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ANALYTIC TREATMENT FOR GENERALIZED $(m+1)$-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, a recently developed semi-analytic technique, so called the residual power series method, is generalized to process higher-dimensional linear and nonlinear partial differential equations. The obtained solution takes a form of an infinite power series which can, in turn, be expressed in a closed exact form. The results show that the proposed generalization is very effective, convenient and simple. This is carried out by dealing the $(m+1)$-dimensional Burger's equation.

1. INTRODUCTION

Over the last four years, a recent developed technique, namely the residual power series method (RPSM), for solving linear and nonlinear differential equations of integer and fractional orders have been proposed [1, 2]. In the current work, we improve this method to process linear and nonlinear partial differential equations (NPDEs) of higher dimensional and orders. Due to their broad variety of relevance, comparative studies, using the Adomian decomposition method [3, 4], homotopy perturbation method [5], variational iteration method [6], differential transform method and its reduction [7, 8], were discussed deeply to tackle NPDEs. In a special case, the higher order Burgers type equations were handled in [9, 10, 11] and references therein.

2. THE GRPSM

Recently, the RPSM has been completely improved by the author to treat NPDEs. The new modification (shortly GRPSM) is successfully applied to the 1D shock-wave equation [12], the mixed hyperbolic-elliptic system of conservation laws [13], the nonlinear diffusion equations [14], and the higher dimensional telegraph equation [15].

The GRPSM method based on constructing term-by-term series solution in a form of Taylor series expansion without need to linearization, discretization, perturbation or unrealistic assumptions. In what follow, description of this Algorithm for solving higher dimensional

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NPDEs will be introduced. For this objective, consider the formal $(m+1)$-dimensional, nth-order NPDE

$$\partial_t^n u\left(x_i,t\right)=F\left(t,x_i;u,\partial_t u,\partial_{x_i}u,\ldots,\partial_t^{n-1}u,\partial_{x_i}^{n-1}u\right),\left(x_i,t\right)\in R^m\times[0,T] \tag{2.1}$$

for $i=1,\ldots,m$, where F is assumed to be sufficiently smooth, and $\partial_t^q u$ represents the qth derivative of the analytic function $u\left(x_i,t\right)$, and in the same way for other independent variables. The GRPSM [12, 13, 14, 15] assumes the solution $u\left(x_i,t\right)$, of Eq.(2.1), in the following form of power series

$$u\left(x_i,t\right)=\sum_{j}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}a_{i_1,\ldots,i_m,j}\left(\prod_{k=1}^{m}\left(x_k-x_{k_0}\right)^{i_k}\right)\left(t-t_0\right)^j \tag{2.2}$$

Using the symbolic notation, Eq.(2.2) can be rewritten in more compact form as

$$u\left(\tilde{\mathbf{x}},t\right)=\sum_{j=0}^{\infty}\xi_j\left(\tilde{\mathbf{x}}\right)\left(t-t_0\right)^j, \tag{2.3}$$

where $\xi_j\left(\tilde{\mathbf{x}}\right)=\sum\limits_{i_1=0}^{\infty}\cdots\sum\limits_{i_m=0}^{\infty}a_{i_1,\ldots,i_m,j}\left(\prod\limits_{p=1}^{m}\left(x_k-x_{k_0}\right)^{i_p}\right)$, and $\tilde{\mathbf{x}}=\left(x_1,\ldots,x_m\right)$.

Subject to the initial data

$$u\left(\tilde{\mathbf{x}},t_0\right)=u_0\left(\tilde{\mathbf{x}}\right),\partial_t u\left(\tilde{\mathbf{x}},t_0\right)=u_1\left(\tilde{\mathbf{x}}\right),\ldots,\partial_t^{n-1}u\left(\tilde{\mathbf{x}},t_0\right)=u_{n-1}\left(\tilde{\mathbf{x}}\right), \tag{2.4}$$

the initial approximation of $u\left(\tilde{\mathbf{x}},t\right)$ will be

$$u_{n-1}\left(\tilde{\mathbf{x}},t\right)=\sum_{j=0}^{n-1}\frac{1}{j!}\partial_t^j\left(u\left(\tilde{\mathbf{x}},t_0\right)\right)\left(t-t_0\right)^j=\sum_{j=0}^{n-1}\xi_j\left(\tilde{\mathbf{x}}\right)\left(t-t_0\right)^j. \tag{2.5}$$

The kth-order approximate solution is defined by the truncated series

$$u_k\left(\tilde{\mathbf{x}},t\right)=u_{n-1}\left(\tilde{\mathbf{x}},t\right)+\sum_{j=n}^{k}\xi_j\left(\tilde{\mathbf{x}}\right)\left(t-t_0\right)^j,k\geq n, \tag{2.6}$$

subject to the well-defined kth- residual function, which represents the basic idea of the GRPSM, given by

$$\mathrm{Res}_k\left(\tilde{\mathbf{x}},t\right)=\partial_t^k u_k\left(\tilde{\mathbf{x}},t\right)-\partial_t^{k-n}F\left(t,\tilde{\mathbf{x}};u,\partial_t u_k,\ldots,\partial_t^{n-1}u_k,\partial_{\tilde{\mathbf{x}}}u,\partial_{\tilde{\mathbf{x}}}^2u,\ldots\right), \tag{2.7}$$

where,

$$\lim_{t\to t_0}\mathrm{Res}_k\left(\tilde{\mathbf{x}},t\right)=0. \tag{2.8}$$

The exact analytic solution of the initial-value problem, Eqs.(2.1)-(2.4), is given by

$$u\left(\tilde{\mathbf{x}},t\right)=\lim_{k\to\infty}u_k\left(\tilde{\mathbf{x}},t\right), \tag{2.9}$$

provided that the series has exact closed form.

3. Convergence Analysis

Following the Taylor's Theorem, an analytic-convergence study of the GRPSM is discussed in this part. The presented scheme, under considerations mentioned in the previous section, approaches the exact analytic solution as more and more terms found.

Theorem 3.1. *If F is assumed to be analytic on an open interval I containing t_0, then the residual function $\mathrm{Res}_k\left(\tilde{\mathbf{x}}, t\right)$ Eq.*(2.7) *vanishes as k approaches the infinity.*

Proof. It is obvious since F is assumed to be analytic as well as for $u\left(\tilde{\mathbf{x}}, t\right)$. ☐

Lemma 3.2. *Suppose that $u\left(\tilde{\mathbf{x}}, t\right) = \sum_{j=0}^{\infty} \xi_j\left(\tilde{\mathbf{x}}\right)\left(t-t_0\right)^j$, then*

$$\left[\partial_t^q u\left(\tilde{\mathbf{x}}, t\right)\right]_{t=t_0} = q!\xi_q\left(\tilde{\mathbf{x}}\right), q \in N. \tag{3.1}$$

Proof. By the continuity of the $\partial_t^q u\left(\tilde{\mathbf{x}}, t\right)$ at $t = t_0$, we get

$$\begin{aligned}
&\left[\partial_t^q u\left(\tilde{\mathbf{x}}, t\right)\right]_{t=t_0} = \lim_{t\to t_0} \partial_t^q u\left(x, t\right) \\
&= \lim_{t\to t_0} \partial_t^q \left(\sum_{i=0}^{\infty} \xi_j\left(\tilde{\mathbf{x}}\right)\left(t-t_0\right)^j\right) \\
&= \lim_{t\to t_0} \left(\sum_{i=0}^{\infty} \xi_j\left(\tilde{\mathbf{x}}\right)\partial_t^q\left(t-t_0\right)^j\right) \\
&= \lim_{t\to t_0} \left(\sum_{i=0}^{\infty} \frac{(q+j)!}{j!}\xi_{q+j}\left(\tilde{\mathbf{x}}\right)\left(t-t_0\right)^j\right) \\
&= \sum_{i=0}^{\infty} \left(\frac{(q+j)!}{j!}\xi_{q+j}\left(\tilde{\mathbf{x}}\right)\lim_{t\to t_0}\left(t-t_0\right)^j\right) \\
&= q!\xi_q\left(\tilde{\mathbf{x}}\right).
\end{aligned}$$

☐

Theorem 3.3. *The approximate truncated series solution $u_k\left(\tilde{\mathbf{x}}, t\right)$ given in Eq.*(2.6) *and obtained by applying the GRPSM for solving the initial-value problem Eqs.*(2.1)-(2.4)*, is the kth Taylor polynomial of $u\left(\tilde{\mathbf{x}}, t\right)$ about $t = t_0$. In general, as $k \to \infty$, the series solution in Eq.*(2.9) *concise the Taylor series expansion of $u\left(\tilde{\mathbf{x}}, t\right)$ centered at $t = t_0$.*

Proof. For $k < n$, it is clear from Eq.(2.5). For $k \geq n$, it suffices to prove that

$$\left[\partial_t^k u\left(\tilde{\mathbf{x}}, t\right)\right]_{t=t_0} = \lim_{t\to t_0} \partial_t^{k-n} F\left(t, \tilde{\mathbf{x}}; u, \partial_t u_k, \ldots, \partial_t^{n-1} u_k, \partial_{\tilde{\mathbf{x}}} u, \partial_{\tilde{\mathbf{x}}}^2 u, \ldots\right).$$

Applying Eq.(2.8) to the kth-order approximate solution given in Eq.(2.6), and using Theorem 3.1, we get

$$\begin{aligned}
0 &= \lim_{k\to\infty} \operatorname{Res}_k(\tilde{\mathbf{x}},t) \\
&= \lim_{t\to t_0}\left(\partial_t^k u_k(\tilde{\mathbf{x}},t) - \partial_t^{k-n} F\left(t,\tilde{\mathbf{x}};u,\partial_t u_k,\ldots,\partial_t^{n-1}u_k,\partial_{\tilde{\mathbf{x}}}u,\partial_{\tilde{\mathbf{x}}}^2 u,\ldots\right)\right) \\
&= \lim_{t\to t_0}\partial_t^k u_k(\tilde{\mathbf{x}},t) - \lim_{t\to t_0}\partial_t^{k-n} F\left(t,\tilde{\mathbf{x}};u,\partial_t u_k,\ldots,\partial_t^{n-1}u_k,\partial_{\tilde{\mathbf{x}}}u,\partial_{\tilde{\mathbf{x}}}^2 u,\ldots\right)
\end{aligned}$$

As a result of Lemma 3.2,

$$0 = q!\xi_q(\tilde{\mathbf{x}}) - \lim_{t\to t_0}\partial_t^{k-n} F\left(t,\tilde{\mathbf{x}};u,\partial_t u_k,\ldots,\partial_t^{n-1}u_k,\partial_{\tilde{\mathbf{x}}}u,\partial_{\tilde{\mathbf{x}}}^2 u,\ldots\right).$$

Which completes the proof. □

Corollary 3.4. *Suppose that the truncated series* $u_k(\tilde{\mathbf{x}},t)$ *Eq.*(2.6) *is used as an approximation to the solution* $u(\tilde{\mathbf{x}},t)$ *of problem Eqs.*(2.1)-(2.4) *on the strip*

$$S = \{(\tilde{\mathbf{x}},t) : \tilde{\mathbf{x}} \in R^m, |t-t_0| < \rho\},$$

then numbers $\eta(t)$*, satisfies* $|\eta(t) - t_0| \le \rho$*, and* $\mu_k > 0$ *exist with*

$$|u(\tilde{\mathbf{x}},t) - u_k(\tilde{\mathbf{x}},t)| \le \frac{\mu_k}{(k+1)!}\rho^{k+1}. \tag{3.2}$$

Proof. Theorem 3.1 implies that

$$u(\tilde{\mathbf{x}},t) - u_k(\tilde{\mathbf{x}},t) = \sum_{j=k+1}^{\infty} \frac{1}{j!}\partial_t^j(u(\tilde{\mathbf{x}},t))(t-t_0)^j.$$

According to the proof of Taylor's Theorem [16], a number $\eta(t) \in (t_0-\rho, t_0+\rho)$ exists with

$$u(\tilde{\mathbf{x}},t) - u_k(\tilde{\mathbf{x}},t) = \frac{\partial_t^{k+1}u(\tilde{\mathbf{x}},\eta(t))}{(k+1)!}(t-t_0)^{k+1}.$$

Since the $(k+1)st$-derivative of the analytic function $u(x,t)$ with respect to t is bounded on S, a number μ_k also exists with $\left|\partial_t^{k+1}u(\tilde{\mathbf{x}},t)\right| \le \mu_k$ for all $t \in [t_0-\rho, t_0+\rho]$. Hence error bound in Eq.(3.2) is obtained. □

While the values of t are of distant from the center t_0 for some positive real number ρ, the error bound increases and series solution may be diverge. In this case, a multistage residual power series algorithm, as was done to the Adomianach decomposition method while treating nonlinear oscillators [17], will reduce the error bound according to required tolerance.

Corollary 3.5. *The GRPSM results the exact analytic solution* $u(\tilde{\mathbf{x}},t)$ *if it is a polynomial of* t*.*

4. Numerical Illustration

To illustrate the technique discussed in Sections 2, we consider the $(m+1)$-dimensional nonlinear Burgers' equation [18, 19]

$$\partial_t u(\tilde{\mathbf{x}}, t) = u(\tilde{\mathbf{x}}, t)\, \partial_{x_1} u(\tilde{\mathbf{x}}, t) + \sum_{j=1}^{m} \partial_{x_j}^2 u(\tilde{\mathbf{x}}, t), |t| < 1, \tag{4.1}$$

subject to

$$u(\tilde{\mathbf{x}}, 0) = \sum_{j=1}^{m} x_j. \tag{4.2}$$

Buregr's equation Eq.(4.1), which is also known as Richard's equation, is used in the study of cellular automata and interacting particle systems. It describes the flow pattern of the particle in a lattice fluid past an impenetrable obstacle; it can be also used as a model to describe the water flow in soils.

Applying the generalized residual power series mechanism to suggested problem, the initial approximation is $u_0(\tilde{\mathbf{x}}, t) = \sum\limits_{j=1}^{m} x_j$, and the kth-order approximate solution has the form

$$u_k(\tilde{\mathbf{x}}, t) = u_0(\tilde{\mathbf{x}}, t) + \sum_{j=n}^{k} \xi_j(\tilde{\mathbf{x}})\, t^j, k \geq 1, \tag{4.3}$$

which satisfies

$$\lim_{t \to 0} \left(\partial_t^k u_k(\tilde{\mathbf{x}}, t) - \partial_t^{k-1} \left(u_k(\tilde{\mathbf{x}}, t)\, \partial_{x_1} u_k(\tilde{\mathbf{x}}, t) + \sum_{j=1}^{m} \partial_{x_j}^2 u_k(\tilde{\mathbf{x}}, t) \right) \right) = 0. \tag{4.4}$$

For $k = 1$, we have $u_1(\tilde{\mathbf{x}}, t) = u_0(\tilde{\mathbf{x}}, t) + \xi_1(\tilde{\mathbf{x}})\, t$ and

$$\lim_{t \to 0} \left(\partial_t u_1(\tilde{\mathbf{x}}, t) - \left(u_1(\tilde{\mathbf{x}}, t)\, \partial_{x_1} u_1(\tilde{\mathbf{x}}, t) + \sum_{j=1}^{m} \partial_{x_j}^2 u_1(\tilde{\mathbf{x}}, t) \right) \right) = 0. \tag{4.5}$$

Hence we get, $\xi_1(\tilde{\mathbf{x}}) - \sum\limits_{j=1}^{m} x_j = 0$ and therefore $\xi_1(\tilde{\mathbf{x}}) = \sum\limits_{j=1}^{m} x_j$. Repeating this procedure for $k = 2, 3, \ldots$, we obtain that $\xi_2(\tilde{\mathbf{x}}) = \xi_3(\tilde{\mathbf{x}}) = \cdots = \sum\limits_{j=1}^{m} x_j$.

As $k \to \infty$, the solution takes the form

$$u(\tilde{\mathbf{x}}, t) = \sum_{j=1}^{\infty} x_j t^j = \frac{\sum\limits_{j=1}^{\infty} x_j}{1 - t}. \tag{4.6}$$

The series solution leads to the exact solution obtained by Taylor's expansion.

5. Conclusions

In this work, we have improved an analytic solution procedure, called the generalized residual power series method, for solving higher dimensional partial differential equations. The results validate the efficiency and reliability of the aforesaid technique that are achieved by handling the $(m+1)$-dimensional Burger's equation. The method is a powerful mathematical tool for solving a wide range of problems arising in engineering and sciences.

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