DENSITY DEPENDENT MORTALITY OF INTERMEDIATE PREDATOR CONTROLS CHAOS-CONCLUSION DRAWN FROM A TRI-TROPHIC FOOD CHAIN

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ABSTRACT. The paper explores a tri-trophic food chain model with density dependent mortality of intermediate predator. To analyze this aspect, we have worked out the local stability of different equilibrium points. We have also derived the conditions for global stability of interior equilibrium point and conditions for persistence of model system. To observe the global behaviour of the system, we performed extensive numerical simulations. Our simulation results reveal that chaotic dynamics is produced for increasing value of half-saturation constant. We have also observed trajectory motions around different equilibrium points. It is noticed that chaotic dynamics has been controlled by increasing value of density dependent mortality parameter. So, we conclude that the density dependent mortality parameter can be used to control chaotic dynamics. We also applied basic tools of nonlinear dynamics such as Poincare section and Lyapunov exponent to investigate chaotic behaviour of the system.

1. INTRODUCTION

In ecology, energy is transferred from one trophic level to another trophic level through the food chain. Chaos in a food chain model reveals that due to irregular and aperiodic population abundance, sustainability of the food chain through predator-prey relation is not preserved. The density-dependent processes occur when population growth rates are regulated by the density of a population. Chaotic dynamics is a very common factor in tri-trophic food chain.

Hastings and Powell [14] produced a chaotic population system in a simple tri-trophic food chain model with type-II functional response. After HP model, many researchers analyzed chaotic dynamics in a food chain model and also studied how to control chaos with the help of different biological parameters. Rai and Sreenivasan [31] presented a new model for a food chain involving three species showing the presence of a period-doubling scenario leading to chaos and discussed the application of this study to aquacultural research.
Ruxton [33, 34] investigated that the system of linked populations or imposition of a population floor on HP model has a stabilizing effect on tri-trophic food chain model. Eisenberg and Maszle [11] revisited the HP model and observed that gradual addition of refugia provides a stabilizing influence for which the chaotic dynamics collapsed to stable limit cycles.

MaCann and Hastings [20] re-investigated the role of omnivory in food webs using a non-equilibrium perspective. Varriale and Gomes [42] studied HP model using two different approaches. Firstly, they observed the asymptotic states of the system resulting from numerical integration of the equations. Secondly, they applied the embedding procedure to extract the relevant dynamical exponents from a time series for only one scalar variable.

Xu and Li [43] investigated the role of intraspecific density dependence (IDD), another important ecological factor, in the dynamics of two versions (deterministic and stochastic) of a food chain. They showed that the addition of IDD to a deterministic three-species food chain model stabilizes the food chain system and the very famous teacup chaotic attractor disappears. They also showed that the addition of IDD to a stochastic one at most results in a reduction in amplitude and frequency of dynamical fluctuations and can never eliminate the stochastic behavior.

Chattopadhyay and Sarkar [7] included the effect of the toxin, produced by toxin producing phytoplankton, as extra mortality term in zooplankton population and observed that chaotic behavior is less likely to occur in a real food chain dynamics. Lonngren et al. [23] also investigated the HP model with different set of parameters by using techniques from modern control theory. They observed that the synchronization of two diverse food chain systems is possible.

Maionchi and Reis [25] proposed a spatial version of the HP model in which predators seek their preys only in a finite neighborhood of their home location. They showed that local predation leads to spontaneous pattern formation and to qualitative changes in the global dynamics of the system. They also observed that the chaotic strange attractor present in the original HP model can be replaced by a stable fixed point or by an attractor of simpler topology depending on the size of the predation neighborhoods.

Bairagi et al. [6] studied the role of harvesting in a predator-prey-parasite system. Using impulsive harvesting effort as control parameter, they showed that it is possible to control the cyclic behavior of the system populations leading to the persistence of all species and other desired stable equilibrium can be obtained.

Younghae Do et al. [10] investigated a three-species food chain system with two types of functional response, Holling type and Beddington-DeAngelis type. Upadhyay and Raw [40] investigated analytically as well as numerically the dynamical complexities of a three species food chain model with Holling type IV predator response.

Zhang et al. [44] studied the effect of feeding selectivity on transfer of methylmercury (MeHg) through the marine food chains. The varying concentration of Hg in the microalgae species as well as the selectivity of zooplankton to the algal diet resulted in varying Hg accumulation in three zooplankton species i.e., feeding selectivity plays a key role in the transfer of MeHg along marine food chains.
Sunaryo et al. [35] studied ecological model with a tri-trophic food chain composed of a classical Lotka-Volterra functional response for prey and predator, and a Holling type-III functional response for predator and superpredator. They found the dynamical behavior of this model is very sensitive to parameter values as well as the parameters of the practical life. Greenhalgh et al. [13] analyzed the features of a predator-prey model with disease amongst the prey and ratio-dependent functional response for both infected and susceptible prey.

The above reviews clearly indicate that a lot of research has already been performed on HP model. As far as our knowledge goes, none of the studies take into account the density dependent mortality of intermediate predator in the system. We like to see how density dependent mortality in intermediate predator influence the dynamics of tri-trophic food chain model proposed by Hastings and Powell [14]. The control of chaotic dynamics with various ecological factors, namely, migration, predation, refuge, omnivory, habitat-heterogeneity is the most challenging task in such studies and thus it receives a good deal of attention from various scientists [7, 23, 42, 43].

To the best of our knowledge, there are few literatures which have considered the inhibitory effect of intra specific density dependence, but it has the ability to regulate the population dynamics significantly [43]. The main objective of the paper is to extend the prey-predator-top predator model by incorporating density dependent death rate of intermediate predator. We have analyzed the local asymptotic stability behaviour of the coexisting equilibrium points. Then, we have obtained the global stability conditions for the system. Finally, extensive numerical simulations are performed to make a further in depth analysis of the system.

The paper is organized as follows. In the section 2, we outline the mathematical model with some basic assumption. We study the stability of the equilibrium points in section 3. We have studied the global stability as well as persistence of the system in section 4. We perform an extensive numerical simulations and discussion in section 5.

2. Model formulation

Incorporating "type II" saturating functional responses in both of the consumer species, the HP model takes the form:

\[
\begin{align*}
\frac{dX}{dT} &= R_0 X \left(1 - \frac{X}{K_0}\right) - C_1 A_1 \frac{Y X}{B_1 + X} \\
\frac{dY}{dT} &= A_1 \frac{Y X}{B_1 + X} - A_2 \frac{YZ}{B_2 + Y} - D_1 Y \\
\frac{dZ}{dT} &= C_2 A_2 \frac{YZ}{B_2 + Y} - D_2 Z.
\end{align*}
\]

Let \(X\) be the numbers of the species at the lowest level of the food chain, \(Y\) the numbers of the species that preys upon \(X\), \(Z\) the numbers of the species that preys upon \(Y\) and \(T\) is time. The constant \(R_0\) is the intrinsic growth rate, and the constant \(K_0\) is the carrying capacity of species \(X\). The constants \(C_1\) and \(C_2\) are conversion rates of prey to predator for species \(Y\) and \(Z\) respectively; \(D_1\) and \(D_2\) are constant death rates for species \(Y\) and \(Z\) respectively. The
constants $A_i$ and $B_i$ for $i = 1, 2$ are maximal predation rate and half saturation constants for $Y$ and $Z$ respectively. Now we introduce density dependent mortality in intermediate predator

$$
\frac{dX}{dt} = R_0X \left(1 - \frac{X}{K_0}\right) - C_1A_1 \frac{XY}{B_1 + X}
$$
$$
\frac{dY}{dt} = A_1 \frac{XY}{B_1 + X} - A_2 \frac{YZ}{B_2 + Y} - D_1Y - D_Y^2
$$
$$
\frac{dZ}{dt} = C_2A_2YZ \frac{B_2 + Y}{B_2 + Y} - D_Z
$$

(2.2)

To reduce the number of parameters and to determine which combinations of parameters control the behaviour of the system, we dimensionalize the system with the following scaling

$$
x = \frac{X}{K_0}, y = \frac{C_1Y}{K_0}, z = \frac{C_1Z}{C_2K_0} \text{ and } t = R_0T
$$

Then system (2.2) takes the form,

$$
\frac{dx}{dt} = x(1 - x) - \frac{a_1xy}{1 + b_1x}
$$
$$
\frac{dy}{dt} = \frac{a_1xy}{1 + b_1x} - \frac{a_2yz}{1 + b_2y} - d_1y - dy^2
$$
$$
\frac{dz}{dt} = \frac{a_2yz}{1 + b_2y} - d_2z
$$

(2.3)

where

$$
a_1 = \frac{A_1K_0}{B_1R_0}, b_1 = \frac{K_0}{B_1}, a_2 = \frac{A_2C_2K_0}{B_2C_1R_0}, b_2 = \frac{K_0}{B_2C_1}, d = \frac{DK_0}{R_0C_1}, d_1 = \frac{D_1}{R_0} \text{ and } d_2 = \frac{D_2}{R_0}
$$

<table>
<thead>
<tr>
<th>Dimentionalised parameters</th>
<th>Biological significance of parameters</th>
</tr>
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<tbody>
<tr>
<td>$a_1$ = $\frac{A_1K_0}{B_1R_0}$</td>
<td>Maximal predation rate of intermediate predator</td>
</tr>
<tr>
<td>$a_2$ = $\frac{A_2C_2K_0}{B_2C_1R_0}$</td>
<td>Maximal predation rate of top predator</td>
</tr>
<tr>
<td>$b_1$ = $\frac{K_0}{B_1}$</td>
<td>Half saturation constant in the functional response for intermediate predator</td>
</tr>
<tr>
<td>$b_2$ = $\frac{K_0}{C_1B_2}$</td>
<td>Half saturation constant in the functional response for top predator</td>
</tr>
<tr>
<td>$d_1$ = $\frac{D_1}{R_0}$</td>
<td>Death rate of intermediate predator</td>
</tr>
<tr>
<td>$d_2$ = $\frac{D_2}{R_0}$</td>
<td>Death rate of top predator</td>
</tr>
<tr>
<td>$d$ = $\frac{DK_0}{R_0C_1}$</td>
<td>Density dependent mortality rate of the intermediate predator</td>
</tr>
</tbody>
</table>
System (2.3) has to be analyzed with the following initial conditions:
\[ x(0) > 0, \ y(0) > 0, \ z(0) > 0. \]

3. Existence and local stability of equilibrium points

The system has four equilibrium points. The trivial equilibrium point \( E_0(0, \ 0, \ 0) \) and the axial equilibrium point \( E_1(1, \ 0, \ 0) \) exist for all parametric values. The top-predator free equilibrium point is \( E_2(\bar{x}, \ \bar{y}, \ 0) \), where \( \bar{x} \) is the positive root of the equation
\[ x^3 + q_1x^2 + q_2x + q_3 = 0 \]

where
\[ q_1 = \frac{(2b_1d - b_1^2d)}{b_1^2d}, q_2 = \frac{(d - 2b_1d + a_1^2 - a_1b_1d_1)}{b_1^3d}, q_3 = -\frac{(a_1d_1 + d)}{b_1^2d} \]
and
\[ \bar{y} = -\frac{d_1}{d} + \frac{a_1\bar{x}}{d(1 + b_1\bar{x})} = \frac{(a_1 - b_1d_1)\bar{x} - d_1}{d(1 + b_1\bar{x})} \]
Here if \( q_1 > 0 \) and \( q_2 > 0 \) i.e., \( (2b_1d - b_1^2d) > 0 \) and \( (d - 2b_1d + a_1^2 - a_1b_1d_1) > 0 \) then by Descartes’ rule of sign, there is one sign change. Therefore the equation has at least one positive root. The positive condition for the existence of \( E_2 \) is \( (a_1 - b_1d_1)\bar{x} - d_1 > 0 \) i.e. \( (a_1 - b_1d_1)\bar{x} > d_1 \). The interior equilibrium point is given by \( E^*(x^*, \ y^*, \ z^*) \), where \( x^* \) is the positive root of the equation
\[ p_1x^2 + p_2x + p_3 = 0 \] \( (3.1) \)
where
\[ p_1 = (b_1b_2d_2 - a_2b_1), \ p_2 = (b_2d_2 + a_2b_1 - a_2 - b_1b_2d_2), \ p_3 = (a_2 - a_1d_2 - b_2d_2) \]
If \( p_1 > 0, \ p_2 > 0 \) and \( p_3 < 0 \) or \( p_1 > 0, \ p_2 < 0 \) and \( p_3 < 0 \) or \( p_1 < 0, \ p_2 < 0 \) and \( p_3 > 0 \) then there is only one sign change. Thus the equation has at least one positive root.
\[ y^* = \frac{d_2}{a_2 - b_2d_2}, \ z^* = \frac{a_1x^*}{(a_2 - b_2d_2)(1 + b_1x^*)} - \frac{d_1}{(a_2 - b_2d_2)} - \frac{dd_2}{(a_2 - b_2d_2)^2} \]
\[ = \frac{(a_2 - b_2d_2)[a_1x^* - d_1(1 + b_1x^*)] - dd_2(1 + b_1x^*)}{(a_2 - b_2d_2)^2(1 + b_1x^*)} \]
The positive conditions for the existence of \( E^* \) are
(i) \( a_2 - b_2d_2 > 0 \)
(ii) \( [a_1x^* - d_1(1 + b_1x^*)] - dd_2(1 + b_1x^*) > 0 \)
i.e., \( [a_1x^* - (d_1 + dd_2)(1 + b_1x^*)] > 0 \).
The characteristic roots of the Jacobian matrix $J$ of the system (2.3) at any arbitrary point $(x, y, z)$ is given by

$$
\begin{bmatrix}
1 - 2x - \frac{a_1 y}{(1 + b_1 x)^2} & \frac{a_1 x}{1 + b_1 x} - \frac{a_1 y}{(1 + b_1 x)^2} & 0 \\
\frac{a_1 y}{(1 + b_1 x)^2} - \frac{a_2 z}{(1 + b_2 y)^2} & -d_1 - 2dy - \frac{a_2 y}{(1 + b_2 y)^2} & -d_2 \\
0 & \frac{a_2 z}{(1 + b_2 y)^2} & -d_2
\end{bmatrix}.
$$

(3.2)

**Theorem 1.** The trivial equilibrium point $E_0$ is always unstable. The axial equilibrium point $E_1$ is locally stable if $R_{01} < 1$ where $R_{01} = \frac{1}{d_1} \left( \frac{a_1}{1 + b_1} \right)$. The top-predator free equilibrium point $E_2$ is locally asymptotically stable if $(1 + b_1 \bar{x})^2 > a_1 b_1 \bar{y}$ and $R_{02} < 1$, where $R_{02} = \frac{1}{d_2} \left( \frac{a_2 \bar{y}}{1 + b_2 \bar{y}} \right)$.

**Proof.** The Jacobian matrix at trivial equilibrium point $E_0$ is given by

$$
J_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}
$$

The characteristic roots of the Jacobian matrix $J_0$ are $1, -d_1$ and $-d_2$.

Since one of the eigenvalues associated with the Jacobian matrix computed around $E_0$ is $1 > 0$, so the equilibrium point $E_0$ is always unstable.

The Jacobian matrix at axial equilibrium point $E_1$ is given by

$$
J_1 = \begin{bmatrix} -1 & -\frac{a_1}{1 + b_1} & 0 \\ 0 & \frac{a_1}{1 + b_1} - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}
$$

The characteristic roots of the Jacobian matrix $J_1$ are $-1, -d_2$ and $\frac{a_1}{1 + b_1} - d_1$

Hence $E_1$ is stable if $\frac{a_1}{1 + b_1} - d_1 < 0$ which implies $R_{01} < 1$ where $R_{01} = \frac{1}{d_1} \left( \frac{a_1}{1 + b_1} \right)$.

The Jacobian matrix at top-predator free equilibrium point $E_2$ is given by

$$
J_2 = \begin{bmatrix} \bar{x} - \frac{a_1 b_1 \bar{x} \bar{y}}{(1 + b_1 \bar{x})^2} + d\bar{y} & -a_1 \bar{x} \bar{y} \bar{z} & 0 \\ \frac{a_1 \bar{y}}{(1 + b_1 \bar{x})^2} - d\bar{y} & -a_2 \bar{y} \bar{z} & -d_2 \\ 0 & \frac{a_2 \bar{y}}{(1 + b_2 \bar{y})^2} & -d_2 \end{bmatrix}
$$

The characteristic roots of the Jacobian matrix $J_2$ are $(\frac{a_2 \bar{y}}{1 + b_2 \bar{y}} - d_2)$ and the roots of the equation

$$
\lambda^2 + \lambda \left( \bar{x} - \frac{a_1 b_1 \bar{x} \bar{y}}{(1 + b_1 \bar{x})^2} + d\bar{y} \right) + d\bar{y} \left( \bar{x} - \frac{a_1 b_1 \bar{x} \bar{y}}{(1 + b_1 \bar{x})^2} \right) + \frac{a_2^2 \bar{x} \bar{y}}{(1 + b_1 \bar{x})^2} = 0
$$

It is clear that $E_2$ is stable if $\left( 1 - \frac{a_1 b_1 \bar{y}}{(1 + b_1 \bar{x})^2} \right) > 0$ i.e. $(1 + b\bar{x})^2 > a_1 b_1 \bar{y}$ and $\frac{a_2 \bar{y}}{1 + b_2 \bar{y}} - d_2 < 0$

i.e. $R_{02} < 1$ where $R_{02} = \frac{1}{d_2} \left( \frac{a_2 \bar{y}}{1 + b_2 \bar{y}} \right)$.

Hence the theorem.
**Theorem 2.** The interior equilibrium point \( E^*(x^*, y^*, z^*) \) of the system (2.3) is locally asymptotically stable if the following conditions hold:

\[
\frac{a_1 b_1 y^*}{(1 + b_1 x^*)^2} < 1 \quad \text{and} \quad \frac{1}{d} \left( \frac{a_2 b_2 z^*}{1 + b_2 y^*} \right) < 1
\]

**Proof.** The Jacobian matrix at the interior point \( E^*(x^*, y^*, z^*) \) is

\[
V = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\]

Where

\[
A_{11} = -x^* + \frac{a_1 b_1 x^* y^*}{(1 + b_1 x^*)^2}, \quad A_{12} = \frac{-a_1 x^*}{1 + b_1 x^*}, \quad A_{13} = 0
\]

\[
A_{21} = \frac{a_1 y^*}{(1 + b_1 x^*)^2}, \quad A_{22} = \frac{a_2 b_2 y^* z^*}{1 + b_2 y^*} - d y^*, \quad A_{23} = -d_2
\]

\[
A_{31} = 0, \quad A_{32} = \frac{a_2 z^*}{(1 + b_2 y^*)^2}, \quad A_{33} = 0
\]

The characteristic equation of Jacobian matrix is given by

\[
\lambda^3 + \sigma_1 \lambda^2 + \sigma_2 \lambda + \sigma_3 = 0
\]

where

\[
\sigma_1 = -(A_{11} + A_{22}), \quad \sigma_2 = A_{11} A_{22} - A_{23} A_{32} - A_{12} A_{21}, \quad \sigma_3 = A_{11} A_{23} A_{32}
\]

\[
\sigma_1 \sigma_2 - \sigma_3 = -A_{11} A_{22} + A_{11} A_{12} A_{21} - A_{11} A_{22}^2 + A_{22} A_{23} A_{32} + A_{22} A_{12} A_{21}
\]

The sufficient conditions for \( \sigma_1 > 0, \sigma_3 > 0 \) and \( \sigma_1 \sigma_2 - \sigma_3 > 0 \) are as follows:

\[
A_{11} < 0 \quad \text{and} \quad A_{22} < 0
\]

which implies the conditions

\[
\frac{a_1 b_1 y^*}{(1 + b_1 x^*)^2} < 1 \quad \text{and} \quad \frac{1}{d} \left( \frac{a_2 b_2 z^*}{1 + b_2 y^*} \right) < 1
\]

Thus, if the condition stated in the theorem holds, then all the Routh-Hurwitz criteria (i) \( \sigma_1 > 0 \) (ii) \( \sigma_1 \sigma_2 - \sigma_3 > 0 \) (iii) \( \sigma_3 > 0 \) are satisfied, and the system (2.3) is locally asymptotically stable around the positive equilibrium point.
4. Global Stability and Persistence

**Theorem 3.** The interior equilibrium \( E_3 \) is globally asymptotically stable if the conditions

(i) \( 1 - \frac{a_1 b_1 y^*}{M} > 0, \)

(ii) \( 1 - \frac{a_1 b_1 y^*}{M} \left( d - \frac{a_2 b_2 z^*}{M'} \right) - \left( \frac{a_1 b_1 x^*}{2M} \right)^2 > 0 \)

hold.

**Proof:**

We consider the following Lyapunov function

\[
V = A(x - x^* - x^* \ln \frac{x}{x^*}) + B(y - y^* - y^* \ln \frac{y}{y^*}) + C(z - z^* - z^* \ln \frac{z}{z^*})
\]

Taking the time derivative along the solution of (3) and after some simplification we get,

\[
\frac{dV}{dt} = -A \left( 1 - \frac{a_1 b_1 y^*}{M} \right) (x - x^*)^2 - B \left( d - \frac{a_2 b_2 z^*}{M'} \right) (y - y^*)^2 - \left( \frac{a_1 b_1 x^*}{2M} \right)^2 \]

where \( M = (1 + b_1 x)(1 + b_1 x^*) \) and \( M' = (1 + b_2 y)(1 + b_2 y^*) \)

Considering \( A=B=C=1 \) the right hand side of the above equation can be written as \(-L^TQL\), where \( L = (x - x^*)(y - y^*)(z - z^*) \) and the symmetric matrix \( Q \) is given by

\[
Q = \begin{bmatrix}
1 - \frac{a_1 b_1 y^*}{M} & \frac{a_1 b_1 x^*}{2M} & 0 \\
\frac{a_1 b_1 x^*}{2M} & d - \frac{a_2 b_2 z^*}{M'} & \frac{a_2 b_2 y^*}{2M'} \\
0 & \frac{a_2 b_2 y^*}{2M'} & 0
\end{bmatrix}
\]

If the symmetric matrix \( Q \) is positive definite, it can be concluded that \( \frac{dV}{dt} < 0 \). \( Q \) is positive definite if

(i) \( 1 - \frac{a_1 b_1 y^*}{M} > 0 \)

(ii) \( 1 - \frac{a_1 b_1 y^*}{M} \left( d - \frac{a_2 b_2 z^*}{M'} \right) - \left( \frac{a_1 b_1 x^*}{2M} \right)^2 > 0 \)

Hence the system around the interior equilibrium point is globally asymptotically stable.

From biological aspect, permanence of a system reveals information about the survival of all populations of the system in a future time. In Mathematics, permanence of a system means that strictly positive solutions do not have omega limit points on the boundary of the non negative cone.
Theorem 4. System (2.3) is permanent around the interior equilibrium point if the conditions

(i) $R_{01} > 1$
(ii) $R_{02} < 1$

hold.

Proof. We consider the average Lyapunov function of the form

$$V(x, y, z) = x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$$

where $\alpha_1$ and $\alpha_2$ are assumed positive and $\alpha_3$ is negative. We have

$$\frac{\dot{V}}{V} = \frac{\alpha_1}{x} \frac{dx}{dt} + \frac{\alpha_2}{y} \frac{dy}{dt} + \frac{\alpha_3}{z} \frac{dz}{dt}$$

$$= \alpha_1 [1 - x - \frac{a_1 y}{1 + b_1 x}] + \alpha_2 [\frac{a_1 x}{1 + b_1 x} - \frac{a_2 z}{1 + b_2 y} - d_1 - dy] + \alpha_3 [\frac{a_2 y}{1 + b_2 y} - d_2]$$

For equilibrium the following conditions should be satisfied

$$\frac{\dot{V}}{V}(E_1) = \alpha_2 [\frac{a_1}{1 + b_1} - d_1] + \alpha_3 [-d_2] > 0$$

$$\frac{\dot{V}}{V}(E_2) = \alpha_3 [\frac{a_2 y}{1 + b_2 y} - d_2] > 0$$

if the conditions

(i) $R_{01} > 1$
(ii) $R_{02} < 1$

hold.

Under these conditions system is permanent around the interior equilibrium point.

5. Numerical results and discussion

In this section we perform extensive numerical experiments to observe the role density dependent mortality of intermediate predator on the dynamics of tri-trophic food chain. For this we have taken a set of parameter values $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $d = 0.002$ and for this set of parameter values we observe the time series chaotic aperiodic solution (Figure(1)).

Hastings and Powell [14] already observed stable focus, limit cycle, period-doubling and chaotic dynamics of the system by changing half saturation constant $b_1$. Our aim is first to observe the exchange of states (stability-limit cycle-period doubling-chaos) as shown in HP model for different values of $b_1$ and subsequently to observe the system under the variation of density dependent mortality in intermediate predator $d$. Thus we will first revisit the dynamics of the system for variation half saturation constant $b_1$.

From Figure(2) we observe that system shows chaotic dynamics for $b_1 = 2.9$ and we also observe the trajectory of this chaotic dynamics in phase plane. This chaotic dynamics in phase
Figure 1. Figure depicts the chaotic dynamics of the system (2.3) for $a_1 = 5.0$, $a_2 = 0.1$, $b_1 = 3.0$, $b_2 = 2.0$, $d_1 = 0.4$, $d_2 = 0.01$, $d = 0.002$.

Figure 2. Figure shows the tea-cup chaotic attractor in phase plane for $b_1 = 2.9$ and other parameter values given in the Figure 1.
plane takes a shape of teacup attractor. To observe the trajectory behaviour of our system around the different equilibrium points for different values of different key parameters, we draw phase diagram of trajectory.

![Phase diagram of trajectory](image_url)

**FIGURE 3.** Figure indicates the motion of trajectory of the system(2.3) around the different equilibrium points corresponding to Figure 1.

From Figure (3), we have observed chaotic trajectory around the interior equilibrium point and the motion of trajectories around the other equilibrium points.

To observe a clear picture we draw a bifurcation diagram for $b_1$ (Figure(4)) and from this diagram we observe that system enters into chaotic dynamics from stable focus. We observe that for $b_1 < 2.1$ the system is stable around the positive steady state and for $2.1 \leq b_1 < 2.3$ it shows limit cycle oscillations. The period doubling is observed at $b_1 = 2.3$. Finally chaotic dynamics is observed for $2.3 < b_1 \leq 3.0$. The dynamics within the attractor is given roughly as follows. Starting in the "handle" of the "teacup", the system moves to the wide part of the teacup and then spirals along the teacup to the narrow end, entering the handle again. In terms of the species behavior, the top predator $Z$ crashes, allowing wide swings in the population levels of $X$ and $Y$. As $Z$ increases in numbers, the swings in $X$ and $Y$ become damped, until $Z$ causes the levels of $Y$ to crash. This leads to a crash in $Z$ and an outbreak in $X$, starting the process again. The sequence of events in terms of species numbers, always follows the same general pattern. What is unpredictable is the timing. One way to express this is that the time between crashes of species $Z$ varies in an erratic fashion. Also, the number of peaks in species $Y$ between major crashes varies and the population size at the peaks varies. Trajectory behaviour around the equilibrium points will be changed for variation of parameter $b_1$. These variations are observed in Figure(5).
From Figure(5a) we have observed limit cycle trajectory around interior equilibrium point and all trajectories around other equilibrium points will settle down into the limit cycle oscillations. From Figure(5b), it is noticed that if we increase the value of $b_1$, limit cycle trajectory behaviour goes into period-doubling trajectory around the interior equilibrium point and other trajectories around the other equilibrium points will converge to period-doubling trajectory. From Figure(5c) we have noticed that the position of different equilibrium points and behaviour of trajectories for further increment of the value of $b_1$. From this figure we have observed chaotic trajectory around interior equilibrium point and other trajectories around the other equilibrium points will flow into chaotic trajectory.

From Figure(6a), it is observed the time series dynamics of Lyapunov exponents and a positive Lyapunov exponent. This positive Lyapunov exponent indicates chaos. Figure(6b) exhibits Poincare map of typical chaotic region on the plane x=0.5. The scattered distribution of the sampling points implies the chaotic behaviour of the system.

In our present study we observe that system shows chaotic dynamics in absence of density dependent mortality in intermediate predator. The terms chaos, strange attractor and fractal are becoming familiar to many, if not all, ecologists [36]. The key feature of chaotic dynamics is
the sensitive dependence on initial conditions. Even a very small change in initial conditions can lead to different results. Indeed, the divergence between results grows exponentially in time for virtually all pairs of starting conditions. Several early mathematical investigations of chaos were of ecological models [27]. An investigation by Gilpin [12] showed that a system of one predator and two competing prey can exhibit chaotic behavior. A number of simple ecological and epidemiological systems with seasonality in contact rates unequivocally demonstrate chaos [37]. Schaffer and Kot [38] and Olsen et al. [29] show that measles in New York, Baltimore and Denmark may be a specific example of this behavior. Tri-trophic food chain model suggests that chaotic behaviour may be much more common in natural systems with interacting producers and consumers (Predators and prey) than Hassel et al. [16] predicted on the basis of
one species discrete time models. It is our interpretation that chaos ultimately arises in this food chain model because of the tendency for predator-prey systems to oscillate. One predator-prey subsystem, for concreteness say $X$ and $Y$ oscillates at one frequency, while another, $Y$ and $Z$ oscillates at a different frequency; the frequencies are determined by the model parameters. In particular, the interaction at the higher trophic levels has a longer natural period because the average lifetime of the top predators is longer than the average lifetime of the consumers at the lower trophic levels. The two systems are coupled through species $Y$ because the predator in one is the prey in the other. We conjecture that when the period of one oscillation is not some multiple of the other frequency (i.e., the frequencies are incommensurate) chaos arises in a manner similar to the occurrence of chaos in periodically forced oscillations. Food webs depict a complex net coupled producer-consumer interactions; accordingly one should not be surprised to find many subsystems within a food web going through the oscillations that are a common feature of all predator-prey systems. It seems unlikely that the frequencies of all the linked subsystems will be commensurate. Accordingly, one expects that at least some, perhaps most of the linked predator-prey subsystems may be simultaneously undergoing chaotic oscillations. It is instructive to compare this view with the three species study (a predator consuming two competing prey) of Gilpin [12], in which chaotic behavior was identified. Typically, one does not associate oscillations with competition models based on results from two species Lotka-Volterra competition [26]. However, once three or more species are allowed, oscillations can occur [28, 30]. We conjecture that food webs containing a number of competing species can also go chaotic in the same way as our consumer-producer food chain does.

To observe the effect of density dependent mortality in intermediate predator, we increase the density dependent mortality parameter $d$ keeping the half-saturation constant $b_1$ fixed at $b_1 = 2.9$. For $d = 0.05$ the system shows chaotic dynamics (Figure(7a)). For $d = 0.08$
the system shows periodic solutions with period four (Figure(7b)) and for $d = 0.1$ the system shows periodic solutions with period three (Figure(7c)). For $d = 0.17$ the system enters into period doubling position from periodic solutions of period three and very famous teacup chaotic attractor is disappeared (Figure(7d)). This shows that the presence of density dependent mortality in intermediate predator prevents the system from entering into chaotic region. The
system goes into the limit cycle oscillation from period-doubling for \( d = 0.24 \) (Figure 7(e)). The system goes into the steady stable position from limit cycle for \( d = 0.5 \) (Figure 7(f)).

![Bifurcation Diagram](image)

**Figure 8.** Figure indicates the bifurcation diagram for \( d \in [0.002, 0.6] \) and also indicates that the system (2.3) ultimately settles down into chaotic dynamics after passing through different dynamics like stable, limit cycle and period-doubling for variation of \( d \) and other parameter values given in Figure 1.

To make it more clear we plot the bifurcation diagram for all the populations with \( d \) as the bifurcating parameter (Figure 8). We observe that the system shows chaotic dynamics in the range \( 0 < d < 0.08 \), periodic solutions of period four in the range \( 0.08 \leq d < 0.1 \), periodic solutions of period three in the range \( 0.1 \leq d < 0.17 \), period-doubling in the range \( 0.17 \leq d < 0.24 \), limit cycle in \( 0.24 \leq d < 0.5 \) and finally system settles down to steady solutions, a stable situation for \( d \geq 0.5 \) keeping other parameters fixed. So it is clear that increasing the strength of density dependent mortality up to a certain level reduce the prevalence of chaos, period-doubling and limit cycle oscillations. This observations indicate that the density dependent mortality factor in intermediate predator could be used as a biological control parameter for persistence of the species. To observe the motion of trajectory around the
different equilibrium points for variation of density dependent mortality parameter, we draw phase diagram of trajectory.

**Figure 9.** (a) Figure indicates the motion of trajectory of the system (2.3) around the different equilibrium points for $d = 0.2$, (b) Figure shows the positions of different equilibrium points for $d = 0.4$, (c) Figure depicts the dynamics of the system (2.3) around the different equilibrium points for $d = 0.55$ and other parameter values given in the Figure 1.

From Figure (9a) we have noticed period-doubling trajectories around the interior equilibrium point and all trajectories around the other equilibrium points go into period-doubling. If we increase the value of density dependent mortality parameter, we have found period-doubling trajectory settles down into limit cycle oscillation around the interior equilibrium point and the position and motion of trajectories around the other equilibrium points (Figure 9b). If we further increase the value of density dependent mortality parameter, the trajectory shows stable focus around the interior equilibrium point and all trajectories around the other equilibrium points converge to stable focus (Figure 9c).
In this paper we have studied the dynamics of HP model by including density dependent mortality in intermediate predator. We first observe the dynamics of original HP model and then we observe the role of density dependent mortality in intermediate predator in controlling chaotic dynamics. In original HP model we observe the chaotic dynamics. Chaotic oscillations is common in food chain models. Oscillating resident communities are particularly worthwhile to study, because prey-predator (or host-parasitoid, plant-herbivore, consumer-resource) systems are well known examples of inherently fluctuating populations [39]. They represent interesting and relevant scenarios in biological control and resource management, because recurrent outbreaks in both terrestrial and aquatic ecosystems pose central problems in ecology (Insect outbreaks and algal blooms, see [1, 8, 17, 19]). Oscillations are also a concern of biological conservation, because populations could reach such small abundances that they are likely to go extinct [32]. However we observe that chaotic oscillations in tri-trophic food chain model is common and control of such oscillations is utmost important from ecological point of view. Our analysis shows that food chain model shows chaotic dynamics in absence of density dependent mortality in intermediate predator. The chaotic oscillatory behavior due to enhancement of half saturation constant can be prevented by increasing the density dependent mortality in intermediate predator. Now we explain the biological reasons of stabilization of chaos for increasing the density dependent mortality. When we increase the density dependent mortality in intermediate predator, the prey species increases due to decrease in intermediate predator and top predator decreases due to low level of food availability. Such type population pressure stabilizes the chaotic dynamics. Thus we may conclude that density dependent mortality in intermediate predator may be used as a biological control for the persistence of the system.

Now we compare our model and results with most of earlier studies. Kooi et al. [21] considered a tri-trophic food chain model consisting of a prey consuming a nutrient, a predator consuming a prey and an omnivore which preys on the predator and the prey. There were two limiting cases where the omnivore is a specialist; a food chain where the omnivore does not consume the prey and competition where the omnivore does not prey on the predator. They used bifurcation analysis to study the long-term dynamic behavior for various degrees of omnivory and observed that attractor can be equilibria, limit cycles or chaotic behavior depending on the control parameters of the chemostat. They also analyzed how the trophic structure of the food web evolves following invasion where a new invader is introduced one at the time. Generally, with an invasion, the invader settles itself and persists with all other species, however, the invader may also replace another species. They observed that the food web model has a global bifurcation, being a heteroclinic connection from a saddle equilibrium to a limit cycle of saddle type. This global bifurcation separates regions in the bifurcation diagram with different attractors to which the system evolves after invasion. Vandermeer [41] considered a tri-trophic food chain model and introduced the concept of omnivory. They observed that omnivory may be either stabilizing or destabilizing depending in part, on the conditions of the parent systems from which it derives. It was further conjectured that the tri-trophic to competition gradient cannot be entirely stable that there must be an instability at some level of intermediate omnivory. In the present study we consider HP model and introduce density dependent mortality in intermediate predator. We observe chaotic dynamic in absence of density dependent mortality
and we also observe that density dependent mortality reduces the chaotic dynamics. In recent time Aziz-Alaouï [3] considered a tri-trophic food-chain model with modified Leslie-Gower type growth rate for top-predator population and established the chaotic dynamics exhibited by the model system for a certain choice of parameter values. Bandyopadhyay et al. [5] had modified the said model by incorporating density dependent death rate for predator population. Their findings revealed the fact that there were two coexisting equilibrium points one of which was a source and the other one was a sink. The positive equilibrium point which was sink is actually globally asymptotically stable under certain parametric conditions. They showed that the model system were capable to produce chaotic dynamics when the rate of intra specific completion was very low and chaotic dynamics disappeared for a certain value of the rate of intra specific completion for predator species. Their results suggested that the consideration of density dependent death rate for predator species had the ability to control the chaotic dynamics.

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