-lived infected and long-lived chronically infected cells with fractions $(1 - q)$ and $q$, respectively, where $q \in (0, 1)$. The parameters $N$ and $M$ are the average number of HIV particles generated in the lifetime of the short-lived and long-lived infected cells, respectively.

The immune response and time delays were neglected in system (1.1)-(1.7) while that assumption is unrealistic where there exists a time lag between the virus contacting the uninfected cells and the time of generating new infectious viruses. Herz et al. [4] presented a first HIV mathematical model with intracellular time delay. Several HIV models with delays have been presented and investigated [6, 7, 8, 9, 10, 11, 12, 15, 18, 19, 20, 22].

The aim of this paper is to propose HIV infection models which improve model (1.1)-(1.7) by taking into account humoral immunity and distributed delays. We consider two types of target cells, CD4$^+$ T cells and macrophages. We derive two threshold parameters and present some mild sufficient conditions for the positivity and global stability of the steady states of the models.

2. HIV DYNAMICS MODEL WITH BILINEAR INCIDENCE RATE

We formulate an HIV dynamics model with bilinear incidence rate taking into account both humoral immunity and distributed delays,

$$
\dot{s}_i(t) = \rho_i - \beta_i s_i(t) - \lambda_i s_i(t)p(t), \quad i = 1, 2, \quad (2.1)
$$

$$
\dot{y}_i(t) = (1 - q_i)\lambda_i \int_0^t f_i(\tau)e^{-m_i\tau}s_i(t - \tau)p(t - \tau)d\tau - \pi_i y_i(t), \quad i = 1, 2, \quad (2.2)
$$
\[ \dot{u}_i(t) = q_i \lambda_i \int_0^{l_i} f_i(\tau)e^{-m_i \tau}s_i(t-\tau)p(t-\tau)d\tau - \omega_i u_i(t), \quad i = 1, 2, \quad (\phi_1(\theta), \phi_2(\theta), \ldots, \phi_8(\theta)) \in C([-\kappa, 0], \mathbb{R}^8_{\geq 0}). \]

Then, the uniqueness of the solution for \( t > 0 \) is guaranteed [32].

where \( z \) represents the concentration of the B cells. Parameters \( b, \nu \) and \( \mu \) represent, respectively, the removal rate constant of the virus due to the humoral immunity, the proliferation rate constant of B cells and the natural death rate constant of B cells. We suppose that, the virus contacts an uninfected target cell at time \( t - \tau \), the cell becomes infected at time \( t \), where \( \tau \) is a random variable taken from a probability distribution function \( f_i(\tau) \) over the time interval \([0, l_i]\) and \( l_i \) is limit superior of this delay period. The factors \( e^{-m_i \tau}, e^{-n_i \tau} \) and \( e^{-r_i \tau} \) account for the loss of target cells, short-lived infected cells and long-lived chronically infected cells during these delay periods, respectively, where \( m_i, n_i \) and \( r_i \) are constants. All the variables and other parameters of the model have the same meanings as given in model (1.1)-(1.7), where \( \lambda_1 = (1 - \varepsilon)\lambda_1, \lambda_2 = (1 - f \varepsilon)\lambda_2 \).

The probability distribution functions \( f_i(\tau), g_i(\tau) \) and \( h_i(\tau) \) are assumed to satisfy \( f_i(\tau) > 0, g_i(\tau) > 0, h_i(\tau) > 0 \) where \( i = 1, 2 \) and

\[
\int_0^{l_i} f_i(\tau)d\tau = \int_0^{e_i} g_i(\tau)d\tau = \int_0^{d_i} h_i(\tau)d\tau = 1, \quad i = 1, 2,
\]

\[
\int_0^{l_i} f_i(\theta)e^{w\theta}d\theta < \infty, \quad \int_0^{e_i} g_i(\theta)e^{w\theta}d\theta < \infty, \quad \int_0^{d_i} h_i(\theta)e^{w\theta}d\theta < \infty, \quad i = 1, 2,
\]

where \( w \) is a positive constant. Let \( \Theta_i(\tau) = f_i(\tau)e^{-m_i \tau}, \Lambda_i(\tau) = g_i(\tau)e^{-n_i \tau}, \Delta_i(\tau) = h_i(\tau)e^{-r_i \tau} \) and

\[
F_i = \int_0^{l_i} \Theta_i(\tau)d\tau, \quad G_i = \int_0^{e_i} \Lambda_i(\tau)d\tau, \quad C_i = \int_0^{d_i} \Delta_i(\tau)d\tau, \quad i = 1, 2,
\]

then \( 0 < F_i, G_i, C_i \leq 1, \quad i = 1, 2, \)

2.1 Preliminaries. Let \( \varrho = \max\{l_1, l_2, e_1, e_2, d_1, d_2\} \) and \( C \) is the Banach space of continuous functions mapping the interval \([-\varrho, 0]\) into \( \mathbb{R}^8_{\geq 0}\). For the model (2.1)-(2.5) we consider initial conditions

\[
\begin{align*}
s_1(\theta) &= \varphi_1(\theta), \quad s_2(\theta) = \varphi_2(\theta), \quad y_1(\theta) = \varphi_3(\theta), \quad y_2(\theta) = \varphi_4(\theta), \\
u_1(\theta) &= \varphi_5(\theta), \quad u_2(\theta) = \varphi_6(\theta), \quad p(\theta) = \varphi_7(\theta), \quad z(\theta) = \varphi_8(\theta)
\end{align*}
\]

(2.6)

where \( (\varphi_1(\theta), \varphi_2(\theta), \ldots, \varphi_8(\theta)) \in C([-\varrho, 0], \mathbb{R}^8_{\geq 0}) \). Then, the uniqueness of the solution for \( t > 0 \) is guaranteed [32].
Lemma 1. The solutions of system (2.1)-(2.5) satisfying the initial conditions (2.6) are non-negative and ultimately bounded for \( t \in [0, \infty) \).

Proof. Let us write system (2.1)-(2.5) in matrix form \( \dot{Q}(t) = J(Q(t)) \), where \( Q = (s_1, s_2, y_1, y_2, u_1, u_2, p, z)^T \), \( J = (J_1, J_2, ..., J_8)^T \) and

\[
J(Q(t)) = \begin{pmatrix}
J_1(Q(t)) \\
J_2(Q(t)) \\
\vdots \\
J_8(Q(t))
\end{pmatrix},
\]

\[
J = \begin{pmatrix}
\rho_1 - \beta_1 s_1(t) - \lambda_1 s_1(t)p(t) \\
\rho_2 - \beta_2 s_2(t) - \lambda_2 s_2(t)p(t) \\
(1 - q_1)\lambda_1 \int_0^t f_1(\tau)e^{-m_1r}s_1(t-\tau)p(t-\tau)d\tau - \pi_1 y_1(t) \\
(1 - q_2)\lambda_2 \int_0^t f_2(\tau)e^{-m_2r}s_2(t-\tau)p(t-\tau)d\tau - \pi_2 y_2(t) \\
q_1 \lambda_1 \int_0^t f_1(\tau)e^{-m_1r}y_1(t-\tau)p(t-\tau)d\tau - \omega_1 u_1(t) \\
q_2 \lambda_2 \int_0^t f_2(\tau)e^{-m_2r}s_2(t-\tau)p(t-\tau)d\tau - \omega_2 u_2(t) \\
\sum_{i=1}^2 \left(N_i \pi_i \int_0^t g_i(\tau)e^{-n_i r_\tau}y_i(t-\tau)d\tau + M_i \omega_i \int_0^t h_i(\tau)e^{-n_i r_\tau}u_i(t-\tau)d\tau\right) \\
-cp(t) - bp(t)z(t) \\
\nu p(t)z(t) - \mu z(t)
\end{pmatrix}.
\]

We have

\[
J_j(Q(t))|_{Q(t) \in \mathbb{R}_{\geq 0}^8} \geq 0, \quad j = 1, ..., 8.
\]  

Using lemma 2 in [33], the solutions of system (2.1)-(2.5) with the initial states (2.6) satisfy \( Q(t) \in \mathbb{R}_{\geq 0}^8 \) for all \( t \geq 0 \). The nonnegativity of the model’s solution implies that

\[
\limsup_{t \to \infty} s_i(t) \leq \frac{\rho_i}{\beta_i}, \quad i = 1, 2.
\]

Let \( T_i(t) = \int_0^t \Theta_i(\tau)s_i(t-\tau)d\tau + y_i(t) + u_i(t), i = 1, 2 \) then:

\[
\dot{T}_i(t) = F_i \rho_i - \beta_i \int_0^t \Theta_i(\tau)s_i(t-\tau)d\tau - \pi_i y_i(t) - \omega_i u_i(t)
\]

\[
\leq F_i \rho_i - \sigma_i \left( \int_0^t \Theta_i(\tau)s_i(t-\tau)d\tau + y_i(t) + u_i(t) \right)
\]

\[
\leq \rho_i - \sigma_i T_i(t),
\]

where \( \sigma_i = \min\{\beta_i, \pi_i, \omega_i\}, \quad i = 1, 2 \). Hence, \( \limsup_{t \to \infty} T_i(t) \leq L_i \), where \( L_i = \rho_i / \sigma_i, \quad i = 1, 2 \). Since \( s_i(t), y_i(t) \) and \( u_i(t) \) are all non-negative, then \( \limsup_{t \to \infty} y_i(t) \leq L_i \) and \( \limsup_{t \to \infty} u_i(t) \leq L_i \) for all \( t \geq 0 \).
Moreover, we let \( T_3(t) = p(t) + \frac{b}{\nu}z(t) \), then:
\[
\dot{T}_3(t) \leq \sum_{i=1}^{2} (N_i \pi_i G_i + M_i \omega_i C_i) L_i - cp + \frac{b\mu}{\nu} z(t)
\]
\[
\leq \sum_{i=1}^{2} (N_i \pi_i G_i + M_i \omega_i C_i) L_i - \sigma_3 T_3(t),
\]
where \( \sigma_3 = \min\{c, \mu\} \).

Hence \( \limsup_{t \to \infty} T_3(t) \leq L_3 \), for all \( t \geq 0 \), where \( L_3 = \sum_{i=1}^{2} \frac{(N_i \pi_i G_i + M_i \omega_i C_i) L_i}{\sigma_3} \). Since \( p(t) \geq 0 \) and \( z(t) \geq 0 \) then, \( \limsup_{t \to \infty} p(t) \leq L_3 \) and \( \limsup_{t \to \infty} z(t) \leq L_4 \) where \( L_4 = \frac{b}{\nu} L_3 \) for all \( t \geq 0 \). Therefore, \( s_i(t), y_i(t), u_i(t), p(t) \) and \( z(t) \) are ultimately bounded, \( i = 1, 2 \).

According to Lemma 1, we can show that the region
\[
\Omega = \{(s_1, y_1, u_1, p, z) \in C^8 : \|s_1\| \leq L_1, \|y\| \leq L_2, \|u\| \leq L_3, \|z\| \leq L_4\},
\]
is positively invariant with respect to system (2.1)-(2.5).

**Lemma 2.** For system (2.1)-(2.5) there exist two bifurcation parameters \( R_0^B \) and \( R_1^B \) with \( R_0^B > R_1^B > 0 \) such that
(i) if \( R_0^B \leq 1 \), then the system has only one nonnegative steady state \( \Pi_0 \),
(ii) if \( R_1^B \leq 1 < R_0^B \), then the system has only two nonnegative steady states \( \Pi_0 \) and \( \Pi_1 \),
(iii) if \( R_1^B > 1 \), then the system has three nonnegative steady states \( \Pi_0, \Pi_1 \) and \( \Pi_2 \).

**Proof.** System (2.1)-(2.5) has the following steady states:
(i) Infection-free steady state \( \Pi_0 = (s_1^0, s_2^0, 0, 0, 0, 0, 0, 0) \) where \( s_i^0 = \rho_i/\beta_i, i = 1, 2 \),
(ii) Humoral-inactivated infection steady state \( \Pi_1 = (\tilde{s}_1, \tilde{s}_2, \tilde{y}_2, \tilde{u}_2, \tilde{p}, 0) \) where
\[
\tilde{s}_i = \frac{s_i^0}{1 + \eta_i \tilde{p}} > 0, \quad \tilde{y}_i = \frac{(1 - q_i) F_i \lambda_i s_i^0}{\pi_i (1 + \eta_i \tilde{p})} \tilde{p} > 0,
\]
\[
\tilde{u}_i = \frac{q_i F_i \lambda_i s_i^0}{\omega_i (1 + \eta_i \tilde{p})} \tilde{p} > 0, \quad \tilde{p} = \frac{-B + \sqrt{B^2 + 4AC}}{2A},
\]
\[
A = \eta_1 \eta_2, \quad B = \eta_1 R_{01}^B + \eta_2 R_{02}^B + (1 - R_0^B)(\eta_1 + \eta_2),
\]
\[
C = R_0^B - 1, \quad \eta_i = \frac{\lambda_i}{\beta_i}, i = 1, 2,
\]
\[
R_0^B = \sum_{i=1}^{2} \frac{\gamma_i \lambda_i s_i^0}{c_i}, \quad \text{represents the basic reproduction number for system (2.1)-(2.5) and } \gamma_i = ((1 - q_i) G_i N_i + q_i C_i M_i) F_i.\]
(iii) Humoral-activated infection steady state \( \Pi_2 = (\bar{s}_1, \bar{s}_2, \bar{y}_1, \bar{y}_2, \bar{u}_i, \bar{u}_2, \bar{p}, \bar{z}) \) where

\[
\bar{s}_i = \frac{\nu \rho_i}{\nu \beta_i + \mu \lambda_i} > 0, \quad \bar{y}_i = \frac{(1 - q_i) F_i \rho_i \lambda_i}{\pi_i (\nu \beta_i + \mu \lambda_i)} > 0, \quad \bar{u}_i = \frac{q_i F_i \rho_i \lambda_i}{\omega_i (\nu \beta_i + \mu \lambda_i)} > 0, \quad i = 1, 2,
\]

\[\bar{p} = \frac{\mu}{\nu} > 0, \quad \bar{z} = \frac{c}{b} (R^B_1 - 1),\]

and \( R^B_1 = \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c (\nu \beta_i + \mu \lambda_i)} = \sum_{i=1}^{2} \frac{R^B_i}{1 + \frac{\nu}{\omega_i}}, \) denotes the humoral immunity activation number for system (2.1)-(2.5).

We will use the following equalities throughout the paper:

\[
\ln \left( \frac{\phi_i(s_i(t-\tau), p(t-\tau))}{\phi_i(s_i, p)} \right) = \ln \left( \frac{\phi_i(s_i^*, p^*)}{\phi_i(s_i, p)} \right) + \ln \left( \frac{y_i^* \phi_i(s_i(t-\tau), p(t-\tau))}{y_i \phi_i(s_i^*, p^*)} \right)
+ \ln \left( \frac{p \phi_i(s_i, p^*)}{p^* \phi_i(s_i, p)} \right) + \ln \left( \frac{p^* y_i}{p y_i^*} \right),
\]

\[
\ln \left( \frac{y_i(t-\tau)}{y_i} \right) = \ln \left( \frac{p \phi_i(s_i, p^*)}{p^* \phi_i(s_i, p)} \right) + \ln \left( \frac{p^* y_i}{p y_i^*} \right),
\]

\[
\ln \left( \frac{\phi_i(s_i(t-\tau), p(t-\tau))}{\phi_i(s_i^*, p^*)} \right) = \ln \left( \frac{\phi_i(s_i^*, p^*)}{\phi_i(s_i, p)} \right) + \ln \left( \frac{u_i^* \phi_i(s_i(t-\tau), p(t-\tau))}{u_i \phi_i(s_i^*, p^*)} \right)
+ \ln \left( \frac{p \phi_i(s_i, p^*)}{p^* \phi_i(s_i, p)} \right) + \ln \left( \frac{p^* u_i}{p u_i^*} \right),
\]

\[
\ln \left( \frac{u_i(t-\tau)}{u_i} \right) = \ln \left( \frac{p \phi_i(s_i, p^*)}{p^* \phi_i(s_i, p)} \right) + \ln \left( \frac{p^* u_i}{p u_i^*} \right). \tag{2.8}
\]

2.2. Global stability analysis. The following theorems investigate the global stability of the steady states of system (2.1)-(2.5). We will use a function \( H : (0, \infty) \rightarrow [0, \infty) \) as: \( H(\nu) = \nu - 1 - \ln \nu \) throughout the paper.

**Theorem 2.1.** For system (2.1)-(2.5), if \( R^B_0 \leq 1 \), then \( \Pi_0 \) is globally asymptotically stable (GAS).

**Proof.** We construct a Lyapunov functional \( V_0 \) as:

\[
V_0 = \sum_{i=1}^{2} \gamma_i \left[ s_i^0 H \left( \frac{s_i}{s_i^0} \right) + \frac{N_i G_i}{\gamma_i} y_i + \frac{M_i C_i}{\gamma_i} u_i + \frac{\lambda_i}{\omega_i} \int_0^{\ell_i} \Theta_i(\tau) \int_0^\tau s_i(t-\theta)p(t-\theta)d\theta d\tau \right]
+ \frac{N_i \pi_i}{\gamma_i} \int_0^{\ell_i} \Lambda_i(\tau) \int_0^\tau y_i(t-\theta)d\theta d\tau + \frac{M_i \omega_i}{\gamma_i} \int_0^{\ell_i} \Delta_i(\tau) \int_0^\tau u_i(t-\theta)d\theta d\tau \right] + p + \frac{b}{\nu}.
\]

We calculate \( \frac{dV_0}{dt} \) along the trajectories of system (2.1)-(2.5) as:

\[
\frac{dV_0}{dt} = \sum_{i=1}^{2} \gamma_i \left[ (1 - \frac{s_i^0}{s_i}) (\rho_i - \beta_i s_i - \lambda_i s_i p) \right]
\]
Proof. Let

$$\frac{dV_1}{dt} = \sum_{i=1}^{2} \gamma_i \int_{0}^{\tau} \Theta_i(\tau) s_i(t - \tau) d\tau - \pi_i y_i(t)$$

Then, we have

$$\frac{dV_1}{dt} = \sum_{i=1}^{2} \gamma_i \int_{0}^{\tau} \Theta_i(\tau) s_i(t - \tau) d\tau - \pi_i y_i(t)$$

collecting Eq. (2.9) we get:

$$\frac{dV_0}{dt} = -\sum_{i=1}^{2} \gamma_i \frac{(s_i - s_i^0)^2}{s_i} + \sum_{i=1}^{2} \gamma_i \lambda_i s_i^0 p - cp - b_\mu z$$

$$= -\sum_{i=1}^{2} \gamma_i \frac{(s_i - s_i^0)^2}{s_i} + \left( \sum_{i=1}^{2} \gamma_i \lambda_i s_i^0 \right) c - 1 \right) cp - b_\mu z$$

$$= -\sum_{i=1}^{2} \gamma_i \frac{(s_i - s_i^0)^2}{s_i} + \left( R_0^B - 1 \right) cp - b_\mu z. \quad (2.10)$$

Therefore, if $R_0^B \leq 1$, then $\frac{dV_0}{dt} \leq 0$ for all $s_1, s_2, p, z > 0$. Clearly, $\frac{dV_0}{dt} = 0$ at $\Pi_0$. Applying LaSalle’s invariance principle (LIP), we get that $\Pi_0$ is GAS.

Theorem 2.2. If $R_0^B \leq 1 < R_0^B$, then $\Pi_1$ is GAS.

Proof. Let

$$V_1 = \sum_{i=1}^{2} \gamma_i \left[ \lambda_i \int_{0}^{\tau} \Theta_i(\tau) H \left( \frac{s_i(t - \tau) p(t - \tau)}{s_i^0} \right) \right] d\tau$$

$$+ \frac{M_i \lambda_i}{\gamma_i} \int_{0}^{\tau} \Theta_i(\tau) H \left( \frac{y_i(t - \tau)}{y_i^0} \right) d\tau$$

$$+ \frac{M_i \omega_i u_i}{\gamma_i} \int_{0}^{\tau} \Delta_i(\tau) H \left( \frac{u_i(t - \tau)}{u_i^0} \right) d\tau + \frac{b_\mu z}{\mu}.$$
Calculating $\frac{dV_i}{dt}$ along the solutions of system (2.1)-(2.5) we obtain:

$$
\frac{dV_i}{dt} = \sum_{i=1}^{2} \gamma_i \left[ \left( 1 - \frac{\tilde{s}_i}{s_i} \right) \left( \rho_i - \beta_i s_i - \lambda_i s_i p \right) + \frac{N_i G_i}{\gamma_i} \left( 1 - \frac{\tilde{y}_i}{y_i} \right) \left( (1 - q_i) \lambda_i \int_{0}^{l_i} \Theta_i(\tau) s_i(t - \tau) p(t - \tau) d\tau - \pi_i y_i \right) + \frac{M_i C_i}{\gamma_i} \left( 1 - \frac{\tilde{u}_i}{u_i} \right) \left( q_i \lambda_i \int_{0}^{l_i} \Theta_i(\tau) s_i(t - \tau) p(t - \tau) d\tau - \omega_i u_i \right) + \frac{\lambda_i \tilde{s}_i \tilde{p}}{F_i} \int_{0}^{l_i} \Theta_i(\tau) \frac{s_i p}{\tilde{s}_i \tilde{p}} - \frac{s_i(t - \tau) p(t - \tau)}{\tilde{s}_i \tilde{p}} + \ln \left( \frac{s_i(t - \tau) p(t - \tau)}{s_i p} \right) \right) d\tau + \frac{\lambda_i \pi_i \tilde{y}_i}{\gamma_i} \int_{0}^{e_i} \Lambda_i(\tau) \left( \frac{y_i}{\tilde{y}_i} - \frac{y_i(t - \tau)}{\tilde{y}_i} + \ln \left( \frac{y_i(t - \tau)}{y_i} \right) \right) d\tau + \frac{M_i \omega_i \tilde{u}_i}{\gamma_i} \int_{0}^{e_i} \Delta_i(\tau) \left( \frac{u_i}{\tilde{u}_i} - \frac{u_i(t - \tau)}{\tilde{u}_i} + \ln \left( \frac{u_i(t - \tau)}{u_i} \right) \right) d\tau + \left( 1 - \frac{\tilde{p}}{p} \right) \left( \sum_{i=1}^{2} (N_i \pi_i \int_{0}^{e_i} \Lambda_i(\tau) y_i(t - \tau) d\tau + M_i \omega_i \int_{0}^{e_i} \Delta_i(\tau) u_i(t - \tau) d\tau) - c p - b p z \right) + \frac{b}{p}(\nu p z - \mu z) \right]

(2.11)

Collecting terms of Eq. (2.11) and using the conditions of the steady state $\Pi_1$

$$
\rho_i = \beta_i \tilde{s}_i + \lambda_i \tilde{s}_i \tilde{p}, \quad (1 - q_i) F_i \lambda_i \tilde{s}_i \tilde{p} = \pi_i \tilde{y}_i, \quad q_i F_i \lambda_i \tilde{s}_i \tilde{p} = \omega_i \tilde{u}_i,
$$

$$
\tilde{p} = \sum_{i=1}^{2} (N_i \pi_i G_i \tilde{y}_i + M_i \omega_i C_i \tilde{u}_i) = \sum_{i=1}^{2} \gamma_i \lambda_i \tilde{s}_i \tilde{p}, \quad c p = \sum_{i=1}^{2} \gamma_i \lambda_i \tilde{s}_i p,
$$

we get

$$
\frac{dV_i}{dt} = \sum_{i=1}^{2} \gamma_i \left[ \left( 1 - \frac{\tilde{s}_i}{s_i} \right) \left( \beta_i \tilde{s}_i - \beta_i s_i \right) + \lambda_i \tilde{s}_i \tilde{p} \left( 1 - \frac{\tilde{s}_i}{s_i} \right) + \frac{2 N_i G_i}{\gamma_i} \int_{0}^{l_i} \Theta_i(\tau) \frac{\tilde{y}_i s_i(t - \tau) p(t - \tau)}{y_i \tilde{s}_i \tilde{p}} d\tau - \frac{M_i C_i \omega_i \tilde{u}_i}{\gamma_i F_i} \int_{0}^{l_i} \Theta_i(\tau) \frac{\tilde{u}_i s_i(t - \tau) p(t - \tau)}{u_i \tilde{s}_i \tilde{p}} d\tau + \frac{2 M_i C_i \omega_i \tilde{u}_i}{\gamma_i} \tilde{u}_i - \frac{N_i \pi_i \tilde{y}_i}{\gamma_i} \int_{0}^{\epsilon_i} \Lambda_i(\tau) \frac{\tilde{p} y_i(t - \tau)}{p \tilde{y}_i} d\tau - \frac{M_i \omega_i \tilde{u}_i}{\gamma_i} \int_{0}^{\epsilon_i} \Delta_i(\tau) \frac{\tilde{p} u_i(t - \tau)}{p \tilde{u}_i} d\tau + \left( \frac{N_i G_i \pi_i \tilde{y}_i + M_i C_i \omega_i \tilde{u}_i}{\gamma_i F_i} \right) \int_{0}^{l_i} \Theta_i(\tau) \ln \left( \frac{s_i(t - \tau) p(t - \tau)}{s_i p} \right) d\tau
$$
\[
\begin{align*}
+ N_i \pi_i \tilde{y}_i \int_0^{\tau_i} \Lambda_i(\tau) \ln \left( \frac{y_i(t - \tau)}{y_i} \right) d\tau \\
+ M_i \omega_i \tilde{u}_i \int_0^{\tau_i} \Delta_i(\tau) \ln \left( \frac{u_i(t - \tau)}{u_i} \right) d\tau + b \left( \tilde{p} - \frac{p}{p} \right) z.
\end{align*}
\]

Using Eqs. (2.8) with \( \phi_i(s_i, p) = \lambda_i s_i p, \ s^*_i = \bar{s}_i, \ y^*_i = \bar{y}_i, \ u^*_i = \bar{u}_i \) and \( p^* = \tilde{p} \), we can obtain

\[
\frac{dV_i}{dt} = \sum_{i=1}^{2} \left[ -\gamma_i \beta_i(s_i - \bar{s}_i)^2 - \gamma_i \lambda_i \bar{s}_i \tilde{p} H \left( \frac{\bar{s}_i}{s_i} \right) \\
- N_i G_i \pi_i \tilde{y}_i \int_0^{\tau_i} \Theta_i(\tau) H \left( \frac{\tilde{y}_i s_i(t - \tau)p(t - \tau)}{y_i s_i \tilde{p}} \right) d\tau \\
- N_i \pi_i \tilde{y}_i \int_0^{\tau_i} \Lambda_i(\tau) H \left( \frac{\tilde{p} y_i(t - \tau)}{y_i \tilde{p}} \right) d\tau \\
- M_i \omega_i \tilde{u}_i \int_0^{\tau_i} \Theta_i(\tau) H \left( \frac{\tilde{u}_i s_i(t - \tau)p(t - \tau)}{u_i \tilde{s}_i \tilde{p}} \right) d\tau \\
- M_i \omega_i \tilde{u}_i \int_0^{\tau_i} \Delta_i(\tau) H \left( \frac{\tilde{p} u_i(t - \tau)}{p \tilde{u}_i} \right) d\tau \right] + b \left( \tilde{p} - \bar{p} \right) z.
\]

From the conditions of the steady state \( \Pi_1 \) we have \( \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c \beta_i(1 + \eta_i \tilde{p})} = 1 \), then

\[
R_i^B - 1 = \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c \beta_i(1 + \eta_i \tilde{p})} - \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c \beta_i(1 + \eta_i \tilde{p})} = \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c \beta_i(1 + \eta_i \tilde{p})} = (\tilde{p} - \bar{p}) \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c \beta_i(1 + \eta_i \tilde{p})(1 + \eta_i \tilde{p})} = \zeta (\tilde{p} - \bar{p})
\]

Eq. (2.12) implies that \( \tilde{p} - \bar{p} = \frac{1}{\zeta} (R_i^B - 1) \), where, \( \zeta = \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho_i}{c \beta_i(1 + \eta_i \tilde{p})(1 + \eta_i \tilde{p})} \). Therefore, \( R_i^B \leq 1 \) ensure \( \frac{dV_i}{dt} \leq 0 \) for all \( s_i, y_i, u_i, p, z > 0 \). It follows that for all \( s_i, y_i, u_i, p, z > 0 \) we have \( \frac{dV_i}{dt} \leq 0 \) and \( \frac{dV_i}{dt} = 0 \) at \( \Pi_1 \). By LIP \( \Pi_1 \) is GAS.

**Theorem 2.3.** If \( R_i^B > 1 \) then \( \Pi_2 \) is GAS.

**Proof.** Consider

\[
V_2 = \sum_{i=1}^{2} \gamma_i \left[ \bar{s}_i H \left( \frac{s_i}{\bar{s}_i} \right) + \frac{N_i G_i}{\gamma_i} \tilde{y}_i H \left( \frac{y_i}{\tilde{y}_i} \right) + \frac{M_i C_i}{\gamma_i} \tilde{u}_i H \left( \frac{u_i}{\tilde{u}_i} \right) \right]
\]
Thus if \( R_1^B > 1 \), then \( \bar{s}_i, \bar{y}_i, \bar{u}_i, \bar{p}, \bar{z} > 0 \). Therefore we get \( \frac{dV_2}{dt} \leq 0 \) and \( \frac{dV_2}{dt} = 0 \) at \( \Pi_2 \). LIP implies that \( \Pi_2 \) is GAS.
3. MODEL WITH SATURATION INCIDENCE RATE

We consider a model with a saturation incidence rate and humoral immunity as:

\[
\begin{align*}
    s_i(t) &= \rho_i - \beta_i s_i(t) - \frac{\lambda_i s_i(t)p(t)}{1 + \alpha_i p(t)} , \quad i = 1, 2, \\
    y_i(t) &= (1 - q_i) \lambda_i \int_0^t \Theta_i(\tau) \frac{s_i(t - \tau)p(t - \tau)}{1 + \alpha_i p(t - \tau)} d\tau - \pi_i y_i(t) , \quad i = 1, 2, \\
    \tilde{u}_i(t) &= q_i \lambda_i \int_0^t \Theta_i(\tau) \frac{s_i(t - \tau)p(t - \tau)}{1 + \alpha_i p(t - \tau)} d\tau - \omega_i u_i(t) , \quad i = 1, 2, \\
    \dot{p}(t) &= \sum_{i=1}^2 \left( N_i \pi_i \int_0^{t_i} \Lambda_i(\tau) y_i(t - \tau) d\tau + M_i \omega_i \int_0^{t_i} \Delta_i(\tau) u_i(t - \tau) d\tau \right) - c p(t) - b p(t) z(t), \\
    \dot{z}(t) &= v p(t) z(t) - \mu z(t).
\end{align*}
\]

where \( \alpha_i > 0 \). As the same to the previous section it’s easy to show the non-negativity and boundedness of the solutions.

**Lemma 3.** For system (3.1)-(3.5) there exist two bifurcation parameters \( R_0^S \) and \( R_1^S \) with \( R_0^S > R_1^S > 0 \) such that

(i) if \( R_0^S \leq 1 \), then the system has only one nonnegative steady state \( \Pi_0 \),

(ii) if \( R_1^S \leq 1 < R_0^S \), then the system has only two nonnegative steady states \( \Pi_0 \) and \( \Pi_1 \),

(iii) if \( R_1^S > 1 \), then the system has three nonnegative steady states \( \Pi_0 \), \( \Pi_1 \) and \( \Pi_2 \).

**Proof.** System (3.1)-(3.5) has the following steady states:

(i) Infection-free steady state \( \Pi_0 = (s_1^0, s_2^0, 0, 0, 0, 0, 0, 0) \) where \( s_i^0 = \frac{\rho_i}{\beta_i}, \quad i = 1, 2 \),

(ii) Humoral-inactivated infection steady state \( \Pi_1 = (\tilde{s}_1, \tilde{s}_2, \tilde{y}_1, \tilde{y}_2, \tilde{u}_1, \tilde{u}_2, \tilde{p}, 0) \) where

\[
\begin{align*}
    \tilde{s}_i &= \frac{s_i^0(1 + \alpha_i \tilde{p})}{1 + \xi_i \tilde{p}}, \\
    \tilde{y}_i &= \frac{(1 - q_i) F_i \lambda_i s_i^0 \tilde{p}}{\pi_i(1 + \xi_i \tilde{p})}, \\
    \tilde{u}_i &= \frac{q_i F_i \lambda_i s_i^0 \tilde{p}}{\omega_i(1 + \xi_i \tilde{p})}, \\
    \tilde{p} &= \frac{-\hat{B} + \sqrt{\hat{B}^2 + 4AC}}{2A},
\end{align*}
\]

where,

\[
\begin{align*}
    \hat{A} &= \xi_1 \xi_2, \quad \hat{B} = \xi_1 R_{01}^S + \xi_2 R_{02}^S + (1 - R_0^S)(\xi_1 + \xi_2), \\
    \hat{C} &= R_0^S - 1, \quad \xi_i = \alpha_i + \frac{\lambda_i}{\beta_i}, \quad i = 1, 2,
\end{align*}
\]

and \( R_0^S = \sum_{i=1}^2 \frac{\gamma_i \lambda_i s_i^0}{\alpha_i} \), is the basic reproduction number for model (3.1)-(3.5).
(iii) Humoral-activated infection steady state \( \Pi_2 = (\bar{s}_1, \bar{s}_2, \bar{y}_1, \bar{y}_2, \bar{u}_1, \bar{u}_2, \bar{p}, \bar{z}) \), where

\[
\bar{s}_i = \frac{\rho_i (\nu + \alpha_i s_i)}{\beta_i (\nu + \mu s_i)} > 0, \quad \bar{y}_i = \frac{(1 - q_i) F_i \rho_i \lambda_i \mu}{\beta_i \pi_i (\nu + \mu s_i)} > 0,
\]

\[
\bar{u}_i = \frac{q_i F_i \rho_i \lambda_i \mu}{\omega_i \beta_i (\nu + \mu s_i)} > 0, \quad i = 1, 2, \quad \bar{p} = \frac{\mu}{\nu} > 0, \quad \bar{z} = \frac{c}{b} (R^S_1 - 1),
\]

and \( R^S_1 = \sum_{i=1}^{2} \frac{\gamma_i \omega_i \beta_i \mu}{c \beta_i (\nu + \mu s_i)} \), is the humoral immunity activation number for system (3.1)-(3.5).

\[\square\]


Theorem 3.1. For system (3.1)-(3.5), if \( R^S_0 \leq 1 \), then \( \Pi_0 \) is GAS.

Proof. We consider a Lyapunov function \( U_0 \) as:

\[
U_0 = \sum_{i=1}^{2} \gamma_i \left[ s_i^0 H \left( \frac{s_i}{s_i^0} \right) + \frac{N_i G_i}{\gamma_i} y_i + \frac{M_i C_i}{\gamma_i} u_i + \frac{\lambda_i}{F_i} \right] \int_{0}^{t_i} \Theta_i (\tau) \int_{0}^{\tau} s_i (t-\theta) p(t-\theta) \frac{1}{1 + \alpha_i p(t-\theta)} d\theta d\tau + \frac{N_i \pi_i}{\gamma_i} \int_{0}^{t_i} \Lambda_i (\tau) \int_{0}^{\tau} u_i (t-\theta) d\theta d\tau + \frac{b}{\nu} z.
\]

Calculating \( \frac{dU_0}{dt} \) along the trajectories of (3.1)-(3.5) we get:

\[
\frac{dU_0}{dt} = -2 \sum_{i=1}^{2} \gamma_i \beta_i \left( \frac{(s_i - s_i^0)^2}{s_i} \right) + \left( \sum_{i=1}^{2} \frac{\gamma_i \lambda_i s_i^0}{c (1 + \alpha_i p)} - 1 \right) cp - \frac{b \mu}{\nu} z
\]

\[
= -2 \sum_{i=1}^{2} \gamma_i \beta_i \left( \frac{(s_i - s_i^0)^2}{s_i} \right) - \sum_{i=1}^{2} \frac{R^S_0 \alpha_i c p^2}{(1 + \alpha_i p)} + (R^S_1 - 1) cp - \frac{b \mu}{\nu} z.
\]

Thus if \( R^S_0 \leq 1 \), then \( \frac{dU_0}{dt} \leq 0 \) for all \( s_1, s_2, p, z > 0 \). Clearly \( \frac{dU_0}{dt} = 0 \) at \( \Pi_0 \). Applying (LIP), we get that \( \Pi_0 \) is GAS.

\[\square\]

Theorem 3.2. For system (3.1)-(3.5) if \( R^S_1 \leq 1 < R^S_0 \), then \( \Pi_1 \) is GAS.

Proof. Construct

\[
U_1 = \sum_{i=1}^{2} \gamma_i \left[ \bar{s}_i H \left( \frac{s_i}{\bar{s}_i} \right) + \frac{N_i G_i}{\gamma_i} \bar{y}_i H \left( \frac{y_i}{\bar{y}_i} \right) + \frac{M_i C_i}{\gamma_i} \bar{u}_i H \left( \frac{u_i}{\bar{u}_i} \right) + \frac{\lambda_i \bar{s}_i \bar{p}}{F_i \left( 1 + \alpha_i \bar{p} \right)} \int_{0}^{t_i} \Theta_i (\tau) \int_{0}^{\tau} H \left( \frac{s_i (t-\theta) p(t-\theta) \left( 1 + \alpha_i \bar{p} \right)}{\bar{s}_i \bar{p} \left( 1 + \alpha_i \bar{p} \right)} \right) d\theta d\tau \right.
\]

\[
+ \frac{N_i \pi_i \bar{y}_i}{\gamma_i} \int_{0}^{t_i} \Lambda_i (\tau) \int_{0}^{\tau} H \left( \frac{y_i(t-\theta)}{\bar{y}_i} \right) d\theta d\tau
\]

\[\square\]
Calculating $\frac{dU}{dt}$ along the solutions of system (3.1)-(3.5), we get

$$
\frac{dU_1}{dt} = \sum_{i=1}^{2} \left[ \left( 1 - \frac{s_i}{s_i} \right) (\rho_i - \beta_i s_i) + \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}} + \frac{N_i G_i \pi_i}{\gamma_i} \tilde{y}_i \right]
+ M_i \omega_i \tilde{u}_i \int_{0}^{\theta_i} \Delta_i(\tau) \int_{0}^{\tau} H \left( \frac{u_i(t - \theta)}{u_i} \right) d\theta d\tau + \tilde{p} H \left( \frac{p}{\tilde{p}} \right) + \frac{b}{\nu} z.
$$

From the steady state conditions of $\Pi_1$:

$$
\rho_i = \beta_i \tilde{s}_i + \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}}, \quad (1 - q_i) F_i \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}} = \pi_i \tilde{y}_i, \quad q_i F_i \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}} = \omega_i \tilde{u}_i,
$$

$$
c\tilde{p} = \sum_{i=1}^{2} (N_i \pi_i G_i \tilde{y}_i + M_i \omega_i C_i \tilde{u}_i) = \sum_{i=1}^{2} \gamma_i \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}}, \quad cp = \frac{p}{\tilde{p}} \sum_{i=1}^{2} \gamma_i \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}},
$$

we obtain:

$$
\frac{dU_1}{dt} = \sum_{i=1}^{2} \left[ \left( 1 - \frac{s_i}{s_i} \right) (\beta_i \tilde{s}_i - \beta_i s_i) + \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}} \left( 1 - \frac{s_i}{s_i} \right) \right]
+ \frac{\lambda_i \tilde{s}_i \tilde{p}}{1 + \alpha_i \tilde{p}} \left( \frac{p(1 + \alpha_i \tilde{p}) - \tilde{p}}{\tilde{p}} \right) + \frac{2N_i \pi_i G_i \tilde{y}_i}{\gamma_i} + \frac{2M_i \omega_i C_i \tilde{u}_i}{\gamma_i}
- \frac{N_i \pi_i G_i \tilde{y}_i}{\gamma_i F_i} \int_{0}^{\theta_i} \Theta_i(\tau) \frac{\tilde{y}_i s_i(t - \tau)p(t - \tau)(1 + \alpha_i \tilde{p})}{\tilde{y}_i \tilde{s}_i \tilde{p}(1 + \alpha_i \tilde{p})} d\tau.
$$
Using Eqs. (2.8) with \( \phi_i(s_i, p) = \frac{\lambda_i s_i p}{1 + \alpha p} \), \( s_i^* = \bar{s}_i \), \( y_i^* = \bar{y}_i \), \( u_i^* = \bar{u}_i \) and \( p^* = \bar{p} \), then we have:

\[
\frac{dU_1}{dt} = \sum_{i=1}^{2} \left[ -\gamma_i \beta_i (s_i - \bar{s}_i)^2 \frac{s_i}{s_i} - \gamma_i \frac{\lambda_i \bar{s}_i \bar{p}}{1 + \alpha_i p} \left( \frac{\alpha_i (p - \bar{p})^2}{\bar{p}(1 + \alpha_i p)(1 + \alpha_i \bar{p})} \right) \right]
- \gamma_i \frac{\lambda_i \bar{s}_i \bar{p}}{1 + \alpha_i p} \left( H \left( \frac{\bar{s}_i}{s_i} \right) + H \left( \frac{1 + \alpha_i p}{1 + \alpha_i \bar{p}} \right) \right)
- N_i \pi_i \bar{y}_i \int_0^{\theta_i} \Theta_i(\tau) H \left( \frac{\bar{p} y_i(t - \tau) p(t - \tau)(1 + \alpha_i \bar{p})}{y_i \bar{s}_i \bar{p}(1 + \alpha_i p(t - \tau))} \right) d\tau
- N_i \pi_i \bar{y}_i \int_0^{\theta_i} \Lambda_i(\tau) H \left( \frac{\bar{p} y_i(t - \tau)}{p y_i} \right) d\tau
- M_i \omega_i \bar{u}_i \int_0^{\theta_i} \Theta_i(\tau) H \left( \frac{\bar{p} u_i(t - \tau)(1 + \alpha_i \bar{p})}{u_i \bar{s}_i \bar{p}(1 + \alpha_i p(t - \tau))} \right) d\tau
- M_i \omega_i \bar{u}_i \int_0^{\theta_i} \Theta_i(\tau) H \left( \frac{\bar{p} u_i(t - \tau)}{p u_i} \right) d\tau + b \left( \bar{p} - \frac{\mu}{\nu} \right) z.
\]

Similar to proof of Eq. (2.12) we can get \( \bar{p} - \bar{p} = \frac{1}{Q}(R_1^S - 1) \) where, \( Q_1 = \sum_{i=1}^{2} \frac{\gamma_i \lambda_i \rho \delta_i}{\varphi_i(1 + \xi_i \bar{p})(1 + \xi_i \bar{p})} \).

Thus, if \( R_1^S \leq 1 \) then \( \bar{p} \leq \frac{\mu}{\nu} = \bar{p} \).

If \( R_1^S \leq 1 \), then \( \frac{dU_1}{dt} \leq 0 \) for all \( s_i, y_i, u_i, p, z > 0 \) where equality occurs at \( \Pi_1 \). LIP implies the global stability of \( \Pi_1 \).

**Theorem 3.3.** For system (3.1)-(3.5) if \( R_1^S > 1 \), then \( \Pi_2 \) is GAS.
Proof. Define:

\[
U_2 = \sum_{i=1}^{2} \gamma_i \left[ \tilde{s}_i H \left( \frac{s_i}{\tilde{s}_i} \right) + \frac{N_i G_i \tilde{y}_i}{\gamma_i} H \left( \frac{y_i}{\tilde{y}_i} \right) + \frac{M_i C_i \bar{u}_i H \left( \frac{u_i}{\bar{u}_i} \right)}{\gamma_i} \right] \\
+ \frac{1}{F_i (1 + \alpha_i \bar{p})} \int_{0}^{\tilde{\tau}_i} \Theta_i(\tau) \int_{0}^{\tau} H \left( \frac{s_i(t - \theta)p(t - \theta)}{\tilde{s}_i \bar{p}(1 + \alpha_i \bar{p}(t - \theta))} \right) \, d\theta \, d\tau \\
+ \frac{N_i \pi_i \tilde{y}_i}{\gamma_i} \int_{0}^{e_i} \Lambda_i(\tau) \int_{0}^{\tau} H \left( \frac{y_i(t - \theta)}{\tilde{y}_i} \right) \, d\theta \, d\tau \\
+ \frac{M_i \omega_i \bar{u}_i}{\gamma_i} \int_{0}^{\tilde{\tau}_i} \Delta_i(\tau) \int_{0}^{\tau} H \left( \frac{u_i(t - \theta)}{\bar{u}_i} \right) \, d\theta \, d\tau + p H \left( \frac{\bar{p}}{\bar{p}} \right) + \frac{b}{\nu} \bar{z} H \left( \frac{\bar{z}}{\bar{z}} \right).
\]

The time derivative of \( U_2 \) along the trajectories of system (3.1)-(3.5) is obtained by:

\[
\frac{dU_2}{dt} = \sum_{i=1}^{2} \gamma_i \left[ \left( 1 - \frac{s_i}{\tilde{s}_i} \right) \left( \rho_i - \beta_i s_i \right) + \frac{\lambda_i \tilde{s}_i \bar{p}}{1 + \alpha_i \bar{p}} + \frac{N_i G_i \pi_i \tilde{y}_i}{\gamma_i} \right] \\
+ \frac{M_i C_i \omega_i \bar{u}_i}{\gamma_i} - \frac{(1 - q_i) N_i G_i \lambda_i}{\gamma_i} \int_{0}^{\tilde{\tau}_i} \Theta_i(\tau) \frac{\tilde{y}_i s_i(t - \tau)p(t - \tau)}{y_i(1 + \alpha_i \bar{p}(t - \tau))} \, d\tau \\
- \frac{q_i M_i C_i \lambda_i}{\gamma_i} \int_{0}^{\tilde{\tau}_i} \Theta_i(\tau) \frac{u_i(t - \tau)p(t - \tau)}{u_i(1 + \alpha_i \bar{p}(t - \tau))} \, d\tau \\
+ \frac{1}{F_i (1 + \alpha_i \bar{p})} \int_{0}^{\tilde{\tau}_i} \Theta_i(\tau) \ln \left( \frac{s_i(t - \tau)p(t - \tau)(1 + \alpha_i \bar{p})}{s_i \bar{p}(1 + \alpha_i \bar{p}(t - \tau))} \right) \, d\tau \\
+ \frac{N_i \pi_i \tilde{y}_i}{\gamma_i} \int_{0}^{e_i} \Lambda_i(\tau) \ln \left( \frac{y_i(t - \tau)}{\tilde{y}_i} \right) \, d\tau \\
+ \frac{M_i \omega_i \bar{u}_i}{\gamma_i} \int_{0}^{\tilde{\tau}_i} \Delta_i(\tau) \ln \left( \frac{u_i(t - \tau)}{\bar{u}_i} \right) \, d\tau - \sum_{i=1}^{2} N_i \pi_i \int_{0}^{e_i} \Lambda_i(\tau) \frac{\tilde{y}_i u_i(t - \tau)}{p} \, d\tau \\
- \sum_{i=1}^{2} M_i \omega_i \int_{0}^{\tilde{\tau}_i} \Delta_i(\tau) \frac{\bar{u}_i(t - \tau)}{p} \, d\tau - cp + c \bar{p} + b \bar{p} \bar{z} - \frac{b \mu}{\nu} \bar{z} + \frac{b \mu}{\nu} \bar{z}. \tag{3.6}
\]

Using the steady state conditions of \( \Pi_2 \):

\[
\rho_i = \beta_i \tilde{s}_i + \frac{\lambda_i \tilde{s}_i \bar{p}}{1 + \alpha_i \bar{p}} (1 - q_i) F_i, \quad \frac{\lambda_i \tilde{s}_i \bar{p}}{1 + \alpha_i \bar{p}} = \pi_i \tilde{y}_i, \quad q_i F_i, \quad \frac{\lambda_i \tilde{s}_i \bar{p}}{1 + \alpha_i \bar{p}} = \omega_i \bar{u}_i, \\
\]

\[
c \bar{p} = \sum_{i=1}^{2} (N_i \pi_i G_i \tilde{y}_i + M_i \omega_i C_i \bar{u}_i) - b \bar{p} \bar{z}, \quad c \bar{p} = \frac{p}{b} \sum_{i=1}^{2} \gamma_i \frac{\lambda_i \tilde{s}_i \bar{p}}{1 + \alpha_i \bar{p}} - b \bar{p} \bar{z}, \quad \bar{p} = \frac{\mu}{\nu},
\]
and applying Eqs. (2.8) with $\phi_i(s_i, p) = \frac{\lambda_i s_i p}{1 + \alpha_i p}$, $s_i^* = \bar{s}_i$, $y_i^* = \bar{y}_i$, $u_i^* = \bar{u}_i$ and $p^* = \bar{p}$, we find:

$$
\frac{dU_2}{dt} = \sum_{i=1}^{2} \left[ -\gamma_i \frac{\lambda_i \bar{s}_i \bar{p}}{s_i} - \gamma_i \frac{\lambda_i \bar{s}_i \bar{p}}{1 + \alpha_i \bar{p}} \left( \frac{\alpha_i (p - \bar{p})^2}{\bar{p}(1 + \alpha_i p)(1 + \alpha_i \bar{p})} \right) 
- \gamma_i \frac{\lambda_i \bar{s}_i \bar{p}}{1 + \alpha_i \bar{p}} \left( H \left( \frac{\bar{s}_i}{s_i} \right) + H \left( \frac{1 + \alpha_i \bar{p}}{1 + \alpha_i p} \right) \right) 
- N_i \pi_i \bar{y}_i \int_{0}^{\varepsilon_i} \Lambda_i(\tau) H \left( \frac{\bar{y}_i \bar{s}_i (t - \tau) p(t - \tau)(1 + \alpha_i p)}{y_i \bar{s}_i \bar{p}(1 + \alpha_i p)(1 + \alpha_i \bar{p})} \right) d\tau 
- \frac{M_i \omega_i \bar{u}_i}{F_i} \int_{0}^{\varepsilon_i} \Theta_i(\tau) H \left( \frac{\bar{u}_i \bar{s}_i (t - \tau) p(t - \tau)(1 + \alpha_i p)}{u_i \bar{s}_i \bar{p}(1 + \alpha_i p)(1 + \alpha_i \bar{p})} \right) d\tau 
- M_i \omega_i \bar{u}_i \int_{0}^{\varepsilon_i} \Delta_i(\tau) H \left( \frac{\bar{u}_i \bar{s}_i (t - \tau)}{\bar{p} u_i} \right) d\tau \right].
$$

Thus, if $R_1^S > 1$ then $\bar{s}_i, \bar{y}_i, \bar{u}_i, \bar{p}, \bar{z} > 0$. Therefore $\frac{dU_2}{dt} \leq 0$. Applying LIP one can show that $\Pi_2$ is GAS.

4. Model with General Incidence Rate

We consider a model with general incidence and neutralization rates as:

$$
\begin{align*}
\dot{s}_i(t) &= \rho_i - \beta_i s_i(t) - \phi_i(s_i(t), p(t)), \quad i = 1, 2, \tag{4.1} \\
\dot{y}_i(t) &= (1 - q_i) \int_{0}^{t_i} \Theta_i(\tau) \phi_i(s_i(t - \tau), p(t - \tau)) d\tau - \pi_i y_i(t), \quad i = 1, 2, \tag{4.2} \\
\dot{u}_i(t) &= q_i \int_{0}^{t_i} \Theta_i(\tau) \phi_i(s_i(t - \tau), p(t - \tau)) d\tau - \omega_i u_i(t), \quad i = 1, 2, \tag{4.3} \\
\dot{p}(t) &= \sum_{i=1}^{2} \left( N_i \pi_i \int_{0}^{\varepsilon_i} \Lambda_i(\tau) y_i(t - \tau) d\tau + M_i \omega_i \int_{0}^{\varepsilon_i} \Delta_i(\tau) u_i(t - \tau) d\tau \right) - c p(t) - b p(t) \psi(z(t)), \tag{4.4} \\
\dot{z}(t) &= \nu p(t) \psi(z(t)) - \mu \psi(z(t)). \tag{4.5}
\end{align*}
$$

All the parameters are positive. Function $\phi_i(s_i, p)$, $i = 1, 2$ represents the incidence rate where, $\phi_1(s_1, p) = (1 - \varepsilon) \phi_1(s_1, p)$, and $\phi_2(s_2, p) = (1 - f \varepsilon) \phi_2(s_2, p)$. Also, $b p \psi(z)$, $\nu p \psi(z)$ and $\mu \psi(z)$, represent the neutralize rate of viruses, the activation rate of B cells and the removal rate of B cells, respectively. For model (4.1)-(4.5) the initial conditions are given by Eq. (2.6).

Suppose that functions $\phi_i$ and $\psi$ are continuously differentiable such that:

**Assumption (A1)** Function $\phi_i$ satisfies:

(i) $\phi_i(s_i, p) > 0$, $\phi_i(s_i, 0) = \phi_i(0, p) = 0$, for all $s_i > 0$, $p > 0$,
Assumption (A2). Function $\phi_i$ satisfies:
(i) $\phi_i(s_i, p) \leq p \phi_i(s_i, 0), \quad \text{for all } p > 0,$
(ii) $\frac{d}{ds_i} \left( \frac{\partial \phi_i(s_i, 0)}{\partial p} \right) > 0$ for all $s_i > 0, i = 1, 2.$

Assumption (A3). Function $\phi_i$ satisfies:
\[
\left( \frac{\phi_i(s_i, p)}{\phi_i(s_i, p^*)} - \frac{p}{p^*} \right) \left( 1 - \frac{\phi_i(s_i, p^*)}{\phi_i(s_i, p)} \right) \leq 0, \quad s_i, p > 0, \quad i = 1, 2, \text{ where } p^* = \bar{p} \text{ or } p^* = \tilde{p}.
\]

Assumption (A4). Function $\psi$ satisfies: (i) $\psi(z) > 0$, for all $z > 0$, $\psi(0) = 0,$
(ii) $\psi'(z) > 0$, for all $z \geq 0$ and
(iii) there is $\overline{z} > 0$ such that $\psi(z) > \overline{z}z$ for all $z > 0.$

The non-negativity of the solutions of system (4.1)-(4.5) can easily be shown. Similar to proof of Lemma 1 we get $\limsup_{t \to \infty} s_i(t) \leq \frac{\beta_i}{\beta_i}, \limsup_{t \to \infty} y_i(t) \leq L_i,$ and $\limsup_{t \to \infty} u_i(t) \leq L_i$ for all $t \geq 0$, and $\sigma_i = \min\{\beta_i, \pi_i, \omega_i\}, i = 1, 2.$ From (A4)(iii), let $T(t) = p(t) + \frac{b}{\psi'(z(t))},$ then
\[
\dot{T}(t) = 2 \left( N_i \pi_i \int_0^{\tau_i} \Delta_i(\tau) y_i(t - \tau) d\tau + M_i \omega_i \int_0^{\tau_i} \Delta_i(\tau) u_i(t - \tau) d\tau \right) - c p(t) - \frac{b \mu}{\nu} \psi(z(t))
\]
\[
\leq 2 \left( N_i \pi_i G_i + M_i \omega_i C_i \right) L_i - \sigma_3 \left( p(t) + \frac{b}{\psi'(z(t))} \right)
\]
where $\sigma_3 = \min\{c, \mu \overline{z}\}.$ Therefore $\limsup_{t \to \infty} T(t) \leq L_3,$ where $L_3 = \sum_{i=1}^{2} \frac{(N_i \pi_i G_i + M_i \omega_i C_i) L_i}{\sigma_3}.$

The non-negativity of $p(t) \geq 0$ and $z(t) \geq 0$ implies that $\limsup_{t \to \infty} p(t) \leq L_3$ and $\limsup_{t \to \infty} z(t) \leq L_4$ where $L_4 = \frac{b}{\nu} L_3$ for all $t \geq 0.$ Hence, $s_i(t), y_i(t), u_i(t), i = 1, 2,$ $p(t)$ and $z(t)$ are ultimately bounded.


Lemma 4. For system (4.1)-(4.5) there exist two bifurcation parameters $R_0^G$ and $R_1^G$ with $R_0^G > R_1^G > 0$ such that
(i) if $R_0^G \leq 1,$ then the system has only one nonnegative steady state $\Pi_0,$
(ii) if $R_1^G \leq 1 < R_0^G,$ then the system has only two nonnegative steady states $\Pi_0$ and $\Pi_1,$
(iii) if $R_1^G > 1,$ then the system has three nonnegative steady states $\Pi_0, \Pi_1$ and $\Pi_2.$

Proof. Let Assumptions (A1)-(A4) are valid, and $\Pi(s_1, s_2, y_1, y_2, u_1, u_2, p, z)$ be any steady state satisfying the following equations:
\[
\rho_i - \beta_i s_i - \phi_i(s_i, p) = 0, \quad i = 1, 2, \quad (4.6)
\]
\[
(1 - q_i) F_i \phi_i(s_i, p) - \pi_i y_i = 0, \quad i = 1, 2, \quad (4.7)
\]
\[
q_i F_i \phi_i(s_i, p) - \omega_i u_i = 0, \quad i = 1, 2, \quad (4.8)
\]
$$\sum_{i=1}^{2} \left( N_i \pi_i G_i y_i + M_i \omega_i C_i u_i \right) - c p \psi(z) = 0,$$

(4.9)

$$\nu_p \psi(z) - \mu \psi(z) = 0. \quad (4.10)$$

From Eq. (4.10) we have $\psi(z) = 0$ or $p = \frac{\mu}{\nu}$. First, we consider the case $\psi(z) = 0$, then from Assumption (A4) we have $z = 0$. Let $z = 0$ in Eqs. (4.6)-(4.9) we have:

$$\sum_{i=1}^{2} \gamma_i \phi_i(s_i, p) - c p = 0. \quad (4.11)$$

Eq. (4.11) has two solutions, $p = 0$ and $p \neq 0$. If $p = 0$ we get $\Pi_0 = (s_1^0, s_2^0, 0, 0, 0, 0, 0)$ where $s_i^0 = \frac{\alpha_i}{\beta_i}, i = 1, 2$. If $p \neq 0$, then we obtain a humoral-inactivated infection steady state $\Pi_1 = (\bar{s}_1, \bar{s}_2, \bar{y}_1, \bar{y}_2, \bar{u}_1, \bar{u}_2, \bar{p}, 0)$ where the coordinates satisfy the equalities:

$$\rho_i = \beta_i \bar{s}_i + \phi_i(\bar{s}_i, \bar{p}), \quad (1 - q_i) F_i \phi_i(\bar{s}_i, \bar{p}) = \pi_i \bar{y}_i;$$

$$c \bar{p} = \sum_{i=1}^{2} \gamma_i \phi_i(\bar{s}_i, \bar{p}), \quad q_i F_i \phi_i(\bar{s}_i, \bar{p}) = \omega_i \bar{u}_i. \quad (4.12)$$

The other solution of Eq. (4.10) is $\bar{p} = \frac{\mu}{\nu}$. Substituting $p = \bar{p}$ in Eq. (4.6) and letting

$$\Psi(s_i) = \rho_i - \beta_i s_i - \phi_i(s_i, \bar{p}) = 0. \quad (4.13)$$

According to Assumption (A1), $\Psi$ is a decreasing function of $s_i$. Besides $\Psi(0) = \rho_i > 0$ and $\Psi(s_i^0) = -\phi_i(s_i^0, \bar{p}) < 0$. Thus, there exists a unique $\bar{s}_i \in (0, s_i^0)$ such that $\Psi(\bar{s}_i) = 0$. It follows from Eqs. (4.7)-(4.9) that:

$$\bar{y}_i = \frac{(1 - q_i) F_i \phi_i(\bar{s}_i, \bar{p})}{\pi_i}, \quad \bar{u}_i = \frac{q_i F_i \phi_i(\bar{s}_i, \bar{p})}{\omega_i},$$

$$\bar{z} = \psi^{-1} \left( \frac{c}{b} \sum_{i=1}^{2} \frac{\gamma_i \phi_i(\bar{s}_i, \bar{p})}{\bar{p}} - 1 \right).$$

Thus, $\bar{z} > 0$ when $\frac{c}{b} \frac{\phi_i(\bar{s}_i, \bar{p})}{\bar{p}} > 1$. Let us define the parameter $R^G_1$ as:

$$R^G_1 = \frac{2}{\sum_{i=1}^{2} \frac{\gamma_i \phi_i(\bar{s}_i, \bar{p})}{\bar{p}}}. \quad \text{If } R^G_1 > 1, \text{ then } \bar{z} = \psi^{-1} \left( \frac{c}{b} \left( R^G_1 - 1 \right) \right) \text{ and there exists a humoral-activated infection steady state } \Pi_2 = (\bar{s}_1, \bar{s}_2, \bar{y}_1, \bar{y}_2, \bar{u}_1, \bar{u}_2, \bar{p}, \bar{z}).
By studying the local stability of $\Pi_0$, we can easily prove that $\Pi_0$ is locally if
\[ R^G_0 = \sum_{i=1}^2 \gamma_i \frac{\partial \phi_i(s_i^0, 0)}{\partial p}. \]

1. Then, the basic reproduction number $R^G_0$ of system (4.1)-(4.5) can be defined as:

\[ R^G_0 = \sum_{i=1}^2 \gamma_i \frac{\partial \phi_i(s_i^0, 0)}{\partial p}. \]

Clearly from Assumptions (A1) and (A2), we have:

\[ R^G_1 = \sum_{i=1}^2 \gamma_i \frac{\partial \phi_i(s_i^0, \bar{p})}{\bar{p}} \leq \sum_{i=1}^2 \gamma_i \frac{\partial \phi_i(s_i^0, 0)}{\partial p} \leq \sum_{i=1}^2 \gamma_i \frac{\partial \phi_i(s_i^0, 0)}{\partial p} = R^G_0. \]

4.2. Global stability analysis.

**Theorem 4.1.** If $R^G_0 \leq 1$ and Assumptions (A1) and (A2) are hold true for system (4.1)-(4.5), then $\Pi_0$ is GAS.

**Proof.** Construct a Lyapunov functional $K_0$ as follows:

\[ K_0 = \sum_{i=1}^2 \gamma_i \left[ s_i - s_i^0 - \int_{s_i^0}^s \left( \lim_{p \to 0+} \frac{\phi_i(s_i^0, p)}{\phi_i(s_i, 0)} \right) d\nu + \frac{N_i G_i}{\gamma_i} y_i + \frac{M_i C_i}{\gamma_i} u_i \right. 
+ \frac{1}{F_i} \int_0^l t_i(\tau) \int_0^\tau \phi_i(s_i(t - \theta), p(t - \theta)) d\theta d\tau 
+ \frac{N_i \pi_i}{\gamma_i} \int_0^l \lambda_i(\tau) \int_0^\tau y_i(t - \theta) d\theta d\tau + \frac{M_i \omega_i}{\gamma_i} \int_0^{t_i} \Delta_i(\tau) \int_0^\tau u_i(t - \theta) d\theta d\tau \right] + p + \frac{b}{\nu} z. \]

We evaluate $\frac{dK_0}{dt}$ along the solutions of (4.1)-(4.5) as:

\[ \frac{dK_0}{dt} = \sum_{i=1}^2 \gamma_i \left[ \left( \frac{1}{\gamma_i} \frac{\phi_i(s_i^0, 0)}{\phi_i(s_i^0, 0)} \right) (\rho_i - \beta_i s_i) + \phi_i(s_i, p) \frac{\partial \phi_i(s_i^0, 0)}{\partial p} \right] - cp - \frac{b \mu}{\nu} \psi(z) \]
\[ = \sum_{i=1}^2 \gamma_i \rho_i \left( 1 - \frac{s_i}{s_i^0} \right) \left( 1 - \frac{\partial \phi_i(s_i^0, 0)}{\partial p} \right) + \sum_{i=1}^2 \gamma_i \phi_i(s_i, p) \frac{\partial \phi_i(s_i^0, 0)}{\partial p} - cp - \frac{b \mu}{\nu} \psi(z) \]
\[ \leq \sum_{i=1}^2 \gamma_i \rho_i \left( 1 - \frac{s_i}{s_i^0} \right) \left( 1 - \frac{\partial \phi_i(s_i^0, 0)}{\partial p} \right) + \sum_{i=1}^2 \gamma_i \phi_i(s_i, p) - cp - \frac{b \mu}{\nu} \psi(z) \]
\[ = \sum_{i=1}^2 \gamma_i \rho_i \left( 1 - \frac{s_i}{s_i^0} \right) \left( 1 - \frac{\partial \phi_i(s_i^0, 0)}{\partial p} \right) + (R^G_0 - 1) cp - \frac{b \mu}{\nu} \psi(z). \]
From Assumptions (A1) and (A2), we have
\[
\left(1 - \frac{s_i}{s_i^0}\right) \left(1 - \frac{\partial \phi_i(s_i^0,0)/\partial p}{\partial \phi_i(s_i,0)/\partial p}\right) \leq 0, \; s_i, \; p > 0, \; i = 1, 2.
\]
Therefore, if \( R_0^G \leq 1 \), then \( \frac{dK_0}{dt} \leq 0 \) and \( \frac{dK_0}{dt} = 0 \) at \( \Pi_0 \). Thus, \( \Pi_0 \) is GAS. 

**Lemma 5.** If \( R_0^G > 1 \) and Assumptions (A1)-(A3) are valid, then:
\[
\text{sgn} (\tilde{s}_i - \hat{s}_i) = \text{sgn} (\tilde{p} - \hat{p}) = \text{sgn} (R_1^G - 1).
\]

**Proof.** Using Assumptions (A1)-(A2), that for \( \tilde{s}_i, \hat{s}_i, \tilde{p}, \hat{p} > 0 \), we find:
\[
(\phi_i(\tilde{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p}))(\tilde{s}_i - \hat{s}_i) > 0, \quad (\phi_i(\tilde{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p}))(\tilde{p} - \hat{p}) > 0. \tag{4.15}
\]
By the inequality (4.15) and Assumption (A3) with \( s_i = \tilde{s}_i \) and \( p = \tilde{p} \) and \( p^* = \hat{p} \) we obtain:
\[
((\phi_i(\tilde{s}_i, \tilde{p})\tilde{p} - \phi_i(\hat{s}_i, \hat{p})) (\tilde{p} - \hat{p}) > 0. \tag{4.16}
\]
Suppose that, \( \text{sgn} (\tilde{s}_i - \hat{s}_i) = \text{sgn} (\tilde{p} - \hat{p}) \). From the conditions of the steady states \( \Pi_1 \) and \( \Pi_2 \) we get:
\[
(\rho_i - \beta_i \tilde{s}_i) - (\rho_i - \beta_i \hat{s}_i) = \phi_i(\tilde{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p})
= \phi_i(\hat{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p}) + \phi_i(\hat{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p}).
\]
Therefore, from inequalities (4.15) we obtain \( \text{sgn} (\tilde{s}_i - \hat{s}_i) = \text{sgn} (\tilde{s}_i - \hat{s}_i) \), which is a contradiction, hence, \( \text{sgn} (\tilde{s}_i - \hat{s}_i) = \text{sgn} (\tilde{p} - \hat{p}) \). Using Eq. (4.12) and the definition of \( R_1^G \) we get
\[
R_1^G - 1 = 2 \sum_{i=1}^{2} \gamma_i \frac{\phi_i(\tilde{s}_i, \tilde{p})}{\tilde{p}} - \frac{\phi_i(\hat{s}_i, \hat{p})}{\hat{p}}
= 2 \sum_{i=1}^{2} \gamma_i \left( \frac{1}{\hat{p}} (\phi_i(\tilde{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p})) + \frac{1}{\hat{p}} (\phi_i(\hat{s}_i, \tilde{p}) - \phi_i(\hat{s}_i, \hat{p})) \right).
\]
Thus, from Eqs. (4.15) and (4.16) we obtain \( \text{sgn} (R_1^G - 1) = \text{sgn} (\tilde{p} - \hat{p}) \). 

**Theorem 4.2.** For system (4.1)-(4.5), if \( R_1^G \leq 1 < R_0^G \) and Assumptions (A1)-(A4) are valid, then \( \Pi_1 \) is GAS.

**Proof.** Consider
\[
K_1 = 2 \sum_{i=1}^{2} \gamma_i \left[ \tilde{s}_i - \hat{s}_i - \int_{\tilde{s}_i}^{\hat{s}_i} \frac{\phi_i(\tilde{s}_i, \tilde{p})}{\phi_i(\nu, \tilde{p})} d\nu + \frac{N G_1}{\gamma_i} \int_{\tilde{y}_i}^{\hat{y}_i} \frac{H(\frac{y_i(t-\theta)}{\hat{y}_i})}{H(\frac{\tilde{y}_i(t-\theta)}{\tilde{y}_i})} d\theta d\tau \right.
+ \frac{1}{F_i} \phi_i(\tilde{s}_i, \tilde{p}) \int_{\tilde{\Theta}_i(\tau)}^{\hat{\Theta}_i(\tau)} \int_{0}^{\tau} H \left( \frac{\phi_i(s_i(t-\theta), p(t-\theta))}{\phi_i(\tilde{s}_i, \tilde{p})} \right) d\theta d\tau
+ \frac{N_i \pi_i \hat{y}_i}{\gamma_i} \int_{0}^{\tilde{\Theta}_i(\tau)} \int_{0}^{\tau} H \left( \frac{y_i(t-\theta)}{\hat{y}_i} \right) d\theta d\tau
\]
\[ + \frac{M_i \omega_i \tilde{u}_i}{\gamma_i} \int_0^{\theta_i} \Delta_i(\tau) \int_0^{\tau} H \left( \frac{u_i(t - \theta)}{\tilde{u}_i} \right) d\theta d\tau \] \[ + \tilde{p} H \left( \frac{p}{\tilde{p}} \right) - \frac{b}{\nu} z. \]

Calculating \( \frac{dK_1}{dt} \) along the solutions of system (4.1)-(4.5) we get:

\[
\frac{dK_1}{dt} = \sum_{i=1}^{2} \left[ \frac{\gamma_i}{\gamma_i} \left( 1 - \frac{\phi_i(s_i, \tilde{p})}{\phi_i(s_i, \tilde{p})} \left( \rho_i - \beta_i s_i \right) + \phi_i(s_i, p) \frac{\phi_i(s_i, \tilde{p})}{\phi_i(s_i, \tilde{p})} \right) \right. \\
- \frac{q_i M_i C_i}{\gamma_i} \int_0^{l_i} \Lambda_i(\tau) \frac{\tilde{y}_i \phi_i(s_i(t - \tau), p(t - \tau))}{y_i} d\tau \\
+ \phi_i(s_i, \tilde{p}) \int_0^{l_i} \Theta_i(\tau) \frac{\tilde{y}_i \phi_i(s_i(t - \tau), p(t - \tau))}{y_i} d\tau \\
+ \frac{N_i \pi_i \tilde{y}_i}{\gamma_i} \int_0^{e_i} \Lambda_i(\tau) \left( 1 - \frac{\phi_i(s_i, \tilde{p})}{\phi_i(s_i, \tilde{p})} \right) d\tau + \frac{M_i \omega_i \tilde{u}_i}{\gamma_i} \int_0^{\theta_i} \Delta_i(\tau) \left( 1 - \frac{u_i(t - \theta)}{\tilde{u}_i} \right) d\tau \\
- \sum_{i=1}^{2} N_i \pi_i \int_0^{e_i} \Lambda_i(\tau) \left( \tilde{y}_i \phi_i(t - \tau, p(t - \tau)) \right) d\tau - \sum_{i=1}^{2} M_i \omega_i \int_0^{\theta_i} \Delta_i(\tau) \left( \tilde{p} u_i(t - \tau) \right) d\tau \\
- cp + \tilde{c} \tilde{p} + b \tilde{p} \psi(z) - \frac{b \mu}{\nu} \psi(z).
\]

From the conditions of the steady state \( \Pi_1 \), we find:

\[ \rho_i = \beta_i \tilde{s}_i + \phi_i(s_i, \tilde{p}), \quad (1 - q_i) F_i \phi_i(s_i, \tilde{p}) = \pi_i \tilde{y}_i, \quad q_i F_i \phi_i(s_i, \tilde{p}) = \omega_i \tilde{u}_i, \]

\[ c \tilde{p} = \sum_{i=1}^{2} (N_i \pi_i G_i \tilde{y}_i + M_i \omega_i C_i \tilde{u}_i) = \sum_{i=1}^{2} \gamma_i \phi_i(s_i, \tilde{p}), \quad c \tilde{p} = \frac{p}{\tilde{p}} \sum_{i=1}^{2} \gamma_i \phi_i(s_i, \tilde{p}), \]

and using inequalities (2.8) with \( s^*_i = \tilde{s}_i, y^*_i = \tilde{y}_i, u^*_i = \tilde{u}_i \) and \( p^* = \tilde{p}, \) we get

\[
\frac{dK_1}{dt} = \sum_{i=1}^{2} \left[ \gamma_i \beta_i \tilde{s}_i \left( 1 - \frac{s_i}{\tilde{s}_i} \right) \left( 1 - \frac{\phi_i(s_i, \tilde{p})}{\phi_i(s_i, \tilde{p})} \right) \right. \\
+ \gamma_i \phi_i(s_i, \tilde{p}) \left( \frac{\phi_i(s_i, p)}{\phi_i(s_i, \tilde{p})} - \frac{p}{\tilde{p}} \right) \left( 1 - \frac{\phi_i(s_i, \tilde{p})}{\phi_i(s_i, \tilde{p})} \right) \\
- \gamma_i \phi_i(s_i, \tilde{p}) \left( H \left( \frac{\phi_i(s_i, \tilde{p})}{\phi_i(s_i, \tilde{p})} \right) + H \left( \frac{p \phi_i(s_i, \tilde{p})}{p \phi_i(s_i, \tilde{p})} \right) \right) \\
- \frac{N_i G_i \pi_i \tilde{y}_i}{F_i} \int_0^{e_i} \Theta_i(\tau) H \left( \frac{\tilde{y}_i \phi_i(t - \tau, p(t - \tau))}{\tilde{y}_i \phi_i(s_i, \tilde{p})} \right) d\tau \\
- \frac{N_i \pi_i \tilde{y}_i}{\gamma_i} \int_0^{e_i} \Lambda_i(\tau) H \left( \frac{\tilde{y}_i \phi_i(t - \tau, p(t - \tau))}{\tilde{y}_i \phi_i(s_i, \tilde{p})} \right) d\tau - \frac{M_i \omega_i \tilde{u}_i}{\gamma_i} \int_0^{\theta_i} \Delta_i(\tau) H \left( \frac{\tilde{p} u_i(t - \tau)}{\tilde{p} u_i} \right) d\tau
\]
\[
\begin{align*}
- \frac{M_iC_i\omega_i\bar{u}_i}{F_i} \int_0^{\tau_i} \Theta_i(\tau)H \left( \frac{\bar{u}_i\phi_i(s_i(t-\tau),p(t-\tau))}{u_i\phi_i(s_i,p)} \right) d\tau + b(\bar{p} - p)\psi(z). \quad (4.17)
\end{align*}
\]

Assumptions (A1), (A4), Lemma 5 and the condition \( R_1^G \leq 1 \) imply that \( \frac{dK_1}{dt} \leq 0 \) for all \( s_i, y_i, u_i, p, z > 0 \) and \( \frac{dK_1}{dt} = 0 \) at \( \Pi_1 \). By LIP \( \Pi_1 \) is GAS. \( \square \)

**Theorem 4.3.** For system (4.1)-(4.5), if \( R_1^G > 1 \) and Assumptions (A1)-(A4) are valid, then \( \Pi_2 \) is GAS.

**Proof.** Constructing a Lyapunov functional as:

\[
K_2 = \sum_{i=1}^{2} \gamma_i \left[ s_i - \bar{s}_i - \int_{s_i}^{\bar{s}_i} \frac{\phi_i(s_i,p)}{\phi_i(s_i,p)} \frac{\phi_i(s_i,p)}{\phi_i(s_i,p)} \right] + \frac{N_iG_i}{\gamma_i} \frac{\bar{u}_i}{u_i} \quad (4.18)
\]

Calculating \( \frac{dK_2}{dt} \) along the solutions of system (4.1)-(4.5) we obtain:

\[
\frac{dK_2}{dt} = \sum_{i=1}^{2} \gamma_i \left[ \left( 1 - \frac{\phi_i(s_i,p)}{\phi_i(s_i,p)} \right) \left( \rho_i - \beta_i s_i \right) + \phi_i(s_i,p) \frac{\phi_i(s_i,p)}{\phi_i(s_i,p)} + \frac{N_iG_i}{\gamma_i} \frac{\bar{u}_i}{u_i} \right]
\]

By using the steady state conditions of \( \Pi_2 \):

\[
\rho_i = \beta_i s_i + \phi_i(s_i,p), \quad (1 - q_i)F_i \phi_i(s_i,p) = \pi_i \bar{y}_i, \quad q_iF_i \phi_i(s_i,p) = \omega_i \bar{u}_i,
\]
\begin{align*}
cp = \sum_{i=1}^{2} (N_i \pi_i G_i \bar{y}_i + M_i \omega_i C_i \bar{u}_i) - b \bar{p} \psi(\bar{z}), \quad cp = \frac{p}{p} \sum_{i=1}^{2} \gamma_i \phi_i(\bar{s}_i, \bar{p}) - b \bar{p} \psi(\bar{z}),
\end{align*}
and the inequalities (2.8) with \( s_i^* = \bar{s}_i, \ y_i^* = \bar{y}_i, \ u_i^* = \bar{u}_i \) and \( p^* = \bar{p} \), we find

\begin{align*}
\frac{dK_2}{dt} = \sum_{i=1}^{2} \left[ \gamma_i \beta_i \bar{s}_i \left( 1 - \frac{s_i}{\bar{s}_i} \right) \left( 1 - \frac{\phi_i(\bar{s}_i, \bar{p})}{\phi_i(s_i, p)} \right) \\
+ \gamma_i \phi_i(\bar{s}_i, \bar{p}) \left( \frac{\phi_i(s_i, p) - p}{p} \right) \left( 1 - \frac{\phi_i(s_i, p)}{\phi_i(s_i, p)} \right) \\
- \gamma_i \phi_i(\bar{s}_i, \bar{p}) \left( H \left( \frac{\phi_i(s_i, p)}{\phi_i(s_i, p)} \right) + H \left( \frac{p \phi_i(s_i, p)}{p \phi_i(s_i, p)} \right) \right) \\
- \frac{N_i G_i \pi_i \bar{y}_i}{F_i} \int_{0}^{L_i} \Theta_i(\tau) H \left( \frac{\bar{y}_i \phi_i(s_i(\tau) - \tau, p(t - \tau))}{y_i \phi_i(s_i, \bar{p})} \right) d\tau \\
- \frac{N_i \pi_i \bar{y}_i}{F_i} \int_{0}^{T_i} \Lambda_i(\tau) H \left( \frac{\bar{y}_i \phi_i(s_i(\tau) - \tau, p(t - \tau))}{y_i \phi_i(s_i, \bar{p})} \right) d\tau \\
- \frac{M_i C_i \omega_i \bar{u}_i}{F_i} \int_{0}^{L_i} \Theta_i(\tau) H \left( \frac{\bar{u}_i \phi_i(s_i(\tau) - \tau, p(t - \tau))}{u_i \phi_i(s_i, \bar{p})} \right) d\tau \\
- \frac{M_i \omega_i \bar{u}_i}{F_i} \int_{0}^{T_i} \Delta_i(\tau) H \left( \frac{\bar{u}_i \phi_i(s_i(\tau) - \tau, p(t - \tau))}{u_i \phi_i(s_i, \bar{p})} \right) d\tau \right].
\end{align*}

According to Assumptions (A1),(A2) and (A4) we get \( \frac{dK_2}{dt} \leq 0 \) and \( \frac{dK_2}{dt} = 0 \) at \( \Pi_2 \). LIP implies that \( \Pi_2 \) is GAS.

5. Numerical simulations

We now perform some computer simulations on the following application. The incidence rate is given by Crowley-Martin (CM) functional response:

\begin{equation}
\phi_i(s_i, p) = \frac{\lambda_i s_i p}{(1 + \mu_i s_i)(1 + \alpha_i p)}, \quad i = 1, 2,
\end{equation}
where \( \mu_i \geq 0, \ i = 1, 2, \) and \( \lambda_1 = (1 - \varepsilon) \lambda_1, \ \lambda_2 = (1 - f \varepsilon) \lambda_2. \) Then, the general model (4.1)-(4.5) with incidence rate given in (5.1) can be described as:

\begin{align}
\dot{s}_i(t) &= \rho_i - \beta_i s_i(t) - \frac{\lambda_i s_i(t)p(t)}{(1 + \mu_i s_i(t))(1 + \alpha_i p(t))}, \quad i = 1, 2, \\
\dot{y}_i(t) &= (1 - q_i) \lambda_i \int_{0}^{L_i} \Theta_i(\tau) \frac{s_i(t - \tau)p(t - \tau)}{(1 + \mu_i s_i(t - \tau)(1 + \alpha_i p(t - \tau))} d\tau - \pi_i y_i(t), \quad i = 1, 2, \\
\dot{u}_i(t) &= q_i \lambda_i \int_{0}^{L_i} \Theta_i(\tau) \frac{\lambda_i s_i(t - \tau)p(t - \tau)}{(1 + \mu_i s_i(t - \tau)(1 + \alpha_i p(t - \tau))} d\tau - \omega_i u_i(t), \quad i = 1, 2,
\end{align}
\[ \dot{p}(t) = \sum_{i=1}^{2} \left( N_i \pi_i \int_0^{\tau_i} \Lambda_i(\tau)y_i(t - \tau_i)d\tau + M_i \omega_i \int_0^{\tau_i} \Delta_i(\tau)u_i(t - \tau_i)d\tau \right) - cp(t) - bp(t)z(t), \]

\[ \dot{z}(t) = \nu p(t)z(t) - \mu z(t). \] (5.5)

To verify Assumptions (A1)-(A4), we have:

\[ \frac{\partial \phi_i(s_i, p)}{\partial p} = \frac{\lambda_i s_i}{(1 + \mu_i s_i)(1 + \alpha_i p)^2} > 0, \quad \frac{\partial \phi_i(s_i, 0)}{\partial p} = \frac{\lambda_i s_i}{1 + \mu_i s_i} > 0, \]

\[ \phi_i(s_i, p) = \frac{\lambda_i s_i p}{(1 + \mu_i s_i)(1 + \alpha_i p)} \leq \frac{\lambda_i s_i p}{1 + \mu_i s_i} = \frac{\partial \phi_i(s_i, 0)}{\partial p}, \]

\[ \left( \frac{\phi_i(s_i, p)}{\phi_i(s_i, p^*)} - \frac{p}{p^*} \right) \left( 1 - \frac{\phi_i(s_i, p^*)}{\phi_i(s_i, p)} \right) = \frac{-\alpha_i (p - p^*)^2}{p^*(1 + \alpha_i p)(1 + \alpha_i p)} \leq 0, \text{ for all } s_i, p > 0. \]

Then Assumptions (A1)-(A3) are valid. Moreover, Assumption (A4) is also valid where \( \psi(z) = z \).

Next, we shall perform simulation studies for model (4.1)-(4.5) with the incidence rate given by Eq. (5.1) and particular form of the probability distributed functions as:

\[ f_i(\tau) = \delta(\tau - \tau_i), \quad g_i(\tau) = \delta(\tau - \tau_i), \quad h_i(\tau) = \delta(\tau - \tau_i), \quad i = 1, 2, \] (5.7)

where \( \delta(.) \) is the Dirac delta function and \( \tau_i \in [0, t_i], \ z_i \in [0, e_i], u_i \in [0, \vartheta_i], i = 1, 2, \) are constants. When \( t_i, e_i \) and \( \vartheta_i \to \infty \), we have:

\[ \int_0^\infty f_i(\tau)d\tau = \int_0^\infty g_i(\tau)d\tau = \int_0^\infty h_i(\tau)d\tau = 1, \quad i = 1, 2. \] (5.8)

Using the properties of Dirac delta function we get:

\[ F_i = \int_0^\infty \delta(\tau - \tau_i)e^{-m_i\tau}d\tau = e^{-m_i\tau_i}, \quad G_i = \int_0^\infty \delta(\tau - z_i)e^{-m_i\tau}d\tau = e^{-m_i\tau_i}, \]

\[ C_i = \int_0^\infty \delta(\tau - u_i)e^{-m_i\tau}d\tau = e^{-m_i\tau_i}, \quad i = 1, 2. \]

Moreover,

\[ \int_0^\infty \delta(\tau - \tau_i)e^{-m_i\tau} \frac{\lambda_i s_i(t - \tau)p(t - \tau)}{(1 + \mu_i s_i(t - \tau))(1 + \alpha_i p(t - \tau))} d\tau = \frac{e^{-m_i\tau_i} \lambda_i s_i(t - \tau_i)p(t - \tau_i)}{(1 + \mu_i s_i(t - \tau_i))(1 + \alpha_i p(t - \tau_i))}, \quad i = 1, 2, \]

\[ \int_0^\infty \delta(\tau - \tau_i)e^{-m_i\tau}y_i(t - \tau)d\tau = e^{-m_i\tau_i}y_i(t - \tau_i), \]
\[
\int_0^\infty \delta(\tau - \iota_i)e^{-r_i\tau}u_i(t - \tau)d\tau = e^{-r_i\kappa_i}u_i(t - \kappa_i), \quad i = 1, 2.
\]

Hence, model (4.1)-(4.5) with incidence rate given by Eq. (5.1), becomes:

\[
\dot{s}_i(t) = \rho_i - \beta_i s_i(t) - \frac{\lambda_i s_i(t)p(t)}{(1 + \mu_i s_i(t))(1 + \alpha_i p(t))}, \quad i = 1, 2, \quad (5.9)
\]

\[
\dot{y}_i(t) = (1 - q_i)e^{-m_i\tau_i} \frac{\lambda_i s_i(t - \tau_i)p(t - \tau_i)}{(1 + \mu_i s_i(t - \tau_i))(1 + \alpha_i p(t - \tau_i))} - \pi_i y_i(t), \quad i = 1, 2, \quad (5.10)
\]

\[
\dot{u}_i(t) = q_i e^{-m_i\tau_i} \frac{\lambda_i s_i(t - \tau_i)p(t - \tau_i)}{(1 + \mu_i s_i(t - \tau_i))(1 + \alpha_i p(t - \tau_i))} - \omega_i u_i(t), \quad i = 1, 2, \quad (5.11)
\]

\[
\dot{p}(t) = \sum_{i=1}^2 (N_i \pi_i e^{-n_i\tau_i}y_i(t - \zeta_i) + M_i \omega_i e^{-r_i\tau_i}u_i(t - \iota_i)) - cp(t) - bp(t)z(t), \quad (5.12)
\]

\[
\dot{z}(t) = \nu p(t)z(t) - \mu z(t). \quad (5.13)
\]

The parameters \( R_0^{CM} \) and \( R_1^{CM} \) will be:

\[
\begin{align*}
R_0^{CM} &= \sum_{i=1}^2 \frac{((1 - q_i)N_i e^{-n_i\tau_i} + q_i M_i e^{-r_i\tau_i}) e^{-m_i\tau_i} \lambda_i s_i^0}{c(1 + c_i s_i^0)}, \\
R_1^{CM} &= \sum_{i=1}^2 \frac{((1 - q_i)N_i e^{-n_i\tau_i} + q_i M_i e^{-r_i\tau_i}) e^{-m_i\tau_i} \lambda_i \bar{s}_i}{c(1 + \mu_i \bar{s}_i)(1 + \alpha_i \bar{p})},
\end{align*}
\]

where

\[
\bar{s}_i = \frac{1}{2\mu_i(1 + \alpha_i \bar{p})} \left[-B + \sqrt{B^2 + 4\mu_i s_i^0 (1 + \alpha_i \bar{p})^2}\right],
\]

\[
\bar{p} = \frac{\mu}{\nu}, \quad \zeta_i = \alpha_i + \frac{\lambda_i}{\beta_i}, \quad i = 1, 2,
\]

\[
B = (1 + \zeta_i \bar{p}) - \mu_i s_i^0(1 + \alpha_i \bar{p}).
\]

Now we are ready to perform some numerical simulations for system (5.9)-(5.13). The data of system (5.9)-(5.13) are provided in Table 1. We let \( \tau_c = \tau_i = \zeta_i = \iota_i, \quad i = 1, 2. \)

### Table 1. Values of some parameters of system (5.9)-(5.13).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \rho_1 )</th>
<th>( \rho_2 )</th>
<th>( m_1 )</th>
<th>( m_2 )</th>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( N_1 )</th>
<th>( N_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>10</td>
<td>0.03198</td>
<td>1</td>
<td>1</td>
<td>0.03</td>
<td>0.01</td>
<td>0.03</td>
<td>0.01</td>
<td>100</td>
<td>30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( \pi_1 )</th>
<th>( \pi_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.01</td>
<td>0.005</td>
<td>1</td>
<td>1</td>
<td>0.25</td>
<td>0.05</td>
<td>1</td>
<td>1</td>
<td>0.3</td>
<td>0.03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( M_1 )</th>
<th>( M_2 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( b )</th>
<th>( f )</th>
<th>( \mu )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>50</td>
<td>10</td>
<td>0.5</td>
<td>0.5</td>
<td>0.01</td>
<td>0.3</td>
<td>0.05</td>
<td>3</td>
</tr>
</tbody>
</table>
Effect of the parameters $\lambda_1$, $\lambda_2$ and $\nu$ on the stability of the steady states: The initial conditions have been considered as: $\varphi_1(\theta) = 600$, $\varphi_3(\theta) = 0.1$, $\varphi_4(\theta) = 0.1$, $\varphi_5(\theta) = 5$, $\varphi_6(\theta) = 0.1$, $\varphi_7(\theta) = 50$, $\varphi_8(\theta) = 60$, $\theta \in [-\tau_e, 0]$. Let us address three cases for the parameters $\lambda_1$, $\lambda_2$ and $\nu$. We assume that $\varepsilon = 0$ (there is no treatment) and $\tau_e = 0.5$.

Figure 1. The concentration of uninfected CD4+ T cells.

Figure 2. The concentration of uninfected macrophages.

Figure 3. The concentration of short-lived infected CD4+ T cells.

Figure 4. The concentration of short-lived infected macrophages.

Case (I): Choose $\lambda_1 = 0.002$, $\lambda_2 = 0.0005$ and $\nu = 0.0005$ which give $R_{CM}^{0} = 0.6007 < 1$ and $R_{CM}^{1} = 0.1481 < 1$. Therefore, based on Lemma 4 and Theorems 4.1 the system has unique steady state, that is $\Pi_0$ and it is GAS. As we can see from Figures 1-8 that the concentration of the uninfected cells is increased and approached its normal value before infection.
that is \( s_0^1 = 1000 \) and \( s_0^2 = 6.396 \) while concentrations of the other compartments converge to zero for the initial condition. As a result, the HIV1 is removed from the plasma.

Case (II): We take the following values: \( \lambda_1 = 0.01, \lambda_2 = 0.001 \) and \( \nu = 0.0005 \). For these values \( R_{CM}^1 = 0.6803 < 1 < R_{CM}^0 = 2.9815 \). Consequently, based on Lemma 4 and Theorems 4.2, the humoral-inactivated infection steady state \( \Pi_1 \) is positive and is GAS. Figures 1-8 confirm that the numerical results support the theoretical results presented in Theorem 4.2. It can be observed that, the variables of the model eventually converge to \( \Pi_1 \) (348.03, 0.76, 6.59, 0.29, 7.91, 0.17, 60.03, 0) for the initial conditions. This case corresponds to a chronic HIV-1 infection in the absence of immune response.
Case (III): $\lambda_1 = 0.01$, $\lambda_2 = 0.001$ and $\nu = 0.002$. Then, we calculate $R_{CM}^0 = 2.9815 > 1$ and $R_{CM}^1 = 1.6544 > 1$. We can see from Figure 1-8 that, there is a consistency between the numerical results and theoretical results of Theorem 4.3.

The states of the system converge to $\Pi_2(550.98,1.29,4.54,0.26,5.45,0.15,25,196.31)$ for the initial conditions. In this case the humoral immune response is activated and can control the disease.

- **Effect of the drug efficacy $\varepsilon$ on the stability of the steady states:** We take $\tau_e = 0.5$, $\lambda_1 = 0.01$, $\lambda_2 = 0.001$ and $\nu = 0.001$. In Figures 9-16 we show the effect of the drug efficacy $\varepsilon$ on the HIV dynamics. Also, we can observe that, as the drug efficacy $\varepsilon$ is increased, the concentration of uninfected cells is increased, while the concentrations of free virus particles and the three types of infected cells are decreased. Table 2 shows that, the values of $R_{CM}^0$ and $R_{CM}^1$ are decreased as $\varepsilon$ is increased. Therefore, the results of Theorems 4.1-4.3 and the results of numerical simulation are compatible. Thus, we can say that treatment with sufficient drug efficacy can success to clear the HIV from the plasma.

### Table 2. Values of steady states, $R_{CM}^0$ and $R_{CM}^1$ for system (5.9)-(5.13) with different values of $\varepsilon$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>steady states $\Pi$</th>
<th>$R_{CM}^0$</th>
<th>$R_{CM}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon = 0.0$</td>
<td>$\Pi_2(386.29,0.84,6.20,0.28,7.44,0.17,50,39.07)$</td>
<td>2.9815</td>
<td>1.1302</td>
</tr>
<tr>
<td>$\varepsilon = 0.12623$</td>
<td>$\Pi_1(457.08,0.87,5.49,0.28,6.59,0.17,50,0)$</td>
<td>2.6064</td>
<td>1.0000</td>
</tr>
<tr>
<td>$\varepsilon = 0.2$</td>
<td>$\Pi_1(525.97,0.96,4.79,0.27,5.75,0.16,43.66,0)$</td>
<td>2.3873</td>
<td>0.9211</td>
</tr>
<tr>
<td>$\varepsilon = 0.4$</td>
<td>$\Pi_1(724.23,1.42,2.79,0.25,3.35,0.15,25.42,0)$</td>
<td>1.7930</td>
<td>0.6996</td>
</tr>
<tr>
<td>$\varepsilon = 0.66908$</td>
<td>$\Pi_0(1000,6.396,0,0,0,0,0,0)$</td>
<td>1.0000</td>
<td>0.3932</td>
</tr>
<tr>
<td>$\varepsilon = 0.8$</td>
<td>$\Pi_0(1000,6.396,0,0,0,0,0,0)$</td>
<td>0.6046</td>
<td>0.2375</td>
</tr>
</tbody>
</table>

### Table 3. Values of steady states, $R_{CM}^0$ and $R_{CM}^1$ for system (5.9)-(5.13) with different values of $\tau_e$.

<table>
<thead>
<tr>
<th>$\tau_e$</th>
<th>steady states $\Pi$</th>
<th>$R_{CM}^0$</th>
<th>$R_{CM}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_e = 0.001$</td>
<td>$\Pi_2(620.40,0.94,6.32,0.45,7.58,0.27,50,269.36)$</td>
<td>4.8642</td>
<td>1.8978</td>
</tr>
<tr>
<td>$\tau_e = 0.1$</td>
<td>$\Pi_2(620.40,0.94,5.72,0.41,6.87,0.25,50,167.08)$</td>
<td>3.9905</td>
<td>1.5570</td>
</tr>
<tr>
<td>$\tau_e = 0.3$</td>
<td>$\Pi_2(620.40,0.94,4.69,0.34,5.62,0.20,50,13.10)$</td>
<td>2.6749</td>
<td>1.0437</td>
</tr>
<tr>
<td>$\tau_e = 0.321363$</td>
<td>$\Pi_1(620.40,0.94,4.59,0.33,5.51,0.20,50,0)$</td>
<td>2.5630</td>
<td>1.0000</td>
</tr>
<tr>
<td>$\tau_e = 0.5$</td>
<td>$\Pi_1(724.23,1.42,2.79,0.25,3.35,0.15,25.42,0)$</td>
<td>1.7930</td>
<td>0.6996</td>
</tr>
<tr>
<td>$\tau_e = 0.791955$</td>
<td>$\Pi_0(1000,6.396,0,0,0,0,0,0)$</td>
<td>1.0000</td>
<td>0.3902</td>
</tr>
<tr>
<td>$\tau_e = 0.9$</td>
<td>$\Pi_0(1000,6.396,0,0,0,0,0,0)$</td>
<td>0.8057</td>
<td>0.3143</td>
</tr>
<tr>
<td>$\tau_e = 1.0$</td>
<td>$\Pi_0(1000,6.396,0,0,0,0,0,0)$</td>
<td>0.6596</td>
<td>0.2574</td>
</tr>
<tr>
<td>$\tau_e = 2.0$</td>
<td>$\Pi_0(1000,6.396,0,0,0,0,0,0)$</td>
<td>0.0892</td>
<td>0.0348</td>
</tr>
</tbody>
</table>
STABILITY OF HIV INFECTION MODELS

5.1. Conclusion. In this paper, we have proposed and analyzed three HIV infection models. We have considered four types of infected cells: short-lived infected CD4+ T cells, long-lived chronically infected CD4+ T cells, short-lived infected macrophages and long-lived chronically infected macrophages.

- **Effect of the time delay on the stability of the system**: Choosing \( \varepsilon = 0.4, \lambda_1 = 0.01, \lambda_2 = 0.001 \) and \( \nu = 0.001 \). Figures 17-24 and Table 3 show the effect of the time delay parameter \( \tau_e \) on the stability of \( \Pi_0, \Pi_1 \) and \( \Pi_2 \). Clearly, the parameter \( \tau_e \) has similar effect as the drug efficacy parameters \( \varepsilon \).
infected macrophages. We have incorporated three distributed time delays into the models. We have represented the HIV-target incidence rate by bilinear and saturation functional response for the first two models while, for the third model, we have considered more general nonlinear functions for both the HIV-target incidence rate and neutralization rate of viruses and we have derived a set of conditions on these general functions. We have proved the nonnegativity and ultimate boundedness of the model’s solutions and the existence and stability of the model’s steady states. We have determined two threshold parameters: the basic reproduction number and the humoral immune response activation number. Using Lyapunov functionals, we have
established the global stability of the three steady states of the models. We have presented an example and performed some numerical simulations to support our theoretical results.
**REFERENCES**


EFFECT OF PERTURBATION IN THE SOLUTION OF FRACTIONAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

MOHAMMED. S. ABDO† AND SATISH. K. PANCHAL

1RESEARCH SCHOLAR AT DEPARTMENT OF MATHEMATICS, DR. BABASAHEB AMBEDKAR MARATHWADA UNIVERSITY, AURANGABAD (M.S), 431004 INDIA
E-mail address: msabdo1977@gmail.com
2DEPARTMENT OF MATHEMATICS, DR. BABASAHEB AMBEDKAR MARATHWADA UNIVERSITY, AURANGABAD (M.S), 431004 INDIA
E-mail address: drpanchalsk@gmail.com

ABSTRACT. In this paper, we study the initial value problem for neutral functional differential equations involving Caputo fractional derivative of order $\alpha \in (0, 1)$ with infinite delay. Some sufficient conditions for the uniqueness and continuous dependence of solutions are established by virtue of fractional calculus and Banach fixed point theorem. Some results obtained showed that the solution was closely related to the conditions of delays and minor changes in the problem. An example is provided to illustrate the main results.

1. INTRODUCTION

This paper is concerned with the uniqueness and continuous dependence of solutions for neutral functional differential equations with fractional order and infinite delay that described by

\[ ^cD^\alpha [x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b], \]

\[ x_0 = \varphi \in \mathcal{B}, \]

where $0 < \alpha < 1$, $^cD^\alpha$ is the standard Caputo fractional derivative of order $\alpha$, $f, g : [0, b] \times \mathcal{B} \rightarrow \mathbb{R}$ are given functions satisfying some assumptions that will be specified in Section 3, and $\mathcal{B}$ the phase space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, which will be specified in Section 2.

Recently, the fractional differential equations became an important branch in mathematics and its applications such as mechanics, physics, chemistry, engineering. So a lot of authors and researchers have contributed in preparing books and papers about this field, for more details,
see the monographs [1, 15, 17, 19, 21], and the papers [7, 8, 11, 16, 20], and the references therein.

The fractional delay of neutral functional differential equations appear frequently in applications as the model of equations, and for this reason, these equations have been extensively studied. Especially, the results dealing with infinite delay has received great attention in the last few years, see for instance. [2, 3, 4, 5, 6, 10, 13, 18, 22, 23, 25]. For example in [5], the authors used the Leray-Schauder type nonlinear alternative and contraction mapping principle to investigate the existence and uniqueness of solutions for the following problem

\[ D^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b], \]
\[ x(t) = \varphi(t), \quad t \in (-\infty, 0], \]

where \(0 < \alpha < 1\), \(D^\alpha\) is the Riemann-Liouville fractional derivative, \(\varphi \in B, \varphi(0) = 0\), \(B\) is a phase space and \(f, g : [0, b] \times B \rightarrow \mathbb{R}\) are suitable functions satisfying some hypotheses with \(g(0, \varphi) = 0\). In [4], the following problem was considered

\[ cD^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [t_0, \infty), \]
\[ x_{t_0} = \varphi \in \mathcal{C}, \]

where \(0 < \alpha < 1\), \(cD^\alpha\) is the Caputo fractional operator, \(f, g : [t_0, \infty) \times \mathcal{C} \rightarrow \mathbb{R}\) are appropriate functions satisfying some hypotheses, and \(\mathcal{C}\) is called a space of continuous functions on \([-\tau, 0]\). The authors employed the Krasnoselskii’s fixed point theorem to study the existence result of the problem (1.3)-(1.4).

The main purpose of this paper is to discuss the uniqueness of solutions and effect of the perturbed data on the solutions by means of the Banach fixed point theorem.

The rest of this paper is organized as follows, In Section 2, we present some preliminary facts and make a list of the hypotheses that will be used throughout this paper. Section 3 is devoted to investigating the uniqueness of solutions of the problem (1.1)-(1.2). In Section 4, we introduce the continuous dependence of solutions to problem (1.1)-(1.2) in the space \(C([a, b])\). Finally, an example to illustrate our results is given in Section 5.

2. Preliminaries

In this section, we present some required notations, definitions and some hypotheses which are used throughout this paper. Let us denote by \(C([0, b], \mathbb{R})\) the Banach spaces of continuous real functions \(h : [0, b] \rightarrow \mathbb{R}\), with the norm \(|h|_{\infty} = \sup\{|h(t)| : t \in [0, b]\}\), \(C^1[0, b]\) the space of all continuously differentiable real functions defined on \([0, b]\), and by \(L^1[0, b]\) the space of all real functions \(h(t)\) such that \(|h(t)|\) is Lebesgue integrable on \([0, b]\). For any continuous function \(x : (-\infty, b) \rightarrow \mathbb{R}\) and for any \(t \in [0, b]\), we denote by \(x_t\) the element of \(B\) defined by \(x_t(s) = x(t + s)\), for \(-\infty < s \leq 0\), we also consider the following space

\[ \Upsilon = \{ x : (-\infty, b) \rightarrow \mathbb{R}; \ x|_{(-\infty,0]} \in B, \ x|[0,b] \in C([0,b], \mathbb{R}) \}, \]

where \(x|_{[0,b]}\) is the restriction of \(x\) to \([0, b]\).
Definition 2.1. ([15]). Let $\alpha > 0$ and $h \in C([0,b])$. The Riemann-Liouville fractional integral of order $\alpha$ for a function $h$ is determined as

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0,b],$$

where $\Gamma(.)$ is the gamma function. Moreover, $I_0^\alpha I_0^\alpha h(t) = I_0^{\alpha+\gamma} h(t)$, for $\delta, \gamma \geq 0$.

Definition 2.2. ([9]) Let $0 < \alpha < 1$, and $h \in C([0,b], \mathbb{R})$. The Riemann-Liouville fractional derivative of order $\alpha$ for a function $h$ is defined by

$$D_0^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{h(s)}{(t-s)^\alpha} ds, \quad t \in [0,b].$$

Further, if $h \in L^1[0,b]$. Then $D_0^\alpha I_0^\alpha h(t) = h(t)$, for $t \in [0,b]$.

Definition 2.3. ([24]). Let $0 < \alpha < 1$ and $h \in C^1([0,b], \mathbb{R})$. The Caputo fractional derivative of order $\alpha$ for a function $h$ is described as

$$c D_0^\alpha h(t) = D_0^\alpha (h(t) - h(0)).$$

Moreover, if $c D_0^\alpha h(t) \in L^1([0,b])$, then

$$I_0^\alpha c D_0^\alpha h(t) = h(t) - h(0).$$

Also, one has

$$c D_0^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{h'(s)}{(t-s)^\alpha} ds, \quad t \in [0,b].$$

Obviously, the Caputo derivative of a constant is equal to zero.

Definition 2.4. A function $x \in \mathcal{Y}$ is said to be a solution of (1.1)–(1.2) if $x$ satisfies the equation

$$c D_0^\alpha [x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0,b],$$

with initial condition $x_0 = \varphi$ and $[x(t) - g(t, x_t)]$ is absolutely continuous on $[0,b]$.

Lemma 2.5. ([24]) (Banach fixed point theorem). Let $K$ be a non-empty closed subset of a Banach space $X$, then each contraction mapping $T : K \to K$ has a unique fixed point.

In this paper, we employ an axiomatic definition for the phase space $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ that is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ and satisfying the following fundamental axioms which is similar to that introduced by Hale and Kato in [12] and exceedingly discussed in [14]:

(H1): If $x : (-\infty, b] \to \mathbb{R}$, such that $x$ is a continuous on $[0,b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0,b]$ the following conditions are satisfied:

(i): $x_t \in \mathcal{B}$;

(ii): $|x(t)| \leq H \|x_t\|_\mathcal{B}$ for some $H > 0$;

(iii): $\|x_t\|_\mathcal{B} \leq K(t) \sup_{0 \leq s \leq t} |x(s)| + M(t) \|x_0\|_\mathcal{B}$, where $M, K : [0, +\infty) \to [0, +\infty)$ with $M$ locally bounded and $K$ continuous, such that $M, K$ are independent of $x(.)$.

(H2): For the function $x(.)$ in (H1), the function $t \to x_t$ is continuous from $[0,b]$ into $\mathcal{B}$. 
(H3): The space $\mathcal{B}$ is complete.

**Remark 2.6.** Note that, the condition (ii) in (H1) is equivalent to $|\varphi(0)| \leq H \|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$.

### 3. Uniqueness Results

In this section, we prove the uniqueness results for the problem (1.1)-(1.2) by means of the Banach fixed point theorem. Before starting and proving this one, we will display the following lemma

**Lemma 3.1.** A function $x \in \Upsilon$ is a solution of the problem (1.1)-(1.2) if and only if $x$ satisfies

$$x(t) = \begin{cases} \varphi(0) - g(0, \varphi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s)ds, & t \in [0, b], \\ \varphi(t), & t \in (-\infty, 0] \end{cases}$$

(3.1)

provided that the integral in (3.1) exists.

**Proof.** The proof is very simple and we can get it according to Definitions 2.1 and 2.3. □

**Theorem 3.2.** Assume that $f, g : [0, b] \times \mathcal{B} \to \mathbb{R}$ are continuous functions. If the following conditions are satisfied

(A1): There exist two functions $\delta_f(t) \in L^1([0, b], \mathbb{R})$ and $\delta_g(t) \in C([0, b], \mathbb{R})$ such that $|f(t, u) - f(t, v)| \leq \delta_f(t) \|u - v\|_{\mathcal{B}}$, and $|g(t, u) - g(t, v)| \leq \delta_g(t) \|u - v\|_{\mathcal{B}}$,

for any $t \in [0, b]$ and for each $u, v \in \mathcal{B}$;

(A2): $\left(\|\delta_g\|_{\infty} + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta\right) K_b < 1$, where $K_b = \sup_{t \in [0, b]} |K(t)|$ and $\eta = \int_0^t \delta_f(s)ds$.

Then there exists a unique solution to (1.1)-(1.2) on $(-\infty, b]$.

**Proof.** According to Lemma 3.1, the problem (1.1)-(1.2) is equivalent to the integral equation

$$x(t) = \varphi(0) - g(0, \varphi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s)ds, \quad t \in [0, b],$$

(3.2)

with $x_0 = \varphi$. Now, we must transform the problem (1.1)-(1.2) to be applicable to fixed point problem. Let the operator $\Pi : \Upsilon \to \Upsilon$ defined by

$$(\Pi x)(t) = \varphi(0) - g(0, \varphi) + g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_s)ds, \quad t \in [0, b],$$

and $(\Pi x)(t) = \varphi(t)$, for $t \in (-\infty, 0]$.

For any continuous function $\varphi \in \mathcal{B}$, let $\tilde{\varphi} : (-\infty, b] \to \mathbb{R}$ be the extension of $\varphi$ such that

$$\tilde{\varphi}(t) = \begin{cases} \varphi(0), & t \in [0, b], \\ \varphi(t), & t \in (-\infty, 0] \end{cases}$$
So, we get \( \tilde{\varphi}_0 = \varphi \). For every function \( z \in C([0, b], \mathbb{R}) \) with \( z(0) = 0 \), let \( \tilde{z} : (-\infty, b] \to \mathbb{R} \) be the extension of \( z \) to \((-\infty, b]\) such that

\[
\tilde{z}(t) = \begin{cases} z(t), & t \in [0, b], \\ 0, & t \in (-\infty, 0]. \end{cases}
\]

If \( x(\cdot) \) satisfies the integral equation (3.2) then, we can analyze \( x(\cdot) \) as follows \( x(t) = \tilde{x}_t(t) + \tilde{z}(t), \ t \in (-\infty, b] \), which allude to \( x_t = \tilde{x}_t + \tilde{z}_t \), for each \( t \in [0, b] \) and the function \( z(\cdot) \) satisfies

\[
z(t) = -g(0, \varphi) + g(t, \tilde{x}_t + \tilde{z}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \tilde{x}_s + \tilde{z}_s) ds, \ t \in [0, b],
\]

with \( \tilde{z}_0 = 0 \). Setting \( \Upsilon_0 = \{ z \in \Upsilon, \text{ such that } z_0 = 0 \} \). For any \( z \in \Upsilon_0 \) and let \( \|z\|_{\Upsilon_0} \) be semi-norm in \( \Upsilon_0 \) determined by

\[
\|z\|_{\Upsilon_0} = \|z_0\|_B + \|z\|_C = \sup_{t \in [0, b]} |z(t)|. 
\]

Then \( (\Upsilon_0, \|z\|_{\Upsilon_0}) \) is a Banach space. Consider the operator \( \Pi_0 : \Upsilon_0 \to \Upsilon_0 \) be defined by

\[
(\Pi_0 z)(t) = -g(0, \varphi) + g(t, \tilde{x}_t + \tilde{z}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \tilde{x}_s + \tilde{z}_s) ds, \ t \in [0, b],
\]

and \( (\Pi_0 z)(t) = 0, t \in (-\infty, 0] \). Hence, we get \( (\Pi_0 z)_0 = 0 \).

Certainly, \( \Pi \) has a fixed point equivalent to \( \Pi_0 \) that has a fixed point too. So, we go ahead to show that \( \Pi_0 \) has a fixed point in \( \Upsilon_0 \) by the Banach fixed point theorem. In order that, we need to prove that the operator \( \Pi_0 : \Upsilon_0 \to \Upsilon_0 \) is a contraction map. In fact, consider \( z_1, z_2 \in \Upsilon_0 \) and \( t \in [0, b] \), then by (3.4), Definition 2.1 and (A1), we have

\[
\| (\Pi_0 z_1)(t) - (\Pi_0 z_2)(t) \| \leq \|g(t, \tilde{x}_t + \tilde{z}_1) - g(t, \tilde{x}_t + \tilde{z}_2)\| \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, \tilde{x}_s + \tilde{z}_1) - f(s, \tilde{x}_s + \tilde{z}_2)\| ds \\
\leq \|g\|_{\infty} \|\tilde{z}_1 - \tilde{z}_2\|_B + \|\tilde{z}_1 - \tilde{z}_2\|_B \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\tilde{x}_s - \tilde{x}_t\| ds \\
= \left( \|g\|_{\infty} + I_0^{\alpha-1} I_0^1 \|\tilde{x}_s - \tilde{x}_t\|_B \right) \|\tilde{z}_1 - \tilde{z}_2\|_B \\
\leq \left[ \|g\|_{\infty} + \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-2} \int_0^s \|\tilde{x}_s - \tilde{x}_t\| ds \right] \|\tilde{z}_1 - \tilde{z}_2\|_B \\
= \left[ \|g\|_{\infty} + \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-2} \int_0^s \|\tilde{x}_s - \tilde{x}_t\| ds \right] \|\tilde{z}_1 - \tilde{z}_2\|_B,
\]

Since

\[
\|\tilde{z}_1 - \tilde{z}_2\|_B \leq K(t) \sup_{0 \leq \tau \leq t} |\tilde{z}_1(\tau) - \tilde{z}_2(\tau)| + M(t) \|\tilde{z}_1 - \tilde{z}_2\|_B \\
\leq K_b \sup_{0 \leq \tau \leq t} |z_1(\tau) - z_2(\tau)|
\]
\[ (\Pi_0 z_1)(t) - (\Pi_0 z_2)(t) \leq \left[ \|\delta g\|_{\infty} + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta \right] K_b \|z_1 - z_2\|_{\Upsilon_0}. \]

Consequently,
\[ \|\Pi_0 z_1 - \Pi_0 z_2\|_{\Upsilon_0} \leq \left[ \|\delta g\|_{\infty} + b^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \right] K_b \|z_1 - z_2\|_{\Upsilon_0}. \]

From the condition (A2), we infer that
\[ \|\Pi_0 z_1 - \Pi_0 z_2\|_{\Upsilon_0} \leq \|z_1 - z_2\|_{\Upsilon_0}. \]

This implies that \( \Pi_0 \) is contraction map. So applying the Lemma 2.5, we can conclude that \( \Pi_0 \) has a fixed point \( z \) in \( \Upsilon_0 \) which is the unique solution to the equation (3.3) on \([0,b]\), what means that the operator \( \Pi \) has a fixed point \( x = \tilde{\varphi} + \tilde{z} \) in \( \Upsilon \) that is the unique solution to the fractional differential equation (1.1)-(1.2) on \( (-\infty,b] \). The proof is completed.

**Remark 3.3.** Note that, in Theorem 3.2, if the functions \( \delta g(t) \) and \( \delta f(t) \) are replaced by two positive constants \( L_g \) and \( L_f \) respectively, the second result follows. \[ \square \]

**Corollary 3.4.** Assume that \( f, g : [0,b] \times B \rightarrow \mathbb{R} \) are continuous functions and if

**B1:** There exist two constants \( L_f, L_g > 0 \) such that
\[ |f(t,u) - f(t,v)| \leq L_f \|u - v\|_B, \quad \text{and} \quad |g(t,u) - g(t,v)| \leq L_f \|u - v\|_B, \]
for any \( t \in [0,b] \) and for each \( u, v \in B \);

**B2:** \[ L_g + \frac{b^{\alpha}}{\Gamma(\alpha+1)} L_f \] \( K_b < 1 \).

Then there exists a unique solution to (1.1)-(1.2) on \( (-\infty,b] \).

\[ \text{Proof. see ([2]).} \]

\[ \square \]

4. **Continuous Dependence**

In this section, we discuss the continuity dependence with respect to parameters \( \varphi, \alpha \).

**Definition 4.1.** ([9]). The solution \( x \in C([0,b]) \) of the problem (1.1)-(1.2) is continuously dependence on initial data if for every \( \varphi, \psi \in \mathcal{B} \),
\[ \|x_\varphi(.) - x_\psi(.)\|_C \leq O(\|\varphi - \psi\|_B), \]
where \( x_\varphi(.) \) is solution of the problem (1.1)-(1.2) and \( x_\psi(.) \) is solution of the following problem
\[ ^cD_0^\alpha [x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0,b], \]
\[ x_0 = \psi \in \mathcal{B}. \]

Firstly, in the following theorem, we investigate the continuous dependence of the solution for the problem (1.1)-(1.2) on the initial value \( \varphi \).
Theorem 4.2. Assume that the hypotheses of Theorem 3.2 are satisfied. Let \( x_\varphi \) and \( x_\psi \) are solutions of (1.1)-(1.2) and (4.1)-(4.2) for \( \varphi, \psi \in \mathcal{B} \), respectively. Then there exists a constant \( \Lambda \) such that

\[
\| x_\varphi(\cdot) - x_\psi(\cdot) \|_C \leq \Lambda \| \varphi - \psi \|_B , \quad \forall \varphi, \psi \in \mathcal{B}.
\]

Proof. In view of Theorem 3.2, we know that for every \( \varphi, \psi \in \mathcal{B} \), the equation (1.1) has solutions \( x_\varphi \) and \( x_\psi \) on \(( -\infty, b]\), respectively. Further, there are \( z_1, z_2 \in C([0, b]) \) such that for all \( t \in [0, b] \), \( x_\varphi(t) = \varphi(0) + z_1(t) \) and \( x_\psi(t) = \psi(0) + z_2(t) \) together with

\[
z_1(t) = -g(0, \varphi) + g(t, \tilde{\varphi}_t + (\tilde{z}_1)_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, \tilde{\varphi}_s + (\tilde{z}_1)_s) \, ds,
\]

\[
z_2(t) = -g(0, \psi) + g(t, \tilde{\psi}_t + (\tilde{z}_2)_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, \tilde{\psi}_s + (\tilde{z}_2)_s) \, ds.
\]

Therefore, by hypotheses of Theorem 3.2, for \( t \in [0, b] \), we have

\[
|x_\varphi(t) - x_\psi(t)| \leq |\varphi(0) - \psi(0)| + |z_1(t) - z_2(t)|
\]

\[
\leq |\varphi(0) - \psi(0)| + |g(t, \tilde{\varphi}_t + (\tilde{z}_1)_t) - g(t, \tilde{\psi}_t + (\tilde{z}_2)_t)|
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |f(s, \tilde{\varphi}_s + (\tilde{z}_1)_s) - f(s, \tilde{\psi}_s + (\tilde{z}_2)_s)| \, ds
\]

\[
\leq H \| \varphi - \psi \|_B + \delta_g(0) \| \varphi - \psi \|_B
\]

\[
+ \delta_g(t) \left[ \| \tilde{\varphi}_t - \tilde{\psi}_t \|_B + \| (\tilde{z}_1)_t - (\tilde{z}_2)_t \|_B \right]
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \delta_f(s) \left[ \| \tilde{\varphi}_s - \tilde{\psi}_s \|_B + \| (\tilde{z}_1)_s - (\tilde{z}_2)_s \|_B \right] \, ds
\]

\[
\leq H \| \varphi - \psi \|_B + \| \delta_g \|_\infty \| \varphi - \psi \|_B
\]

\[
+ \| \delta_g \|_\infty \left[ \| \tilde{\varphi}_t - \tilde{\psi}_t \|_B + \| (\tilde{z}_1)_t - (\tilde{z}_2)_t \|_B \right]
\]

\[
+ \eta \left[ \| \tilde{\varphi}_t - \tilde{\psi}_t \|_B + \| (\tilde{z}_1)_t - (\tilde{z}_2)_t \|_B \right] \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha - 2} \, ds.
\]

In a similar way to (3.5), we can deduce that

\[
\| (\tilde{z}_1)_t - (\tilde{z}_2)_t \|_B \leq K_b \| z_1 - z_2 \|_C . \tag{4.3}
\]

Also, we have

\[
\left\| \tilde{\varphi}_t - \tilde{\psi}_t \right\|_B \leq K(t) \sup_{0 \leq \tau \leq t} \left| \tilde{\varphi}(\tau) - \tilde{\psi}(\tau) \right| + M(t) \left\| \tilde{\varphi}_0 - \tilde{\psi}_0 \right\|_B
\]

\[
\leq K_b \| \varphi(0) - \psi(0) \| + M_b \| \varphi - \psi \|_B
\]

\[
\leq (K_bH + M_b) \| \varphi - \psi \|_B ,
\]
where $M_b = \sup \{ M(t) : t \in [0,b] \}$. From preceding processes, we can conclude that

$$|x_\alpha(t) - x_\beta(t)|$$

$$\leq \left(H + \|\delta_y\|_\infty \|\varphi - \psi\|_B + \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right](K_b H + M_b) \|\varphi - \psi\|_B \right.$$ 

$$+ \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right]K_b \|z_1 - z_2\|_C$$

$$\leq \left(H + \|\delta_y\|_\infty \|\varphi - \psi\|_B + \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right](K_b H + M_b) \|\varphi - \psi\|_B \right.$$ 

$$+ \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right]K_b \|x_\alpha(.) - x_\beta(.)\|_C + H \|\varphi - \psi\|_B \right).$$

Accordingly, and by (A2), we obtain

$$\|x_\alpha(.) - x_\beta(.)\|_C \leq \left( H(1 + K_b) + \|\delta_y\|_\infty \right) \|\varphi - \psi\|_B$$

$$+ \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right] \left( K_b H + M_b \right) \|\varphi - \psi\|_B$$

$$+ \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right] K_b \|x_\alpha(.) - x_\beta(.)\|_C \right.$$ 

$$\leq \left[\frac{\theta}{1 - \Theta} \right] \|\varphi - \psi\|_B,$$

where $\theta = \left[H(1 + K_b) + \|\delta_y\|_\infty + \frac{\Theta}{K_b}(K_b H + M_b)\right]$ and $\Theta = \left[\|\delta_y\|_\infty + \frac{\eta^{-1}}{\Gamma(\alpha)}\right] K_b$.

Take $\Lambda = \left[\frac{\theta}{1 - \Theta} \right]$, we get

$$\|x_\alpha(.) - x_\beta(.)\|_C \leq \Lambda \|\varphi - \psi\|_B.$$

This proves the desired. \(\square\)

Next, we discuss the consequences of modification of the order to the fractional differential equation.

**Definition 4.3.** The solution $x \in C([0,b])$ of the problem (1.1)-(1.2) is continuously dependence on order $\alpha$ to fractional differential equations if $0 < \alpha < \tilde{\alpha} < 1$,

$$\|x_\alpha(.) - x_\tilde{\alpha}(.)\|_C \leq O |\tilde{\alpha} - \alpha|,$$

where $x_\alpha(.)$ is solution of the problem (1.1)-(1.2) and $x_\tilde{\alpha}(.)$ is solution of the following problem

$$^cD_0^\tilde{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0,b],$$

$$x_0 = \varphi \in \mathcal{B}.$$

**Theorem 4.4.** Assume that the assumptions of Theorem 3.2 hold. Then there exists a constant $\lambda$ such that

$$\|x_\alpha(.) - x_\tilde{\alpha}(.)\|_C \leq \lambda |\tilde{\alpha} - \alpha|, \quad 0 < \alpha < \tilde{\alpha} < 1.$$
Proof. The uniqueness of the solution can be concluded as above. Let \( z_1, z_2 \in C([0, b]) \) be such that \( x_\alpha(t) = \varphi(0) + z_1(t) \) and \( x_\tilde{\alpha}(t) = \varphi(0) + z_2(t) \), for all \( t \in [0, b] \). Then for \( t \in [0, b] \), \( z_1 \) and \( z_2 \) satisfy

\[
z_1(t) = -g(t, \varphi) + g(t, \tilde{\varphi}_1 + (\tilde{z}_1)_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \tilde{\varphi}_s + (\tilde{z}_1)_s) \, ds,
\]

\[
z_2(t) = -g(t, \varphi) + g(t, \tilde{\varphi}_1 + (\tilde{z}_2)_t) + \frac{1}{\Gamma(\tilde{\alpha})} \int_0^t (t-s)^{\tilde{\alpha}-1} f(s, \tilde{\varphi}_s + (\tilde{z}_2)_s) \, ds.
\]

Consequently, by hypotheses of Theorem 3.2 and (4.3), we have

\[
|x_\alpha(t) - x_{\tilde{\alpha}}(t)| = |z_1(t) - z_2(t)|
\]

\[
\leq |g(t, \tilde{\varphi}_1 + (\tilde{z}_1)_t) - g(t, \tilde{\varphi}_1 + (\tilde{z}_2)_t)|
\]

\[
+ \int_0^t \left( \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right) |f(s, \tilde{\varphi}_s + (\tilde{z}_1)_s) - f(s, \tilde{\varphi}_s + (\tilde{z}_2)_s)| \, ds
\]

\[
\leq \left( |\delta_\varphi|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \right) \| \tilde{z}_1_t - \tilde{z}_2_t \|_B
\]

\[
+ \int_0^t \left( \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right) |f(s, \tilde{\varphi}_s + (\tilde{z}_1)_s)| \, ds
\]

\[
\leq \left( |\delta_\varphi|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \right) K_b \| z_1 - z_2 \|_C
\]

\[
+ \sup_{(t, u) \in [0, b] \times B} |f(t, u)| \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right| \, ds.
\]

Now, we estimate the integral in the right-hand side of the inequalities above

\[
\int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right| \, ds
\]

\[
\leq \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right| \, ds + \int_0^t \left| \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right| \, ds
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{t^{\alpha-1} - t^{\tilde{\alpha}-1}}{t^{\alpha-1} - t^{\tilde{\alpha}-1}} \, dt + \frac{1}{\Gamma(\tilde{\alpha})} t^{\tilde{\alpha}-1} \, dt
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \left[ \frac{b^\alpha}{\alpha} - \frac{b^{\tilde{\alpha}}}{\tilde{\alpha}} \right] + \left| \frac{\Gamma(\tilde{\alpha}) - \Gamma(\alpha)}{\Gamma(\alpha) \Gamma(\tilde{\alpha})} \right| \frac{b^{\tilde{\alpha}}}{\alpha}.
\]

Since \( \alpha < \tilde{\alpha} \) and by Mean Value Theorem of classical differential calculus, we get

\[
\int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\tilde{\alpha}-1}}{\Gamma(\tilde{\alpha})} \right| \, ds
\]

\[
\leq \left| \frac{\tilde{\alpha} - \alpha}{\alpha + 1} \right| \frac{b^{\tilde{\alpha}}}{\Gamma(\alpha + 1) \alpha} + \left| \frac{\tilde{\alpha} - \alpha}{\alpha + 1} \right| \frac{b^{\tilde{\alpha}}}{\Gamma(\alpha + 1) \alpha}.
\]
\[
\leq \frac{|\bar{\alpha} - \alpha|}{\Gamma(\alpha + 1)\alpha} b \left[ 1 + \frac{\Gamma(\bar{\alpha}) - \Gamma(\alpha)}{(\bar{\alpha} - \alpha)} \right]
\]

\[
= \frac{|\bar{\alpha} - \alpha|}{\Gamma(\alpha + 1)\alpha} \left[ b(1 + |\Gamma'(\bar{\alpha} - \theta(\bar{\alpha} - \alpha))|) \right]
\]

\[
= \frac{|\bar{\alpha} - \alpha|}{\Gamma(\alpha + 1)\alpha} R
\]

for some \( \theta \) such that \( 0 < \alpha < \theta < \bar{\alpha} < 1 \) and \( R = \frac{|b(1+|\Gamma'(\bar{\alpha} - \theta(\bar{\alpha} - \alpha))|)|}{\Gamma(\alpha + 1)\alpha} \).

In the end, by (A2), we arrive at the following

\[
\frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)\alpha} R \leq |\bar{\alpha} - \alpha| \left[ b(1 + |\Gamma'(\bar{\alpha} - \theta(\bar{\alpha} - \alpha))|) \right]
\]

\[
= |\bar{\alpha} - \alpha| R,
\]

This proves the required. \( \square \)

5. AN EXAMPLE

Firstly, Let \( \rho \) be a positive real constant and we define the functional space \( \mathbb{B}_\rho \) by

\[
\mathbb{B}_\rho = \{ x \in C((\infty, 0], \mathbb{R}) : \lim_{s \to -\infty} e^{\rho s} x(s) \text{ exist in } \mathbb{R} \},
\]

endowed with the following norm \( \|x\|_\rho = \sup \{ e^{\rho s} |x(s)| : -\infty < s \leq 0 \} \).

Then \( \mathbb{B}_\rho \) satisfies axioms (H1), (H2) and (H3) with \( K(t) = M(t) = 1 \) and \( H = 1 \). (see [5]).

Next, we consider the fractional neutral functional differential equation

\[
^cD_0^\alpha x(t) = \frac{e^{-\rho t}}{(2 + e^{-2t})} \frac{\|x_t\|}{1 + \|x_t\|}, \quad t \in [0, b],
\]

with initial condition

\[
x_0 = \varphi, \text{ on } (-\infty, 0],
\]

where \( \varphi \in \mathbb{B}_\rho \).

Finally, we applying Theorem 3.2. Consider \( f(t, u) = \frac{e^{-\rho t}}{(1+t)} \frac{u}{1+u} \) and \( g(t, u) = \frac{e^{-\rho t}}{(2+e^{-2t})} \frac{u}{1+u} \), for \( (t, u) \in [0, b] \times \mathbb{B}_\rho \). Let \( x, y \in \mathbb{B}_\rho \), and \( t \in [0, b] \). Then

\[
|f(t, x) - f(t, y)| = \frac{e^{-\rho t}}{(1+t)} \frac{x}{1+x} - \frac{y}{1+y}
\]
\[
\frac{e^{-\rho t}}{(1 + t)} \left| \frac{x - y}{(1 + x)(1 + y)} \right| \\
\leq \frac{e^{-\rho t}}{(1 + t)} |x - y| \\
\leq \frac{1}{(1 + t)} \|x - y\|_\rho
\]

and
\[
|g(t, x) - g(t, y)| = \frac{e^{-\rho t}}{(2 + e^{-2t})} \left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| \\
= \frac{e^{-\rho t}}{(2 + e^{-2t})} \left| \frac{x - y}{(1 + x)(1 + y)} \right| \\
\leq \frac{1}{(2 + e^{-2t})} |x - y| \\
\leq \frac{1}{(2 + e^{-2t})} \|x - y\|_\rho.
\]

Hence, the condition (A1) holds with \(\delta_f(t) = \frac{1}{(1 + t)} \in L^1([0, b], \mathbb{R})\) and \(\delta_g(t) = \frac{1}{(2 + e^{-2t})} \in C([0, b], \mathbb{R})\). It can be checked that condition (A2) is satisfied with \(K_b = 1\), \(\alpha = 1/4\) and \(b = 1\).

Indeed, since \(\|\delta_g\|_\infty = \sup\{|\delta_g(t)| : t \in [0, 1]\} = \sup\{|1/(2 + e^{-2t})| : t \in [0, 1]\} = \frac{1}{2}\), \(\eta = \int_0^1 \delta_f(s)ds = \int_0^1 \frac{1}{1 + s}ds = \ln 2\) and \(\frac{1}{\Gamma(\frac{3}{4})} \ln 2 < \frac{1}{2}\). Then, \(K_b \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta\right] < 1\).

According to Theorem 3.2, then problem (5.1)-(5.2) has a unique solution on \((-\infty, 1]\).

6. Conclusions

This paper has investigated the uniqueness and continuous dependence of solutions for the fractional neutral functional differential equation (1.1)-(1.2) with infinite delay which includes the Caputo fractional derivative. Firstly, we introduced some properties of fractional calculus with mention useful definitions and lemmas related to the fixed point theory, phase space. Further, we offered the list of appropriate conditions for \(f\) and \(g\). Secondly, the uniqueness results are investigated by utilizing the Banach fixed point theorem. Moreover, the continuous dependence of solutions to the problem (1.1)-(1.2) is discussed in the space \(C[0, b]\). Finally, an example is provided to show the effectiveness of the proposed results.

Acknowledgment

The authors express their deep gratitude to the referees for their valuable suggestions and comments for improvement of the paper.

References


COMPARATIVE STUDY OF NUMERICAL ALGORITHMS FOR THE ARITHMETIC ASIAN OPTION

JIAN WANG\textsuperscript{1} AND JUNGYUP BAN\textsuperscript{2}, SEONGJIN LEE\textsuperscript{2}, CHANGWOO YOO\textsuperscript{2,†}

\textsuperscript{1} DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 02841, KOREA
\textsuperscript{2} DEPARTMENT OF FINANCIAL ENGINEERING, KOREA UNIVERSITY, SEOUL 02841, KOREA
E-mail address: coreapoa@korea.ac.kr

ABSTRACT. This paper presents the numerical valuation of the arithmetic Asian option by using the operator-splitting method (OSM). Since there is no closed-form solution for the arithmetic Asian option, finding a good numerical algorithm to value the arithmetic Asian option is important. In this paper, we focus on a two-dimensional PDE. The OSM is famous for dealing with plural-dimensional PDE using finite difference discretization. We provide a detailed numerical algorithm and compare results with MCS method to show the performance of the method.

I. INTRODUCTION

We consider an efficient and accurate finite difference method \cite{1} and Monte Carlo simulation \cite{2} for an arithmetic Asian option. The Asian option is a contract that gives the holder the right to buy an asset based on its average price over some prescribed period of time \cite{3}. There are two types of Asian options such as arithmetic Asian option and geometric Asian option. Geometric Asian option with payoff which depends on geometric mean of underlying asset over time interval has a closed form solution. On the other hand, arithmetic Asian option with payoff which depends on arithmetic mean of underlying asset over time interval does not have a closed form solution. Thus, finding a good numerical algorithm to value arithmetic Asian option is important.

A lot of previous studies for the Asian option have been implemented. There are a number of studies to approximate this option \cite{4, 5, 6, 7}. However, those approximation formula are only suitable for a simple type of Asian option, i.e., European type of Asian option. In general, the value of derivative can be found by solving PDE \cite{8}. There are five representative forms of PDE for Asian option. Two of them have two spatial dimensions and they are derived by Ingersoll

\begin{itemize}
\item Received by the editors February 28 2018; Accepted March 13 2018; Published online March 16 2018.
\item 2000 Mathematics Subject Classification. 93B05.
\item Key words and phrases. Arithmetic Asian option, operator splitting method, finite difference method, Black–Scholes equation.
\item † Corresponding author.
\end{itemize}
in [9] and Duffy in [1]. They are derived by using same hedging argument underlying Black–Scholes in [10]. In this paper, we are going to deal with PDE derived by Ingersoll. Rest of them have one spatial dimension which is derived by change of numeraire technique. Ingersoll set \( R = S/I \) and Duffy set \( R = I/S \) where \( S \) is stock process and \( I = \int S \, dt \). And they work some stochastic calculus to derived deterministic PDE in [1, 9, 11]. Turning upside down of numerator and denominator and some work of stochastic calculus can make PDE more simpler. However, these PDEs are only available for floating strike type of Asian option. In 1995, Rogers and Shi [12] derived one spatial dimensional PDE which is available for both floating strike type and fixed strike type of Asian options by setting \( x = [K - \int_0^t S(\tau) \mu(\tau) \, d\tau]/S_t \). Floating and fixed strikes are as follows: \((A(T) - K)^+, (K - A(T))^+, (S(T) - A(T))^+, \) and \((A(T) - S(T))^+\), respectively, for fixed strike call, fixed strike put, floating strike call, and floating strike put, where \( A(T) = I(T)/T \).

Reduction of spatial dimension for Asian option makes PDE more simpler. Also, when we solve Asian option numerically, it reduces computational cost and increases accuracy of the numerical solution. However, it may not always be possible to find a similarity solution [1] and derived one spatial dimensional PDEs are not suitable for barrier type of Asian option and local volatility model [13]. Also, combining other properties of other exotic option with Asian option will make problems more complicated. It is hard to find boundary conditions since one spatial dimensional PDE is not intuitive than two spatial dimensional PDE. Thus, there is necessity of good numerical algorithm to solve two spatial dimensional PDE in recent complicated financial market situation. We are going to adapt OSM and method of characteristic for one of spatial dimension which has no diffusion term to solve two spatial dimensional PDE. This concept was introduced by Duffy in [1] but was not described in detail. To show performance of the proposed algorithm, we will compare with MCS method.

Section 2 presents arithmetic Asian option and PDE derived by Ingersoll in [9]. Section 3 describes numerical solution algorithm using finite difference scheme and OSM with method of characteristic and presents a numerical algorithm to price arithmetic Asian option by using MCS method. Section 4 presents the computational results showing the performances of the MCS method and FDM using OSM, that is, comparative study. Conclusions are presented in Section 5.

2. ARITHMETIC ASIAN OPTION PDE BY INGERSOLL

We adapt the classical Geometric Brownian Motion (GBM) for stock process \( S(t) \):

\[
dS(t) = rS(t)dt + \sigma S(t)dW(t)
\]  

where \( r \) is risk-neutral constant interest rate, \( \sigma \) is constant volatility and \( dW(t) \) is standard Brownian Motion. \( V(T, S(T), I(T)) \) denote payoff function of Asian option where

\[
I(t) = \int_0^t S(\theta) d\theta.
\]
By definition, dynamics of variable $I$ is

$$dI(t) = S(t)dt$$

which will be used for Itô’s lemma. There must exist some function $v(t, S(t), I(t))$ such that

$$v(t, S(t), I(t)) = \mathbb{E} \left[ \frac{D(T)}{D(t)} V(T, S(T), I(T)) | F(t) \right]$$

$$= \mathbb{E} \left[ e^{-r(T-t)} V(T, S(T), I(T)) | F(t) \right]$$

(2.2)

where $\mathbb{E}$ represent risk-neutral expectation and $D(t)$ denote discount factor. Since we assumed constant interest rate, discount factor is $D(t) = e^{-rt}$. Price function $v(t, S(t), I(t))$ can represent any strike type of Asian option. By property of risk-neutral pricing, dynamics of $D(t)v(t, S(t), I(t))$ becomes

$$d(e^{-rt}v(t, S(t), I(t))) = e^{-rt} \left[ -rvdt + v_t dt + v_{S} dS + v_{I} dI + \frac{1}{2} v_{SS} dS dS \right]$$

$$= e^{-rt} \left[ -rv + v_t + rSv_S + Sv_I + \frac{1}{2} \sigma^2 v_{SS} \right] dt + e^{-rt} \sigma S v_S \tilde{d}W(t)$$

where $v_x$ denote differential of $v$ on x-direction and $\tilde{d}W(t)$ is risk-neutral Brownian motion. For this process to be martingale, $dt$ term of this process must be zero. Thus, we derived Ingersoll’s 2-spatial dimensional PDE as follows

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} + S \frac{\partial V}{\partial I} - rV = 0.$$  

(2.3)

To demonstrate following numerical algorithms, we set payoff function as fixed strike call

$$V(T, S(T), I(T)) \Lambda(T) = (A(T) - K)^+.$$  

(2.4)

Without loss of generality, other payoff function also can be adapted in those algorithms by changing some boundary conditions and initial conditions.

3. Numerical methods

In this section, we describe the numerical discretization of Eq. (2.3). We also present the operator-splitting algorithm in detail. Particularly, we are going to use method of characteristic for one of spatial dimension which does not have diffusion term.

3.1. Finite difference discretization. Let $L_{BS}$ be the operator

$$L_{BS} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS \frac{\partial v}{\partial S} + \frac{s}{\partial I} - rv$$
and $\tau$ be the remaining time to maturity such that $\tau = T - t$ where $T$ is maturity. Then, the two spatial dimensional Ingersoll’s equation can be rewritten as

$$\frac{\partial v}{\partial \tau} = \mathcal{L}_{BS}, \quad \text{for } (\tau, s, i) \in [0, T] \times \Omega,$$

Value of this derivative is defined on unbounded domain such that $\{(\tau, s, i) \mid \tau \in [0, T], s \geq 0, \ i \in \mathbb{R}\}$ as you can see in [14]. To compute price of option in computer, we truncate this domain as finite computational domain such that $\{(\tau, s, i) \mid \tau \in [0, T], 0 \leq s \leq M_s, 0 \leq i \leq M_i\}$, where $M_s$ and $M_i$ are large enough number. In general, $M_s$ and $M_i$ could be set as two or three times of strike price $K$. Then, error of the price driven by truncation becomes ignorable [15]. Note that setting of domain $i$ is $\{i \mid 0 \leq i \leq M_i\}$. Although the original domain for $i$ is $\{i \mid i \in \mathbb{R}\}$, we assumed that $i$ is almost surely positive because $I$ is defined as $I(t) = \int_0^t S(\theta) d\theta$ and classically defined GBM stock process $S(t)$ is positive on any time. We have Dirichlet boundary condition when $s = 0$ such that

$$v(\tau, 0, i) = e^{-\mu \tau} \left(\frac{i}{T} - K\right)^+$$

for $\tau \in [0, T]$ and $0 \leq i \leq M_i$. Also, for rest of three artificial boundaries, we have linear boundary condition [16, 17, 18, 19] such that

$$\frac{\partial^2 v}{\partial i^2} v(\tau, s, 0) = \frac{\partial^2 u}{\partial i^2} u(\tau, s, M_i) = \frac{\partial^2 u}{\partial s^2} u(\tau, M_s, i) = 0$$

for $\tau \in [0, T], 0 \leq s \leq M_s$ and $0 \leq i \leq M_i$.

Let $N_s, N_i$ and $N_\tau$ denote the numbers of grid points for $s$-, $i$- and $\tau$-directions, respectively. We are going to adapt uniform grid as $h = M_s/N_s = M_i/N_i$ and $\Delta \tau = T/N_\tau$ which discretize the settled computational domain $[0, T] \times \Omega$ where $\Omega = (0, M_s) \times (0, M_i)$. Figure 1 illustrates the 2-dimensional uniform grid with a spatial step size $h$.

According to discretization, $v^n_{jk}$ denote approximated numerical solution such that

$$v^n_{jk} = v(\tau^n, s_j, i_k) = v(n \Delta \tau, jh, kh)$$

for $n = 0, \ldots, N_\tau, j = 1, \ldots, N_s - 1$ and $k = 1, \ldots, N_i - 1$. For Dirichlet boundary condition when $s = 0$, numerical approximation in each time steps defined as

$$v^n_{0k} = v(n \Delta \tau, 0, kh) = e^{-\mu n \Delta \tau} \left(\frac{kh}{T} - K\right)^+.$$  

Since other directions are used linear boundary condition and formula used linear boundary condition do not depend on time step, for all time steps, numerical approximation can be denoted as

$$v_{j0} = 2v_{j1} - v_{j2}, \quad v_{j,N_i} = 2v_{j,N_i-1} - v_{j,N_i-2}$$

for $j = 1, \cdots, N_s - 1$ and

$$v_{N_s,k} = 2v_{N_s-1,k} - v_{N_s-2,k}$$

for $k = 1, \cdots, N_i - 1$. 


3.2. **Operator-splitting method.** The concept of operator-splitting method is to transfer problem of multi-dimensional PDE to problem of multiple one-dimensional problems [1, 20]. In this case, we are going to use method of characteristic for i-direction whose solution could be represented as analytic solution by solving simple partial differential equation. Simple partial differential equation is

\[ V_\tau - SV_I = 0. \]  

(3.1)

We can find analytic solution by some work with Eq.(3.1) such that

\[ V(S, I, \tau) = V(S, I(\tau - \tau^n), \tau^n) \]  

for \( \tau \geq \tau^n \)

Verification of solution is as follows:

Since \( V_\tau = SV'(S, I + S(\tau - \tau^n), \tau^n) \) and \( V_I = V'(S, I + S(\tau - \tau^n), \tau^n) \),

hence \( V_\tau - SV_I = SV' - SV' = 0 \).

Adapting our discretized scheme to this solution, solution for each time step could be represented as

\[ V(S, I, \tau^{n+1}) = V(S, I + S\Delta\tau, \tau^n). \]  

(3.2)

Also, because solution could be in between points that we have discretized, we are going to use linear interpolation for computing this problem. In general, the basic operator-splitting scheme for the spatial two-dimensional Ingersoll’s PDE as follows:

\[ \frac{v_{jk}^{n+1} - v_{jk}^n}{\Delta\tau} = L_{ic}^i v_{jk}^{n+1} + L_{ic}^s v_{jk}^{n+1}, \]  

(3.3)
where the operator $L^i_{IG}$ and $L^s_{IG}$ defined by
\[
L^i_{IG} v_{n+\frac{1}{2}jk} = \frac{v_{j,k+1}^n - v_{j,k}^n}{h} s_j
\]
\[
L^s_{IG} v_{n+1}^{jk} = \frac{(\sigma s_j)^2}{2} v_{j-1,k}^{n+1} - 2v_{j,k}^{n+1} + v_{j+1,k}^{n+1} \frac{v_{j+1,k}^{n+1} - v_{j,k}^{n+1}}{h} + r s_j.
\]

The first step is using analytic solution in the i-direction. Next step is solving implicitly in the s-direction. We set temporary time step $n + \frac{1}{2}$ for compute OS-scheme which does not exist in real. This OSM reduce 2-dimensional problem into two 1-dimensional problem.

By Eq.(3.2), discrete solution is
\[
v^n_{n+\frac{1}{2}jk} = v_{j,k}^n + 1 - \frac{v_{j,k}^n}{h} s_j \Delta \tau + v_{j,k}^n.
\]

This implies discrete difference i-direction operator using method of characteristic $L^i_{IG}$ is defined by
\[
\frac{v^{n+\frac{1}{2}}_{j,k} - v_{j,k}^n}{\Delta \tau} = L^i_{IG} v_{n+\frac{1}{2}jk}.
\]

Also, implicitly approximating s-direction operator $L^s_{IG}$ is defined by
\[
\frac{v_{n+1}^{jk} - v^{n+\frac{1}{2}}_{j,k}}{\Delta \tau} = L^s_{IG} v_{n+1}^{jk}.
\]

Note that sum of two Eqs.(3.5) and (3.6) is Eq.(3.3). Before iterate Algorithm’s logic we set $v^n_{0jk} = \Lambda (T)$. An algorithm of the OSM is as follows:

**Algorithm OS**

- **Step 1**

  Initialize $v^n_{0jk} = e^{-rn\Delta \tau (\frac{kh}{T} - K)^+}$ in every time step.

- **Step 2**

  To describe as general form of OSM, Eq. (3.5) is used. However, to compute i-direction solution, Eq. (3.5) is rewritten as Eq.(3.4). For each $j$, we have

  \[
  v^{n+\frac{1}{2}}_{j,k} = \frac{v_{j,k+1}^n - v_{j,k}^n}{h} s_j \Delta \tau + v_{j,k}^n.
  \]

  The first step of the OS method is then implemented in a loop over the s-direction:

  for $j = 0 : N_s$
  
  for $k = 1 : N_i - 1$

  Solve $v^{n+\frac{1}{2}}_{j,k}$ by Eq. (3.7) (see Fig.2(a))

  end
Use boundary condition
\[
v_{j0}^{n+\frac{1}{2}} = 2v_{j1}^{n+\frac{1}{2}} - v_{j2}^{n+\frac{1}{2}}, \quad v_{j,N_i}^{n+\frac{1}{2}} = 2v_{j,N_i-1}^{n+\frac{1}{2}} - v_{j,N_i-2}^{n+\frac{1}{2}}
\]
end

Note that \(v_{0,k}^{n+\frac{1}{2}} = v_{0k}^n\) for \(k = 0, \ldots, N_i\) because \(s_0 = 0\). Thus, Step 2 do nothing for \(v_{0,0,N_i}\).

\[\begin{align*}
\text{Figure 2. Two steps of the OSM.}
\end{align*}\]

- Step 3

Unlike in Step 1, Eq. (3.6) is rewritten as follows:
\[
\alpha_j v_{j-1,k}^{n+1} + \beta_j v_{j,k}^{n+1} + \gamma_j v_{j+1,k}^{n+1} = f_{jk},
\]
where
\[
\begin{align*}
\alpha_j &= -\frac{1}{2} \sigma^2 s_j^2 - \frac{\sigma^2 s_j^2}{h^2} + \frac{r s_j}{h} + r, \\
\beta_j &= \frac{1}{\Delta \tau} + \frac{\sigma^2 s_j^2}{h^2} + \frac{r s_j}{h} + r, \\
\gamma_j &= -\frac{1}{2} \sigma^2 s_j^2 - \frac{r s_j}{h}, \text{ for } k = 1, \ldots, N_i - 1
\end{align*}
\]
and
\[
\begin{align*}
f_{1k} &= \frac{v_{1k}^{n+\frac{1}{2}}}{\Delta \tau} - \alpha_1 e^{-rn \Delta \tau} (\frac{k h}{T} - K)^+, \\
f_{jk} &= \frac{v_{jk}^{n+\frac{1}{2}}}{\Delta \tau} \text{ for } j = 2, \ldots, N_s - 1.
\end{align*}
\]

Note that we adapt Dirichlet boundary condition to get the coefficients.

As with Step 2, Step 3 is then operated in a loop over the \(i\)-direction.
for $k = 0 : N_i$
for $j = 1 : N_y - 1$
Set $f_{jk}$ by Eq. (3.10)
end
Solve $A_s u^{n+1}_{1:N_s-1,k} = f_{1:N_s-1,k}$ by using Thomas algorithm (see Fig. 2(b))
end

Here $A_s$ is tridiagonal matrix constructed from Eq. (3.8) with a Dirichlet boundary condition for $j = 0$ and a linear boundary condition for $j = N_s$

$$A_s = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \ldots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \ldots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \beta_{N_y-1} & \gamma_{N_y-1} \\ 0 & 0 & 0 & \alpha_{N_y} & \gamma_{N_y} & \beta_{N_y} + 2\gamma_{N_y} \end{pmatrix}.$$}

• **Step 4**

Iterate Step 1 to 3 for all time step.

The information of OSM in this paper was based in [1]. For more information on OSM, please refer to [1].

3.3. **Monte Carlo simulation method.** To describe general Monte Carlo simulation, we also adapt Eq.(2.1). Integrate both side of Eq.(2.1) and rearrange by $S(t)$ equation becomes

$$S(t) = S(0)e^{(r-0.5\sigma^2)t + \sigma W(t)}.$$ Let $\Delta t$ be small time increment and $S(n \Delta t)$ be denoted by $S^n$. Then, discretized stock process is defined as

$$S^{n+1} = S^n e^{(r-0.5\sigma^2)\Delta t + \sigma \sqrt{\Delta t} Z^n} \quad \text{for} \quad 0 \leq n \leq \frac{T}{\Delta t} - 1$$ (3.11)

where $n$ is integer and random variable $Z^n$ is independent and identically distributed (i.i.d) with standard Normal distribution which is denoted as $N(0,1)$. According to Eq.(3.11), we generate enough large path to satisfy consistency of estimation. (See Fig. 3.) Let $M$ denote the number of iteration and $\Omega$ denote set of all parameters and sample space. Then,

$$\hat{v}(0, S(0), I(0)) = \frac{1}{M} e^{-rT} \sum_{i=1}^{M} \Lambda(T)$$ (3.12)
becomes a good estimation of $v(0, S(0), I(0))$ by Eq. (2.2) where $A(T)$ represented as Eq. (2.4).

Figure 3. Schematic of path generation of MCS.

More information of Monte Carlo simulation method in finance is introduced in [2].

4. Computational results

4.1. Convergence test of FDM and MCS. To demonstrate consistency of the numerical algorithms, we perform convergence tests. We set some parameters as $r = 0.03, S(0) = 100, K = 100, T = 1, \sigma = 0.3$. We test numerical scheme Eq. (3.3) with respect to $h$ and $\Delta \tau$ for FDM. For truncated computational domain, we also set $M_s = M_i = 300$ and we are going to use these values unless otherwise specified. Table 1 show the convergence of Asian fixed strike type call option prices at $(S(0), I(0), 0)$ as we refine $h$ and $\Delta \tau$. As you see, tendency of prices flow with respect to $h$ and $\Delta \tau$ seem to be converges to a certain value.

<table>
<thead>
<tr>
<th>Case</th>
<th>$h = 4$</th>
<th>$h = 2$</th>
<th>$h = 1$</th>
<th>$h = 0.5$</th>
<th>$h = 0.25$</th>
<th>$h = 0.125$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \tau = 1/90$</td>
<td>10.250442</td>
<td>8.496569</td>
<td>7.677030</td>
<td>7.610064</td>
<td>7.579022</td>
<td>7.569500</td>
</tr>
<tr>
<td>$\Delta \tau = 1/180$</td>
<td>10.666462</td>
<td>9.004364</td>
<td>8.024811</td>
<td>7.598022</td>
<td>7.564093</td>
<td>7.548436</td>
</tr>
<tr>
<td>$\Delta \tau = 1/360$</td>
<td>10.867678</td>
<td>9.246414</td>
<td>8.299454</td>
<td>7.776086</td>
<td>7.558052</td>
<td>7.540982</td>
</tr>
<tr>
<td>$\Delta \tau = 1/720$</td>
<td>10.966702</td>
<td>9.364765</td>
<td>8.432927</td>
<td>7.919678</td>
<td>7.648182</td>
<td>7.537956</td>
</tr>
</tbody>
</table>

Table 1. Convergence test for European call option values by FDM.
Since Arithmetic Asian option does not have analytic solution, we need to set reference solution. We set \( V(S(0), I(0), 0) = 7.529395 \) as reference solution by FDM very slight grid such as \( h = 0.0625 \) and \( \Delta \tau = 1/720 \).

For MCS, we calculate estimation Eq.(3.12) with respect to the number of iteration. Let \( N_{it} \) denote the number of iteration. Figure 4 shows convergence of MCS method. As \( N_{it} \) increases, numerical solution of MCS also seem to converge to a certain value. (See figure 4 (a).) We compute prices by 5000 interval of iteration. Figure 4 (b) shows convergence with respect to \( \Delta t \). To approximate integration of stock process, that is, \( \int_{0}^{T} S(t)dt \approx \sum_{n=0}^{T/\Delta t} S_n \Delta t \), we set \( \Delta t = 1/10000 \) unless otherwise specified so that the error of numerical integration becomes negligible. Table 2 represent numerical solutions by 50000 interval of iteration.

![Figure 4. Convergence test of MCS.](image)

<table>
<thead>
<tr>
<th>( N_{it} )</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>50000</td>
<td>7.498124</td>
</tr>
<tr>
<td>100000</td>
<td>7.515509</td>
</tr>
<tr>
<td>150000</td>
<td>7.494195</td>
</tr>
<tr>
<td>200000</td>
<td>7.523239</td>
</tr>
<tr>
<td>250000</td>
<td>7.551128</td>
</tr>
<tr>
<td>300000</td>
<td>7.527592</td>
</tr>
</tbody>
</table>

Our test could be enough evidence that both FDM and MCS converges to a certain value.

In this section, we show the performance of the proposed numerical algorithms by numerical experiments. All performance were computed on a 2.7 GHz Intel PC 8 GB of RAM loaded with MATLAB 2016a [21]. We consider three case with respect to money-ness such as in the money, at the money and out of the money. Table 3 represent the parameters we set. In this section, we also set reference solution by FDM with fine grid size such as \( h = 0.0625 \) and \( \Delta \tau = 1/720 \). For at the money price, we use 7.529395 which is used for convergence test. Other values 21.423097 and 1.743159 are in the money and out of the money reference solution each.
4.2. **Computational results of Finite Difference Method.** We perform results of FDM with respect to $h$ and $\Delta \tau$. In general, spatial grid $h$ could be set as 0.5 or 1 for spatially one-dimensional PDE. This value could be considered as enough small value. However, since proposed algorithm is spatially two-dimensional PDE, results shows that spatial grid size should be smaller. Thus, we set spatial grid $h$ as 0.25 unless otherwise specified. Table 4, 5 and 6 shows numerical results.

**Table 4.** In the money results of FDM with $h = 0.25$ and Iteration number ($1/\Delta \tau$)

<table>
<thead>
<tr>
<th>Strike</th>
<th>Iteration Number</th>
<th>Reference Price</th>
<th>Numerical Price</th>
<th>Error (%)</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>90</td>
<td>21.459686</td>
<td>0.171</td>
<td>0.403min</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>180</td>
<td>21.446979</td>
<td>0.111</td>
<td>0.812min</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>360</td>
<td>21.441516</td>
<td>0.086</td>
<td>1.574min</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>720</td>
<td>21.489467</td>
<td>0.310</td>
<td>3.145min</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5.** At the money results of FDM with $h = 0.25$ and Iteration number ($1/\Delta \tau$)

<table>
<thead>
<tr>
<th>Strike</th>
<th>Iteration Number</th>
<th>Reference Price</th>
<th>Numerical Price</th>
<th>Error (%)</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>90</td>
<td>7.579022</td>
<td>0.659</td>
<td>0.400min</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>180</td>
<td>7.564093</td>
<td>0.461</td>
<td>0.780min</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>360</td>
<td>7.558052</td>
<td>0.381</td>
<td>1.577min</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>720</td>
<td>7.648182</td>
<td>1.578</td>
<td>3.086min</td>
<td></td>
</tr>
</tbody>
</table>

**Table 6.** Out of the money results of FDM with $h = 0.25$ and Iteration number ($1/\Delta \tau$)

<table>
<thead>
<tr>
<th>Strike</th>
<th>Iteration Number</th>
<th>Reference Price</th>
<th>Numerical Price</th>
<th>Error (%)</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>90</td>
<td>1.784897</td>
<td>2.394</td>
<td>0.391min</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>180</td>
<td>1.768427</td>
<td>1.450</td>
<td>0.793min</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>360</td>
<td>1.770094</td>
<td>1.545</td>
<td>1.559min</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>720</td>
<td>1.792457</td>
<td>2.828</td>
<td>3.091min</td>
<td></td>
</tr>
</tbody>
</table>

As you see the tables, numerical solutions seem to converge to reference solution except the case that $\Delta \tau = 1/720$. We could interpret this results as time error become negligible against spatial error by the fact that FDM has spatial error $O(h^2)$ and time error $O(\Delta \tau)$.
4.3. **Computational results of Monte Carlo Simulation.** Table 7, 8 and 9 are numerical results of MCS. We formed the following tables like FDM solution tables to compare results.

**Table 7. In the money results of MCS**

<table>
<thead>
<tr>
<th>Strike</th>
<th>Iteration Number</th>
<th>Reference Price</th>
<th>Numerical Price</th>
<th>Error (%)</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>5000</td>
<td>21.423097</td>
<td>21.275702</td>
<td>0.688</td>
<td>0.019min</td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td></td>
<td>21.734840</td>
<td>1.455</td>
<td>0.047min</td>
</tr>
<tr>
<td></td>
<td>50000</td>
<td></td>
<td>21.347946</td>
<td>0.351</td>
<td>2.272min</td>
</tr>
<tr>
<td></td>
<td>100000</td>
<td></td>
<td>21.474813</td>
<td>0.241</td>
<td>11.461min</td>
</tr>
</tbody>
</table>

**Table 8. At the money results of MCS**

<table>
<thead>
<tr>
<th>Strike</th>
<th>Iteration Number</th>
<th>Reference Price</th>
<th>Numerical Price</th>
<th>Error (%)</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>5000</td>
<td>7.529395</td>
<td>7.284143</td>
<td>3.257</td>
<td>0.021min</td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td></td>
<td>7.638652</td>
<td>1.451</td>
<td>0.043min</td>
</tr>
<tr>
<td></td>
<td>50000</td>
<td></td>
<td>7.546381</td>
<td>0.226</td>
<td>2.210min</td>
</tr>
<tr>
<td></td>
<td>100000</td>
<td></td>
<td>7.517309</td>
<td>0.161</td>
<td>10.942min</td>
</tr>
</tbody>
</table>

**Table 9. Out of the money results of MCS**

<table>
<thead>
<tr>
<th>Strike</th>
<th>Iteration Number</th>
<th>Reference Price</th>
<th>Numerical Price</th>
<th>Error (%)</th>
<th>Computational Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>5000</td>
<td>1.743159</td>
<td>1.535304</td>
<td>11.924</td>
<td>0.019min</td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td></td>
<td>1.635084</td>
<td>6.200</td>
<td>0.041min</td>
</tr>
<tr>
<td></td>
<td>50000</td>
<td></td>
<td>1.715868</td>
<td>1.566</td>
<td>2.245min</td>
</tr>
<tr>
<td></td>
<td>100000</td>
<td></td>
<td>1.737488</td>
<td>0.325</td>
<td>11.147min</td>
</tr>
</tbody>
</table>

As you see the tables, solutions of MCS seem to converge to reference solution. However, MCS solution could not monotonously converge to reference solution, whereas FDM solution tends to go to reference solution monotonously as \(N_m\) increase except what we have mentioned previous subsection. Figure 5 shows this fact. In practical field, the number of simulation \(N_m\) is set as 60000 which is assumed to be enough number to get consistent estimator with various technique such as quasi random number [22], Brownian bridge [23] and variance reduction [24, 25]. In this paper, we set the maximum number of iteration as 100000 without any technique.

5. **Conclusions**

In this paper, we presented a numerical algorithm for arithmetic Asian option by using the OSM and method of characteristic with 2-dimensional PDE. We construct the algorithm to approximate the value of Asian option which does not have analytic solution. We modeled Ingersoll’s spatially 2-dimensional PDE by adapting FDM with OSM and method of characteristic. We describe a detailed numerical algorithm and computational results illustrating
performance of proposed algorithms. As we mentioned previously, there are many techniques to evolve MCS. There are also various techniques to evolve FDM such as adapting adaptive schemes and high order schemes, various ways to deal with boundary condition to reduce time and etc. In this paper, we compared pure FDM and MCS so that we introduce simple basic algorithm to be understood easily and be adapted easily to various ways which evolve algorithms more efficient. Fig. 5 (a) show that FDM is superior than MCS. Fig. 5 (b) and (c) see also our proposed FDM algorithm is good except extraordinary price which we have mentioned in section of results of FDM. Indeed, our proposed FDM algorithm is competitive even though we consider extraordinary price made by fact that spatial error covers time error.
ACKNOWLEDGMENTS

The first author (Jian Wang) was supported by the Brain Korea 21 Project from the Ministry of Education of Korea. This paper was based on master’s thesis of the corresponding author (Changwoo Yoo).

REFERENCES


Instruction for Authors

Manuscript Preparation
Beginning January 1, 2008, Journal of the Korean Society for Industrial and Applied Mathematics (J.KSIAM) uses a new paper format. The templates for the new paper format (MS word template and LaTeX template) can be downloaded from the KSIAM website (URL: http://www.ksiam.org/).

To keep the review time as short as possible, J.KSIAM requests all authors to submit their manuscripts online via the journal Online Submission System, except for very special circumstances that J.KSIAM recognizes.

All manuscripts submitted to J.KSIAM should follow the instructions for authors and use the template files. Manuscripts should be written in English and typed on A4-sized papers. The first page of the manuscript must include (1) the title which should be short, descriptive and informative, (2) the name(s) and address(es) of the author(s) along with e-mail address(es), (3) the abstract which should summarize the manuscript and be at least one complete sentence and at most 300 words. The manuscript should also include as a footnote the 2000 Mathematics Subject Classification by the American Mathematical Society, followed by a list of key words and phrases.

Final Manuscript Submission
To facilitate prompt turnarounds of author proofs and early placements of the accepted manuscripts in the future issues, J.KSIAM requests all authors of the accepted manuscripts to submit the electronic text/graphics file(s). The final manuscript must be submitted in either the LaTeX format or the MS word format. We recommend that all authors use the template files which can be downloaded at http://www.ksiam.org/jksiam. In the case that the corresponding author does not submit all relevant electronic text/graphics, further producing process regarding his/her manuscript would be stopped. The final manuscripts for publication should be sent to an appropriate managing editor via the journal Online Submission System.

Please note that the submission of the final manuscript implies that the author(s) are agreeing to be bound by the KSIAM Provisions on Copyright. Author check list and Copyright transfer can be found during the submission process via homepage.

Offprints
J. KSIAM supplies a final version of published paper in PDF form to a corresponding author. Authors are allowed to reproduce their own papers from the PDF file. Reprints of published paper can be also supplied at the approximate cost of production on request. For details, please contact one of managing editors.

Peer Review
Suitability for publication in J.KSIAM is judged by the editorial policy of the Editorial Board.

A submitted paper is allocated to one of the Managing Editors, who has full directorship for the reviewing process of the submission. The managing editor selects one of Associate Editors, and the selected Associate Editor chooses two qualified referees to review the paper. The Associate Editor him/herself may act as a referee. The managing Editor handles all correspondence with the author(s). The anonymity of the reviewers is always preserved. The reviewers should examine the paper and return it with their reports to the Associate Editor within 4 weeks from the date of the initial review request. The reviewers recommend acceptance, rejection or revision through a report. Based on the reports, the Associate Editor makes a recommendation to the managing Editor whether the paper should be accepted, rejected or needs to be returned to the author(s) for revision. Papers needing revision will be returned to the author(s), and the author(s) must submit a revised manuscript to the online system within one month from the date of the revision request; otherwise it will be assumed that the paper has been withdrawn. The revision is assigned for review to the managing Editor who handled its initial submission. The managing Editor sends the revised manuscript to the original Associate Editor to check if the manuscript is revised as suggested by the reviewers in the previous review. The Associate Editor makes a recommendation for the revision to the managing Editor within 3 weeks from the date of the Associate Editor assignment.
Final decision by the managing Editor is usually made within 3 months from the time of initial submission. The length of time from initial submission to final decision may vary, depending on the time spent for review and revision. A letter announcing a publication date is sent to author(s) after a manuscript has been accepted by the managing Editor.

Research and Publication Ethnic
Research published in Journal of KSIAM must have followed institutional, national and international guidelines. For the policies on the research and publication ethics that are not stated in these instructions, the Guidelines on Good Publication (http://www.publicationethics.org.uk/guidelines) can be applied.

Publication Charge
The publication fee is US $200, and extra charge could be necessary for color printings. The publication cost is subjected to change according to the society’s financial situation

Forms of Publication
- Original papers: this form of publication represents original research articles on research findings.
- Review article: this form does not cover original research but rather accumulate the results of many different articles on a particular topic
- Erratum/Revision/Retraction: these kinds of editorial notice may be published.

Format of References
The Vancouver style system is the preferred reference system for stipulations not otherwise described below: References should be listed at the end of the paper and conforming to current Journal style. Corresponding bracketed numbers are used to cite references in the text [1]. For multiple citations, separate reference numbers with commas [2, 3], or use a dash to show a range [4-7]. The DOI (Digital Object Identifier) should be incorporated in every reference for which it is available (see the last reference example).

- Journal

- Book

- Proceedings
Copyright Transfer Agreement

Copyright in the unpublished and original article, including the abstract, entitled -

__________________________________________________________ (Title of Article)

Submitted by the following author(s)

__________________________________________________________ (Names of Authors)

is hereby assigned and transferred to Korean Society for Industrial and Applied Mathematics (KSIAM) for the full term thereof throughout the world, subject to the term of this Agreement and to acceptance of the Article for publication in Journal of Korean Society for Industrial and Applied Mathematics.

KSIAM shall have the right to publish the Article in any medium or form, or by any means, now known or later developed.

KSIAM shall have the right to register copyright to the Article in its name as claimant whether separately or as part of the journal issue or other medium in which the Article is included. The Author(s) reserve all proprietary right other than copyright, such as patent rights. The Author(s) represent and warrant:

(1) that the Article is original with them;
(2) that the Article does not infringe any copyright or other rights in any other work, or violate any other rights;
(3) that the Author(s) own the copyright in the Article or are authorized to transfer it;
(4) that all copies of the Article the Author(s) make or authorize will include a proper notice of copyright in KSIAM’s name;

If each Author’s signature does not appear below, the signing Author(s) represent that they sign this Agreement as authorized agents for and on behalf of all the Authors, and that this Agreement and authorization is made on behalf of all the Authors.

Authors

__________________________________________________________

Signature ______________________ Date _________________
Author’s checklist

☐ This manuscript has never been submitted to or published in other journals.

☐ All citation references are correct and meet the submission rule.

☐ I will supply a manuscript in either TeX or MS-Word format as well as all artworks once the article is accepted.

☐ I filled out and signed the proper copyright transfer agreement.

☐ Research published in the Journal has been followed institutional, national, and international guidelines of ethnics.

☐ I checked that the manuscript contained all authors’ names, affiliation, full address, e-mail. And also contained an abstract and keywords.


Paper title________________________________________________________

Authors__________________________________________________________

Signature_________________________ Date__________________________
The Korean Society for Industrial and Applied Mathematics

Contents

VOL. 22   No. 1            March 2018

AN OPTIMAL BOOSTING ALGORITHM BASED ON NONLINEAR CONJUGATE
GRADIENT METHOD
JOOYEON CHOI, BORA JEONG, YESOM PARK, JIWON SEO,
CHOHONG MIN ................................................................. 1

ACCELERATION OF MACHINE LEARNING ALGORITHMS BY TCHEBYCHEV
ITERATION TECHNIQUE
MIKHAIL P. LEVIN ................................................................. 15

STABILITY OF DELAY-DISTRIBUTED HIV INFECTION MODELS WITH
MULTIPLE VIRAL PRODUCER CELLS
A. M. ELAIW, E. KH. ELNAHARY, A. M. SHEHATA, M. ABUL-EZ ........... 29

EFFECT OF PERTURBATION IN THE SOLUTION OF FRACTIONAL NEUTRAL
FUNCTIONAL DIFFERENTIAL EQUATIONS
MOHAMMED. S. ABDO, SATISH. K. PANCHAL ............................ 63

COMPARATIVE STUDY OF NUMERICAL ALGORITHMS FOR THE ARITHMETIC
ASIAN OPTION
JIAN WANG, JUNGYUP BAN, SEONGJIN LEE, CHANGWOO YOO ........... 75

ISSN 1226-9433(print)
ISSN 1229-0645(electronic)