FINITE DIFFERENCE SCHEME FOR SINGULARLY PERTURBED SYSTEM OF
DELAY DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY
CONDITIONS

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ABSTRACT. In this paper we consider a class of singularly perturbed system of delay differential
equations of convection diffusion type with integral boundary conditions. A finite difference
scheme on an appropriate piecewise Shishkin type mesh is suggested to solve the problem. We
prove that the method is of almost first order convergent. An error estimate is derived in the
discrete maximum norm. Numerical experiments support our theoretical results.

1. INTRODUCTION

In natural or technological control problems, a controller monitors the state of the system,
and makes adjustments to the system based on its observations. Since these adjustments can
never be made instantaneously, a delay arises between the observation and the control action.
This kind of systems are governed by differential equations with delay arguments. A sub-
class of these equations consists of singularly perturbed differential equations with delay are
typically characterized by the presence of small positive parameter \( \varepsilon \) multiplying some or all
of the highest derivatives present in the differential equation. These equations arise in math-
ematical models of biological science and engineering. Differential equations with integral
boundary conditions have plenty of applications. A Parabolic equation with nonlocal bound-
ary conditions arising from electro chemistry is well studied by Choi and Chan [7]. In [8],
Day have discussed Parabolic equations and thermodynamics. Cannon [6] have worked for
the solution of the heat equation subject to the specification of energy, etc. The authors of
[4, 9, 14] have proved that the problem of differential equations with integral boundary condi-
tions is well posed. The authors of [1, 5, 16, 20, 21] have developed various numerical schemes
on uniform meshes for singularly perturbed first and second order differential equations with
integral boundary conditions. The standard numerical methods used for solving singularly per-
turbed differential equation are some time ill posed and fail to give analytical solution when
the perturbation parameter \( \varepsilon \) is small. Therefore, it is necessary to improve suitable numerical

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methods which are uniformly convergent to solve this type of differential equations. Many authors have worked on singularly perturbed differential equations with small and large delay using uniformly convergent numerical methods. In [13], Lange and Miura have discussed singularly perturbed linear second order differential-difference equations with small delay. In [10, 11, 12, 17, 18, 19] finite difference and finite element methods are proposed to solve these kind of equations with large and small shifts.

In the present paper, motivated by the works of [1, 2, 3], we analyze a fitted finite difference scheme on a piecewise uniform mesh for the numerical solution of a class of second order singularly perturbed system of delay differential equations with integral boundary conditions. The present paper is arranged as follows. Statement of the problem is given in section 2. In section 3, maximum principle, stability result and appropriate bounds for the derivatives of the solution of the problem are presented. Section 4 describes the numerical method. Error analysis for approximate solution is given section 5. Numerical results are given in section 6. Section 7 includes the conclusion part. Throughout our analysis we use the following notations: \( \Omega = [0, 2] \), \( \Omega = (0, 2) \), \( \Omega_1 = (0, 1) \), \( \Omega_2 = (1, 2) \), \( \Omega^x = \Omega_1 \cup \Omega_2 \). \( \Omega^{2N} = \{0, 1, 2, \ldots, 2N\} \), \( \Omega_1^{2N} = \{1, 2, \ldots, N - 1\} \), \( \Omega_2^{2N} = \{N + 1, \ldots, 2N - 1\} \). \( C, C_1 \) are generic positive constants that are independent of parameter \( \varepsilon \) and \( 2N \) mesh points. We assume that \( \varepsilon \leq CN^{-1} \). The supremum norm used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem is \( \|u\|_\Omega = \sup_{x \in \Omega} |u(x)| \).

### 2. Statement of the Problem

Find \( \bar{u} = (u_1, u_2)^T \), \( u_1, u_2 \in Y = C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2) \) such that

\[
\begin{aligned}
&-\varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) + c_{11}(x)u_1(x-1) \\
&+ c_{12}(x)u_2(x-1) = f_1(x), \ x \in \Omega_1 \cup \Omega_2, \\
&-\varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) + c_{21}(x)u_1(x-1) \\
&+ c_{22}(x)u_2(x-1) = f_2(x), \ x \in \Omega_1 \cup \Omega_2, \\
&u_1(x) = \phi_1(x), \ x \in [-1, 0], \ K_1u_1(2) = u_1(2) - \varepsilon \int_0^2 g_1(x)u_1(x)dx = l_1, \\
&u_2(x) = \phi_2(x), \ x \in [-1, 0], \ K_2u_2(2) = u_2(2) - \varepsilon \int_0^2 g_2(x)u_2(x)dx = l_2.
\end{aligned}
\]

(2.1)

where \( 0 < \varepsilon \ll 1 \) is a small positive parameter, the functions \( a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), c_{11}(x), c_{12}(x), c_{21}(x), c_{22}(x), f_1(x), f_2(x) \) are sufficiently smooth on \( \bar{\Omega} = [0, 2] \) and satisfy the following assumptions.

\[
\begin{aligned}
a_1(x) &\geq a_1 > 0, \ a_2(x) \geq a_2 > 0, \ 0 < \alpha < \min\{a_1, a_2\}, \ x \in \bar{\Omega} \\
b_{11}(x) &\geq 0, \ b_{12}(x) \leq 0, \ b_{21}(x) \leq 0, \ b_{22}(x) \geq 0, \\
b_{11}(x) + b_{12}(x) &\geq \beta_1 \geq 0, \ b_{21}(x) + b_{22}(x) \geq \beta_2 \geq 0, \\
c_{11}(x) &\leq 0, \ c_{12}(x) \leq 0, \ c_{21}(x) \leq 0, \ c_{22}(x) \geq 0, \\
c_{11}(x) + c_{12}(x) &\geq \gamma_1 \geq 0, \ c_{21}(x) + c_{22}(x) \geq \gamma_2 \geq 0, \\
g_i &\text{ are nonnegative and } 1 - \int_0^2 g_i(x)dx > 0, \ i = 1, 2.
\end{aligned}
\]
Let then there exists at least one point

The problem (2.1) can be rewritten as,

\[
\mathcal{L}_1 \tilde{u}(x) = \begin{cases} 
-\varepsilon u''_1(x) + a_1(x)u'_1(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) \\
= f_1(x) - c_{11}(x)\phi_1(x - 1) - c_{12}(x)\phi_2(x - 1), & x \in \Omega_1, \\
-\varepsilon u''_1(x) + a_1(x)u'_1(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) + c_{11}(x)u_1(x - 1) \\
+ c_{12}(x)u_2(x - 1) = f_1(x), & x \in \Omega_2,
\end{cases}
\]

(2.2)

\[
\mathcal{L}_2 \tilde{u}(x) = \begin{cases} 
-\varepsilon u''_2(x) + a_2(x)u'_2(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) \\
= f_2(x) - c_{21}(x)\phi_1(x - 1) - c_{22}(x)\phi_2(x - 1), & x \in \Omega_1, \\
-\varepsilon u''_2(x) + a_2(x)u'_2(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) + c_{21}(x)u_1(x - 1) \\
+ c_{22}(x)u_2(x - 1) = f_2(x), & x \in \Omega_2,
\end{cases}
\]

(2.3)

\[
\begin{align*}
&u_1(0) = \phi_1(0), \quad u_1(1-) = u_1(1+), \quad u'_1(1-) = u'_1(1+), \\
&K_1 u_1(2) = u_1(2) - \varepsilon \int_0^1 g_1(x)u_1(x)dx = l_1, \\
&u_2(0) = \phi_2(0), \quad u_2(1-) = u_2(1+), \quad u'_2(1-) = u'_2(1+), \\
&K_2 u_2(2) = u_2(2) - \varepsilon \int_0^1 g_2(x)u_2(x)dx = l_2,
\end{align*}
\]

(2.4)

3. THE CONTINUOUS PROBLEM

Theorem 3.1. (Maximum Principle) Let \( \bar{w} = (w_1, w_2)^T, w_1, w_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega_1 \cup \Omega_2) \) be any function satisfying \( w_i(0) \geq 0, \ i = 1, 2, K_i w_i(2) \geq 0, \ i = 1, 2, \mathcal{L}_i \bar{w}(x) \geq 0, \ \forall x \in \Omega_1 \cup \Omega_2, \ i = 1, 2 \) and \( w'_i(1+) - w'_i(1-) = [w'_i(1)] \leq 0, \ i = 1, 2. \) Then \( w_i(x) \geq 0, \ \forall x \in \Omega, \ i = 1, 2. \)

Proof. Let \( \bar{s} = (s_1, s_2)^T \) be a function defined by

\[
s_i(x) = \begin{cases} 
\frac{1}{2} + \frac{x}{2}, & x \in [0, 1] \\
\frac{1}{2} + \frac{x}{4}, & x \in [1, 2],
\end{cases}
\]

(3.1)

It is easy to see that, \( s_i(x) > 0, \ \forall x \in \bar{\Omega}, \ i = 1, 2, \mathcal{L}_i \bar{s}(x) > 0, \ \forall x \in \Omega_1 \cup \Omega_2, \ i = 1, 2 \) and \( [s'_i](1) < 0, \ i = 1, 2. \) Let

\[
\mu = \max \left\{ \max_{\Omega} \left\{ \frac{-w_1(x)}{s_1(x)} \right\}, \max_{\Omega} \left\{ \frac{-w_2(x)}{s_2(x)} \right\} \right\}.
\]

Then there exists at least one point \( x_0 \in \bar{\Omega} \) such that \( w_1(x_0) + \mu s_1(x_0) = 0 \) or \( w_2(x_0) + \mu s_2(x_0) = 0 \) or both and \( w_i(x) + \mu s_i(x) \geq 0, \ \forall x \in \bar{\Omega}, \ i = 1, 2. \) With out loss of generality we assume that \( w_1(x_0) + \mu s_1(x_0) = 0. \) Therefore the function \( (w_1 + \mu s_1) \) attains its minimum at \( x = x_0. \) Suppose the theorem does not hold true, then \( \mu > 0. \)

Case(i): \( x_0 \in \Omega_1 \)

\[
0 < \mathcal{L}_1(\bar{w} + \mu \bar{s})(x_0) = -\varepsilon (w_1 + \mu s_1)''(x_0) + a_1(x_0)(w_1 + \mu s_1)'(x_0) \\
+ b_{11}(x_0)(w_1 + \mu s_1)(x_0) + b_{12}(x_0)(w_2 + \mu s_2)(x_0) \leq 0.
\]
Define Using Corollary 3.2 and applying arguments as given in [15] this lemma gets proved.

Proof.

Case (ii): $x_0 = 1$

$$0 \leq [(\bar{w} + \mu \bar{s})'](1) = [\bar{w}'](1) + \mu [\bar{s}'](1) < 0$$

Case (iii): $x_0 \in \Omega_2$

$$0 < L_1(\bar{w} + \mu \bar{s})(x_0) = -\varepsilon (w_1 + \mu s_1)'(x_0) + a_1(x_0)(w_1 + \mu s_1)'(x_0)$$
$$+ b_{11}(x_0)(w_1 + \mu s_1)(x_0) + b_{12}(x_0)(w_2 + \mu s_2)(x_0)$$
$$+ c_{11}(x_0)(w_1 + \mu s_1)(x_0 - 1) + c_{12}(x_0)(w_2 + \mu s_2)(x_0 - 1) \leq 0.$$

Case (iv): $x_0 = 2$

$$0 < K_1(\bar{w} + \mu \bar{s})(2) = (w_1 + \mu s_1)(2) - \varepsilon \int_0^2 g_1(x)(w_1 + \mu s_1)(x)dx \leq 0.$$

Observe that in all the four cases we arrived a contradiction. Therefore $\mu > 0$ is not possible. This shows that $w_1(x) \geq 0$. Similarly $w_2(x) \geq 0$.

Corollary 3.2. (Stability Result) Let $\bar{u} = (u_1, u_2)^T, u_1, u_2 \in Y$ be any function. Then,

$$|u_i(x)| \leq C \max \left\{ \max_{j=1,2} \{|u_j(0)|\}, \max_{j=1,2} \{|K_j u_j(2)|\}, \max_{j=1,2} \{\sup_{x \in \Omega_1 \cup \Omega_2} |L_j \bar{u}(x)|\} \right\},$$

$$\forall x \in \Omega, i = 1, 2.$$

Proof. Define $\psi^\pm(x) = (\psi_1^\pm(x), \psi_2^\pm(x))^T$, $x \in \Omega$, where

$$\psi_i^\pm(x) = CMS_i(x) \pm u_i(x), x \in \Omega, i = 1, 2,$$

$$M = \max_{j=1,2} \{\max_{j=1,2} \{|u_j(0)|\}, \max_{j=1,2} \{|K_j u_j(2)|\}, \max_{j=1,2} \{\sup_{x \in \Omega_1 \cup \Omega_2} |L_j \bar{u}(x)|\} \}$$

and $\bar{s}$ is defined by (3.1). Using the above barrier functions $\psi^\pm(x)$ and Theorem 3.1, this corollary can be proved easily.

Bounds for the derivatives of $\bar{u}(x)$ are given in the following lemma.

Lemma 3.3. Let $\bar{u}(x)$ be the solution of (1). Then, for $1 \leq k \leq 3$,

$$|u_j^{(k)}(x)| \leq C\varepsilon^{-k}, j = 1, 2.$$


The uniform error estimates can be derived using the sharper bounds on the derivatives of the solution $\bar{u}(x)$. To get sharper bounds we write the analytical solution $\bar{u}(x)$ in the form $\bar{u}(x) = \bar{v}(x) + \bar{w}(x)$, where $\bar{v}(x) = (v_1(x), v_2(x))^T$ and $\bar{w}(x) = (w_1(x), w_2(x))^T$. The
regular component $\bar{v}(x)$ can be written as $\bar{v}(x) = \bar{v}_0(x) + \varepsilon \bar{v}_1(x) + \varepsilon^2 \bar{v}_2(x)$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$, and $\bar{v}_2 = (v_{21}, v_{22})^T$ satisfy the following equations:

$$\begin{cases} a_1(x) v'_{01}(x) + b_{11}(x) v_{01}(x) + b_{12}(x) v_{02}(x) + c_{11}(x) v_{01}(x-1) \\
+ c_{12}(x) v_{02}(x-1) = f_1(x), \\
\end{cases}$$

$$\begin{cases} a_2(x) v'_{02}(x) + b_{21}(x) v_{01}(x) + b_{22}(x) v_{02}(x) + c_{21}(x) v_{01}(x-1) \\
+ c_{22}(x) v_{02}(x-1) = f_2(x), \\
\end{cases}$$

$$v_{01}(x) = u_1(x), \quad v_{02}(x) = u_2(x), \quad x \in [-1, 0]$$

Thus the regular component $\bar{v}(x)$ is the solution of

$$\begin{cases} L_1 v(x) = f_1(x), \quad v(x) = u_1(x), \quad x \in [-1, 0], \\
K_1 v_1(2) = K_1 v_{01}(2) + \varepsilon K_1 v_{11}(2) + \varepsilon^2 K_1 v_{21}(2), \\
L_2 v(x) = f_2(x), \quad v(x) = u_2(x), \quad x \in [-1, 0], \\
K_2 v_2(2) = K_2 v_{02}(2) + \varepsilon K_2 v_{12}(2) + \varepsilon^2 K_2 v_{22}(2), \\
\bar{v}(1) = \bar{v}_0(1) + \varepsilon \bar{v}_1(1) + \varepsilon^2 \bar{v}_2(1). \\
\end{cases}$$

and $\bar{w}(x)$ is the solution of

$$\begin{cases} L_1 w(x) = 0, \quad w(x) = 0, \quad x \in [-1, 0], \\
K_1 w_1(2) = K_1 u_1(2) - K_1 v_1(2), \\
L_2 w(x) = 0, \quad w(x) = 0, \quad x \in [-1, 0], \\
K_2 w_2(2) = K_2 u_2(2) - K_2 v_2(2), \\
[\bar{w}'](1) = -[\bar{v}'](1). \\
\end{cases}$$

We further decompose $\bar{w}(x)$ as $\bar{w}(x) = \bar{w}_B(x) + \bar{w}_I(x)$, where the function $\bar{w}_B(x)$ is boundary layer component and $\bar{w}_I(x)$ is interior layer component, which are the solution of the following problems respectively:

Find $\bar{w}_B(x) \in X$ such that

$$\begin{cases} L_1 w_{B1}(x) = 0, \quad w_{B1}(x) = 0, \quad x \in [-1, 0], \quad K_1 w_{B1}(2) = K_1 u_1(2) - K_1 v_1(2), \\
L_2 w_{B2}(x) = 0, \quad w_{B2}(x) = 0, \quad x \in [-1, 0], \quad K_2 w_{B2}(2) = K_2 u_2(2) - K_2 v_2(2). \\
\end{cases}$$

$$\begin{cases} L_1 w_{I1}(x) = 0, \quad w_{I1}(x) = 0, \quad x \in [-1, 0], \quad K_1 w_{I1}(2) = K_1 u_1(2) - K_1 v_1(2), \\
L_2 w_{I2}(x) = 0, \quad w_{I2}(x) = 0, \quad x \in [-1, 0], \quad K_2 w_{I2}(2) = K_2 u_2(2) - K_2 v_2(2). \\
\end{cases}$$
Find \( w_I(x) \in C^0(\Omega) \cap C^2(\Omega^*) \) such that
\[
\begin{align*}
\mathcal{L}_1 w_{I1}(x) &= 0, \quad w_{I1}(x) = 0, \quad x \in [-1, 0], \quad [w'_{I1}](1) = -[v'](1), \quad \mathcal{K}w_{I1}(2) = 0, \\
\mathcal{L}_2 w_{I2}(x) &= 0, \quad w_{I2}(x) = 0, \quad x \in [-1, 0], \quad [w'_{I2}](1) = -[v'](1), \quad \mathcal{K}w_{I2}(2) = 0.
\end{align*}
\] (3.8)

**Theorem 3.4.** Let \( \bar{u}(x) \) be the solution of the problem (1) and \( \bar{v}_0(x) \) be its reduced problem solution defined in (6). Then
\[
|u_j(x) - v_{0j}(x)| \leq C(\varepsilon + e^{-\alpha(2-x)/\varepsilon}), \quad x \in \Omega, \quad j = 1, 2.
\]

**Proof.** Consider the barrier functions \( \bar{v}^\pm(x) = (\psi_1^\pm(x), \psi_2^\pm(x))^T \), where
\[
\psi_1^\pm(x) = C(\varepsilon s_j(x) + e^{-\alpha(2-x)/\varepsilon}) \pm (u_j(x) - v_{0j}(x)), \quad x \in \Omega, \quad j = 1, 2.
\]

Note that \( \psi_1^\pm(x) \in C^0(\Omega) \cap C^2(\Omega) \). It is easy to see that, \( \psi_1^\pm(0) \geq 0 \) for a suitable choice of \( C > 0 \). Further
\[
\mathcal{K}_1\psi_1^\pm(2) = \psi_1^\pm(2) - \varepsilon \int_0^2 g_1(x)\psi_1^\pm(x)dx
\]
\[
\geq C(2\varepsilon + 1) - 2C\varepsilon \int_0^2 g_1(x)dx - C\varepsilon \int_0^2 g_1(x)dx \pm \mathcal{K}_1(u_1 - v_{01})(2) \geq 0
\]
for a suitable choice of \( C > 0 \).

Let \( x \in (0, 1) \). Then
\[
\mathcal{L}_1\psi_1^\pm(x) = C\varepsilon[a_1(x)s'_1(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x)]
\]
\[
+ C\varepsilon [\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x) + b_{12}(x)]e^{-\alpha(1-x)/\varepsilon}
\]
\[
\pm \mathcal{L}_1(\bar{u} - \bar{v}_0)(x) \geq 0,
\]
by a proper choice of \( C > 0 \). Let \( x \in \Omega_2 \). Then
\[
\mathcal{L}_1\psi_1^\pm(x) = C\left[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x) + (c_{11}(x)s_1(x) + c_{12}(x)s_2(x))
\exp(-\frac{\alpha}{\varepsilon}) \exp(-\frac{\alpha(2-x)}{\varepsilon}) + \varepsilon(a_1(x) + b_{11}(x)s_1(x) + b_{12}(x)s_2(x)
\]
\[
+ (c_{11}(x)s_1(x-1) + c_{12}(x)s_2(x-1)) \right] \pm \varepsilon v_{01}''(x),
\]
\[
\geq 0.
\]
for a suitable choice of \( C > 0 \).

Similarly one can prove that \( \mathcal{L}_2\psi_1^\pm(x) \geq 0 \) and \( \mathcal{K}_2\psi_1^\pm(2) \geq 0 \). Then by maximum principle we have \( \psi_1^\pm(x) \geq 0, \quad x \in \Omega, \quad i = 1, 2. \)

**Lemma 3.5.** The regular component \( \bar{v}(x) \) and the singular component \( \bar{w}(x) \) of the solution \( \bar{u}(x) \) satisfy the following bounds,
\[
||w_j^k(x)||_{\Omega^*} \leq C(1 + \varepsilon^{2-k}), \quad \text{for } k = 0, 1, 2, 3
\] (3.9)
\begin{align}
|w_{B_j}^k(x)| &\leq C\varepsilon^{-k}\exp\left(-\frac{\alpha(2-x)}{\varepsilon}\right), \quad x \in \Omega^*, \ k = 0, 1, 2, 3 \quad (3.10)
\\
|w_{f_j}^k(x)| &\leq C\begin{cases}
\varepsilon^{1-k} \exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right), &x \in \Omega_1, \\
\varepsilon^{1-k}, &x \in \Omega_2,
\end{cases} \quad k = 0, 1, 2, 3, \quad (3.11)
\end{align}

where \( j = 1, 2 \)

**Proof.** Integrating (3.2) and (3.4) and using the stability result, the inequalities (3.9) can be proved easily. To prove the inequalities (3.11), consider the barrier functions

\[ \Phi_j^\pm(x) = C_1(\exp(-\frac{\alpha(2-x)}{\varepsilon})) \pm w_{B_j}(x), \ x \in \Omega, \ j = 1, 2 \]

It is easy to see that \( \Phi_j^\pm(0) \geq 0 \).

Further,

\[ K_1\Phi_1^+(2) = \Phi_1^+(2) - \varepsilon \int_0^2 g_1(x)\Phi_1^+(x)dx \]

\[ = C[1 - \varepsilon \int_0^2 g_1(x)\exp(-\frac{\alpha(2-x)}{\varepsilon})dx] \pm K_1w_{B1}(2) \geq 0 \]

Also

\[ \mathcal{L}\Phi_j^\pm(x) = C_1\left[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x) + b_{12}(x) + (c_{11}(x) + c_{12}(x))\exp(-\frac{\alpha}{\varepsilon})\right] \exp\left(-\frac{\alpha(2-x)}{\varepsilon}\right) \pm \mathcal{L}w_{B1}(x) \]

\[ \geq C_1\left[\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta + \gamma \exp(-\frac{\alpha}{\varepsilon})\right] \exp\left(-\frac{\alpha(2-x)}{\varepsilon}\right) \pm 0 \]

\[ \geq 0. \]

By the Theorem 3.1,

\[ |w_{B1}(x)| \leq C\exp(-\frac{\alpha(2-x)}{\varepsilon}). \]

Integration of (3.7) yields the estimates of \( |w_{B1}(x)| \). From the differential equations (3.6), one can derive the rest of the derivative estimates (3.11). Similarly it is easy to prove \( w_{B2} \) is bounded. To prove the inequalities (3.11), consider the barrier functions

\[ \Phi_j^+(x) = C_1\varepsilon(\exp(-\alpha(1-x)/\varepsilon)) \pm w_{Hj}(x), \ x \in [0, 1], \ j = 1, 2. \]

Clearly, \( \Phi_j^+(0) \geq 0 \) and also \( \mathcal{L}\Phi_j(x) \geq 0, \ j = 1, 2 \) easily proves the first inequality.

Similarly, consider the following barrier functions

\[ \Phi_j^+(x) = C_1x\varepsilon \pm w_{Hj}(x), \ x \in [1, 2]. \]

Note that

\[ K_j\Phi_j^+(2) = \Phi_j^+(2) - \varepsilon \int_0^2 g_j(x)\Phi_j^+(x)dx \]
\[
\begin{align*}
\text{Hence the proof.} & \\
\text{Note: From the above lemma, it is not difficult to prove} &
\end{align*}
\]

\[
|u_j(x) - v_j(x)| \leq C \begin{cases} 
\varepsilon \exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right) + \exp\left(-\frac{\alpha(2-x)}{\varepsilon}\right), & x \in \Omega_1 \\
\varepsilon + \exp\left(-\frac{\alpha(2-x)}{\varepsilon}\right), & x \in \Omega_2.
\end{cases}
\]

\section{The discrete problem}

The BVP (2.1) exhibits strong boundary layer at \(x = 2\) and interior layer at \(x = 1\).

The interval \([0, 1]\) is partitioned into \([0, 1-\sigma]\) and \([1-\sigma, 1]\) and the interval \([1, 2]\) is partitioned as \([1, 2-\sigma]\) and \([2-\sigma, 2]\), where \(\sigma\) is transition parameter for this mesh defined by

\[
\sigma = \min\left\{\frac{1}{2}, 2\frac{\varepsilon}{\alpha} \ln N\right\}.
\]

The mesh \(\Omega_{2N} = \{x_0, x_1, \ldots, x_{2N}\}\) is defined by

\[
x_0 = 0, \quad x_i = x_0 + iH, \quad i = 1 \text{ to } N, \quad x_{i+N} = x_0 + iH, \quad i = 1 \text{ to } \frac{N}{2}, \quad x_{i+N} = x_N + ih, \quad i = 1 \text{ to } \frac{N}{2} \text{ where } h = \frac{2\sigma}{N}, \quad H = \frac{2(1-\sigma)}{N}.
\]

The discrete problem corresponding to (2.2)-(2.4) is: Find \(\hat{U}(x) = (U_1(x_1), U_2(x_2))^T\) such that

\[
\begin{align*}
\mathcal{L}_1^N \hat{U}(x) &= -\varepsilon \delta^2 U_1(x_i) + a_1(x_i) D^- U_1(x_i) + b_1(x_i) U_1(x_i) + b_{12}(x_i) U_2(x_i) + c_11(x_i) U_1(x_{i-N}) + c_12(x_i) U_2(x_{i-N}) = f_1(x_i), \quad \forall x_i \in \Omega_{2N} \\
\mathcal{L}_2^N \hat{U}(x) &= -\varepsilon \delta^2 U_2(x_i) + a_2(x_i) D^- U_2(x_i) + b_21(x_i) U_1(x_i) + b_{22}(x_i) U_2(x_i) + c_21(x_i) U_1(x_{i-N}) + c_22(x_i) U_2(x_{i-N}) = f_2(x_i), \quad \forall x_i \in \Omega_{2N},
\end{align*}
\]

\[
\begin{align*}
U_j(x_i) &= \phi_j(x_i), \quad i = -N, -N + 1, \ldots, 0, \\
K_j^N U_j(x_N) &= U_j(x_N) - \varepsilon \sum_{i=1}^{2N} g_i(x_{i-1}) U_i(x_{i-1}) h_i = l_j, \quad \forall x_i \in \Omega_{2N} \\
D^- U_j(x_N) &= D^+ U_j(x_N),
\end{align*}
\]

where

\[
\begin{align*}
\delta^2 U_j(x_i) &= \frac{2}{h_{i+1} + h_i} \left( \frac{U_j(x_{i+1}) - U_j(x_i)}{h_{i+1}} - \frac{U_j(x_i) - U_j(x_{i-1})}{h_i} \right), \\
D^- U_j(x_i) &= \frac{U_j(x_i) - U_j(x_{i-1})}{h_{i-1}}, \quad j = 1, 2.
\end{align*}
\]
Theorem 4.1. (Discrete Maximum Principle) Let $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))^T$ be the mesh function satisfying $\Psi_1(x_0) \geq 0, \Psi_2(x_0) \geq 0, \kappa_1^N \Psi_1(x_{2N}) \geq 0, \kappa_2^N \Psi_2(x_{2N}) \geq 0, \mathcal{L}_1^N \bar{\Psi}(x_i) \geq 0, \mathcal{L}_2^N \bar{\Psi}(x_i) \geq 0$ and $[D]U_j(x_N)) \leq 0, j = 1, 2$. Then $\bar{\Psi}(x_i) \geq 0, x_i \in \bar{\Omega}^{2N}$.

**Proof.** Define $\bar{S}_j(x_i) = (S_1(x_i), S_2(x_i))^T$,

$$S_j(x_i) = \begin{cases} \frac{1+x_i}{2}, & x_i \in [0, 1] \cap \bar{\Omega}^{2N}, \\ \frac{3-2x_i}{4}, & x_i \in [1, 2] \cap \bar{\Omega}^{2N}, \end{cases} j = 1, 2.$$  

Note that $\bar{S}_j(x_i) > 0, \forall x_i \in \bar{\Omega}^{2N}, \kappa_1^N S_1(x_{2N}) > 0, \kappa_2^N S_2(x_{2N}) > 0, \mathcal{L}_1^N \bar{S}_1(x_i) > 0$ and $\mathcal{L}_2^N \bar{S}_2(x_i) > 0, \forall x_i \in \bar{\Omega}^{2N}$. Let

$$\mu = \max \left\{ \max_{x_i \in \Omega^{2N}} \left( \frac{-\Psi_1(x_i)}{S_1(x_i)} \right), \ \max_{x_i \in \Omega^{2N}} \left( \frac{-\Psi_2(x_i)}{S_2(x_i)} \right) \right\}.$$  

Then there exists one $x_k \in \bar{\Omega}^{2N}$ such that $\Psi_1(x_k) + \mu S_1(x_k) = 0$ or $\Psi_2(x_k) + \mu S_2(x_k) = 0$ or both. We have $\Psi_j(x_i) + \mu S_j(x_i) \geq 0, x_i \in \bar{\Omega}^{2N}, j = 1, 2$. Therefore either $(\Psi_1 + \mu S_1)$ or $(\Psi_2 + \mu S_2)$ attains minimum at $x_i = x_k$. Suppose the theorem does not hold true, then $\mu > 0$.

Case (i): $x_k = x_0$

$$0 < (\Psi_j + \mu S_j)(x_0) = 0$$  

It is a contradiction.

Case (ii): $x_k \in \Omega_1^{2N}$

$$0 < \mathcal{L}_j^N (\Psi_j + \mu S_j)(x_k) \leq 0, j = 1, 2.$$  

It is a contradiction.

Case (iii): $x_k = x_N$

$$0 \leq [D_j(\Psi_j + \mu S_j)](x_N) < 0, j = 1, 2.$$  

It is a contradiction.

Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < \mathcal{L}_j^N (\Psi_j + \mu S_j)(x_k) \leq 0, j = 1, 2.$$  

It is a contradiction.

Case (v): $x_k = x_{2N}$

$$0 < \kappa_j^N (\Psi_j + \mu S_j)x_{2N} \leq 0, j = 1, 2$$  

It is a contradiction. Hence the proof of the theorem. \square

Lemma 4.2. (Discrete Stability Result) Let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ be any mesh function. Then

$$|U_k(x)| \leq C \max \left\{ \max_{j=1,2} \{ |U_j(0)| \}, \ \max_{j=1,2} \{ |\kappa_j U_j(2)| \}, \ \max_{j=1,2} \{ \sup_{x_i \in \Omega_1 \cup \Omega_2} |\mathcal{L}_j \bar{U}(x_i)| \} \right\},$$  

$$\forall x_i \in \bar{\Omega}^{2N}, k = 1, 2.$$  

**Proof.** By choosing suitable barrier functions and using Theorem 4.1, one can establish the above inequality. \square
Analogous to the continuous case, the discrete solution $\tilde{U}(x_i)$ can be decomposed as

$$\tilde{U}(x_i) = \tilde{V}(x_i) + \tilde{W}(x_i),$$

where $\tilde{V}(x_i)$ and $\tilde{W}(x_i)$ are respectively, the solutions of the problems:

\[
\begin{align*}
\mathcal{L}^N_1 V_1(x_i) &= f_1(x_i), \quad x_i \in \Omega_1^{2N}, \quad V_1(x_0) = v_1(0), \\
[D]V_1(x_N) &= [v'_{1}](1), \quad \mathcal{K}_1^N V_1(x_{2N}) = \mathcal{K}_1 v_1(2) \\
\mathcal{L}^N_2 V_2(x_i) &= f_2(x_i), \quad x_i \in \Omega_2^{2N}, \quad V_2(x_0) = v_2(0), \\
[D]V_2(x_N) &= [v'_{2}](1), \quad \mathcal{K}_2^N V_2(x_{2N}) = \mathcal{K}_2 v_2(2). \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}^N W_1 &= 0, \quad x_i \in \Omega_1^{2N}, \quad W_1(x_0) = w_1(0), \\
[D]W_1(x_N) &= -[D]V_1(x_N), \quad \mathcal{K}_1^N W_1(x_{2N}) = \mathcal{K}_1 w_1(2) \\
\mathcal{L}^N W_2 &= 0, \quad x_i \in \Omega_2^{2N}, \quad W_2(x_0) = w_2(0), \\
[D]W_2(x_N) &= -[D]V_2(x_N), \quad \mathcal{K}_2^N W_2(x_{2N}) = \mathcal{K}_2 w_1(2). \\
\end{align*}
\]

The following theorem gives an estimate for the difference of the solutions of (4.1) – (4.2) and (4.3).

**Theorem 4.3.** Let $\tilde{U}(x_i)$ be a numerical solution of (2.2) – (2.4) defined by (4.1) – (4.2) and $\tilde{V}(x_i)$ be a numerical solution of (3.5) defined by (4.3). Then

$$|\tilde{U}_j(x_i) - \tilde{V}_j(x_i)| \leq C \begin{cases} N^{-1}, & i = 0, 1, \ldots, \frac{3N}{2} \\
N^{-1} + |l_j - \mathcal{K}_j^N V_j(x_{2N})| & i = \frac{3N}{2} + 1, \ldots, 2N \end{cases} \quad j = 1, 2,$$

**Proof.** Consider a mesh function $\tilde{\Psi}^\pm(x_i) = (\Psi_{i,j}^+(x_i), \Psi_{i,j}^-(x_i))^T$, where

$$\begin{align*}
\Psi_{i,j}^+(x_i) &= CN^{-1} S_j(x_i) + C x_i \varphi(x_i) + (U_j(x_i) - V_j(x_i)), \\
\varphi(x_i) &= \begin{cases} 0, & i = 0, 1, \ldots, \frac{3N}{2} \\
|l_j - \mathcal{K}_j^N V_j(x_{2N})| & i = \frac{3N}{2} + 1, \ldots, 2N \end{cases} \quad j = 1, 2.
\end{align*}$$

It is clear that $\Psi^\pm(x_0) \geq 0$ and $\mathcal{K}\Psi^\pm(x_{2N}) \geq 0$. If $\forall \ x_i \in \Omega_1^{2N}$

$$\mathcal{L}_j^N \Psi^\pm(x_i) \geq 0, \quad j = 1, 2,$$

If $\forall \ x_i \in \Omega_2^{2N}$

$$\mathcal{L}_j^N \Psi^\pm(x_i) \geq 0, \quad j = 1, 2$$

$[D]^+ \Psi_{i,j}^\pm(x_N) < 0, j=1,2$, for a suitable choice of $C_1 > 0$. By Theorem 4.1, this theorem gets proved. \[\square\]
5. Error estimates for the solution

We obtain separate error estimates for each component of the numerical solution.

**Theorem 5.1.** Let \( \tilde{V}(x_i) \) be a numerical solution of (3.5) defined by (4.3). Then
\[
|v_j(x_i) - V_j(x_i)| \leq CN^{-1}, \quad x_i \in \Omega^{2N}, \text{ where } j = 1, 2.
\]

**Proof.** If \( x_i \in \Omega^{2N} \) and \( x_i \in \Omega_1^{2N} \) then by [15], we have
\[
|CN(v_j(x_i) - V_j(x_i))| \leq CN^{-1}, \quad i \in \Omega_1^{2N} \cup \Omega_2^{2N}.
\]

By the Lemma 4.2, we have
\[
|v_j(x_i) - V_j(x_i)| \leq CN^{-1}, \quad i \in \Omega_1^{2N} \cup \Omega_2^{2N}.
\]

At the point \( x_i = x_{2N} \),
\[
\mathcal{K}^N_j(V_j - v_j)(x_{2N}) = \mathcal{K}^N_j V_j(x_{2N}) - \mathcal{K}^N_j v_j(x_{2N})
\]
\[
= I - \mathcal{K}^N_j v_j(x_{2N})
\]
\[
= \mathcal{K}_j v_j(x_{2N}) - \mathcal{K}^N_j v_j(x_{2N})
\]
\[
= v_j(x_{2N}) - \int_{x_0}^{x_{2N}} g_j(x)v(x)dx - v_j(x_{2N}) + \sum_{i=1}^{2N} \frac{g_i - 1v_i + g_iv_i}{2}h_i
\]
\[
|\mathcal{K}^N_j(V_j - v_j)(x_{2N})| \leq C\varepsilon(h_i^3 + \cdots + h_{2N}^3)
\]
\[
\leq CN^{-2}
\]
\[
\leq CN^{-1}, \quad \text{where } x_{i-1} \leq x_i \leq x_j, \quad j = 1, 2, \quad 1 \leq i \leq 2N.
\]

Applying Lemma 4.2, we have \(|(V_j - v_j)(x_{2N})| \leq CN^{-1}.
Hence \(|v_j(x_i) - V_j(x_i)| \leq CN^{-1}, \quad i \in \Omega_1^{2N}, \text{ where } j = 1, 2. \]

**Theorem 5.2.** Let \( W(x_i) \) be a numerical solution of (3.6) defined by (4.4). Then
\[
|w_j(x_i) - W_j(x_i)| \leq CN^{-1}ln^2N, \quad x_i \in \Omega_1^{2N}, \text{ where } j = 1, 2.
\]

**Proof.** Note that
\[
|w_j(x_i) - W_j(x_i)| \leq |u_j(x_i) - U_j(x_i)| + |v_j(x_i) - V_j(x_i)|
\]

Then by (3.12), Theorem 3.4 and Theorem 4.3, we have
\[
|u_j(x_i) - U_j(x_i)| \leq |U_j(x_i) - V_j(x_i)| + |v_j(x_i) - V_j(x_i)| + |u_j(x_i) - v_j(x_i)|.
\]

Now,
\[
|w_j(x_i) - W_j(x_i)| \leq |U_j(x_i) - V_j(x_i)| + 2|v_j(x_i) - V_j(x_i)| + |u_j(x_i) - v_j(x_i)|
\]
\[
\leq C_1N^{-1} + C_1 \exp\left(\frac{-\alpha(2 - x)}{\varepsilon}\right) + \varepsilon
\]
Consider mesh functions
\[
\phi_j^\pm(x_i) = C_1N^{-1}\hat{s}(x_i) + C_1N^{-1}\frac{\sigma}{\varepsilon^2}(x_i - (2 - \sigma)) \pm (w_j(x_i) - W_j(x_i)) x_i \in [2 - \sigma, 2] \cap \Omega^{2N}.
\]
From (5.1), it is easy to check \( \phi_j^\pm(x_{iN}) \geq 0 \) and \( K_j \phi_j^\pm(x_{2N}) \geq 0 \), for a suitable choice of \( C_1 > 0 \).

\[
\mathcal{L}_j^N \phi_j^\pm(x_i) \geq C_1N^{-1}[\beta_j + \gamma_j] + C_1N^{-1}\frac{\sigma}{\varepsilon^2}[\alpha + \beta_j(x_i + \sigma - 2) + \gamma_j(x_i + \sigma + \sigma - 2)] \\
\geq 0
\]
Then by the Theorem 5.1, we have \( \phi_j^\pm(x_i) \geq 0 \), \( x_i \in \bar{\Omega}^{2N} \). Therefore
\[
|w_j(x_i) - W_j(x_i)| \leq C N^{-1} \ln^2 N, \quad x_i \in \bar{\Omega}^{2N}, \text{ where } j = 1, 2.
\]
Hence the proof. \( \square \)

**Theorem 5.3.** Let \( \bar{U}(x_i) \) be the solution of (2.2) \( -(2.4) \) defined in (4.1) \( -(4.2) \). Then
\[
|u_j(x_i) - U_j(x_i)|_{\Omega^{2N}} \leq C N^{-1} (\ln N)^2, \text{ where } j = 1, 2.
\]

**Proof.** Combining Theorem 5.1 and Theorem 5.2, the proof gets completed. \( \square \)

6. **Numerical Results**

**Example 6.1.**

\[
\begin{cases}
-\varepsilon u_1''(x) + 11u'(x) + 10u_1(x) - 2u_2(x) - x^2u_1(x - 1) - xu_2(x - 1) = e^x, \quad x \in \Omega^* \\
-\varepsilon u_1''(x) + 16u'(x) - 2u_1(x) + 10u_2(x) - xu_1(x - 1) - xu_2(x - 1) = e^{2x}, \quad x \in \Omega^*
\end{cases}
\]
with the boundary conditions
\[
\begin{align*}
&u_1(0) = 1, \quad u_1(2) - \varepsilon \int_0^2 \frac{2}{3} u_1(x)\,dx = 2, \quad x \in \bar{\Omega} \\
&u_2(0) = 1, \quad u_2(2) - \varepsilon \int_0^2 \frac{2}{3} u_2(x)\,dx = 2, \quad x \in \bar{\Omega}.
\end{align*}
\]

**Example 6.2.**

\[
\begin{cases}
-\varepsilon u_1''(x) + 11u'(x) + 6u_1(x) - 2u_2(x) - u_1(x) = 0, \quad x \in \Omega^* \\
-\varepsilon u_1''(x) + 16u'(x) - 2u_1(x) + 5u_2(x) - u_2(x) = 0, \quad x \in \Omega^*
\end{cases}
\]
with the boundary conditions
\[
\begin{align*}
&u_1(0) = 1, \quad u_1(2) - \varepsilon \int_0^2 \frac{2}{3} u_1(x)\,dx = 2, \quad x \in \bar{\Omega} \\
&u_2(0) = 1, \quad u_2(2) - \varepsilon \int_0^2 \frac{2}{3} u_2(x)\,dx = 2, \quad x \in \bar{\Omega}.
\end{align*}
\]
TABLE 1. Maximum pointwise errors and order of convergence for Example 6.1

<table>
<thead>
<tr>
<th>Number of mesh points $2N$</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1^N$</td>
<td>6.0996e-03</td>
<td>3.0853e-03</td>
<td>1.5512e-03</td>
<td>7.7754e-04</td>
<td>3.8912e-04</td>
<td>1.9459e-04</td>
</tr>
<tr>
<td>$P_1^N$</td>
<td>9.8329e-01</td>
<td>9.9200e-01</td>
<td>9.9644e-01</td>
<td>9.9869e-01</td>
<td>9.9981e-01</td>
<td>-</td>
</tr>
<tr>
<td>$D_2^N$</td>
<td>4.5859e-03</td>
<td>2.3027e-03</td>
<td>1.1532e-03</td>
<td>5.7681e-04</td>
<td>2.8833e-04</td>
<td>1.4408e-04</td>
</tr>
<tr>
<td>$P_2^N$</td>
<td>9.9391e-01</td>
<td>9.9764e-01</td>
<td>9.9949e-01</td>
<td>1.0004e+00</td>
<td>1.0009e+00</td>
<td>-</td>
</tr>
</tbody>
</table>

TABLE 2. Maximum pointwise errors and order of convergence for Example 6.2

<table>
<thead>
<tr>
<th>Number of mesh points $2N$</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1^N$</td>
<td>4.3386e-03</td>
<td>2.2283e-03</td>
<td>1.1292e-03</td>
<td>5.6836e-04</td>
<td>2.8512e-04</td>
<td>1.4283e-04</td>
</tr>
<tr>
<td>$D_2^N$</td>
<td>4.4291e-03</td>
<td>2.2517e-03</td>
<td>1.1349e-03</td>
<td>5.6958e-04</td>
<td>2.8527e-04</td>
<td>1.4276e-04</td>
</tr>
</tbody>
</table>

The analytical solution of the above example are not available. Therefore, we estimate the error using double mesh principle which is defined as $D_1^N = \max_{x_i \in \Omega_{2N}} |U_1^N(x_i) - U^{2N}(x_i)|$ and $D_2^N = \max_{x_i \in \Omega_{2N}} D_1^N$ where $U_1^N(x_i)$ and $U^{2N}(x_i)$ denote the numerical solution computed using $N$ and $2N$ mesh points. From these quantities the order of convergence is defined as $P_1^N = \log_2 \left( \frac{D_2^N}{D_1^N} \right)$. In Tables 1 and 2, $D_1^N$ and $D_2^N$ denote the maximum pointwise errors of $U_1$ and $U_2$ respectively, $P_1^N$ and $P_2^N$ denote the order of convergence with respect to $U_1$ and $U_2$ respectively.

**Figure 1.** Graph of the numerical solution of Example 6.1 for $n = 128$ and $\varepsilon = 2^{-8}$.
The numerical solution of Example (6.2) is plotted in Figure 2.

![Graph of the numerical solution](image)

**Figure 2.** Graph of the numerical solution of Example 6.2 for \( n = 128 \) and \( \varepsilon = 2^{-12} \).

7. Conclusion

We have solved a class of system of singularly perturbed boundary value problem (2.1) with integral boundary conditions, using a finite difference method on Shishkin mesh. The method is shown to be of order \( O(N^{-1} \ln^2 N) \), that is, the method has almost first order convergence with respect to \( \varepsilon \). Two examples are given to illustrate the numerical method. Our numerical results reflect the theoretical estimates. Maximum pointwise errors and order of convergence of the Examples (6.1) and (6.2) are given in Table 1 and 2 respectively. The numerical solution of Example (6.1) is plotted in Figure 1.

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References