A Stabilizing Scheme for the Dynamic Analyses of Slack Cables

Yeun Chul Park [1] and Hae Sung Lee[2]

1) *Korea Bridge Design & Engineering Research Center, Seoul 151-742, KOREA*
2) *Department of Civil, Urban and Geosystem Engineering, Seoul National University, Seoul151-742, Korea*

Corresponding Author: Hae Sung Lee, chslee@snu.ac.kr

ABSTRACT

An numerical method is presented for the dynamic analysis of slack cables. The numerical solution strategy is based on finite element approximation of differential equation. For perfectly flexible cable, numerical instability occurs by negative tension. We can obtain stable solution for slack cables by combining beam element and cable element as the stabilizing scheme. Numerical examples that involve free vibration with initial displacement is presented to verify validity of proposed method.

INTRODUCTION

Most applications call for a taut cable because it has very desirable features as a structural mechanism [2, 3, 4, 5]. There are, however, other applications where it is desirable to have cables under very low tension: for example, small tethered under water vehicles, marine neutrally buoyant cables supporting hydrophones, and conductor cable connecting electronic equipments. The behavior of slack tension cables is intrinsically non-linear and faced with the distinct possibility of negative tension. Irrespective of the question of what the actual physical behavior of a cable is when the tension becomes negative, and we must also address the question of how the specific model employed behaves in such situation.

DYNAMIC EQUILIBRIUM E EQUATION OF CABLES

We consider a perfectly flexible cable under self weight $\mathbf{q} = (0, q)$ and distributed forces of each direction $\mathbf{f} = (f_x\ ,\ f_y)$ and derive the equation of motion of the cable. Let T denotes the tension of cable during vibration, p the strained length of cable, $\mathbf{x} = (x\ ,\ y)$ the position of a particle on the cable, ρ mass per unit length and $\mathbf{F} = (F_x\ ,\ F_y)$ the end force. The Lagrangian co-ordinate along the cable length is given by unstrained length s. Therefore,

$$T\frac{d\mathbf{x}}{dp} + \mathbf{F} + \mathbf{q}s - \int_0^s \rho\ddot{\mathbf{x}}ds + \int_0^s \mathbf{f}ds = 0 \tag{1}$$

together with the relation

$$p(s) = \int_0^s ((\frac{dx}{ds})^2 + (\frac{dy}{ds})^2)^{0.5} ds \tag{2}$$

By the small deformation assumption, the strain of cable is defined as follows,

$$\varepsilon = \frac{dp^2 - ds^2}{2ds^2} \cong \frac{dp - ds}{ds} = \frac{dp}{ds} - 1 \tag{3}$$

where ε is axial strain of cable. The tension along the cable is derived by the shape of cable.

$$T(s) = EA\varepsilon = EA(\frac{dp}{ds} - 1) = EA(x'^2(s) + y'^2(s))^{0.5} - 1) \tag{4}$$

where EA is the axial stiffness of cable and $(\ \ )'$ denotes the differential with respect to the unstrained cable length s. By using equation (3) and (4), we can derive the derivative of s with respect to p.

$$T(s) = EA\varepsilon = EA(\frac{dp}{ds} - 1) = EA(x'^2(s) + y'^2(s))^{0.5} - 1) \tag{5}$$

After differentiate equation (1) with respect to s and substitute equation (5) into equation (1), the equation of motion becomes

$$\frac{d}{ds}(\frac{T}{1 + T/EA}\frac{d\mathbf{x}}{ds}) + (-\rho\ddot{\mathbf{x}} + \mathbf{q}) + \mathbf{f} = 0 \tag{6}$$

By multiplying virtual position and integrating over the unstrained length, we may obtain the weak form of equation of motion of cable.

$$\int_{l_0} \delta\mathbf{x}(\frac{d}{ds}(\frac{T}{1 + T/EA}\frac{d\mathbf{x}}{ds}) - \rho\ddot{\mathbf{x}} + \mathbf{q} + \mathbf{f}) = 0 \tag{7}$$

Final expression is obtained from Equation (7) by integration by parts. If we assume that both ends of cable are supported by supports, virtual positions of both ends are always zero. Therefore, the boundary terms are zero.

$$\int_{l_0} (\delta\mathbf{x})^{\mathrm{T}}\ddot{\mathbf{x}}ds + \int_{l_0} (\delta\mathbf{x}')^{\mathrm{T}}\frac{T}{1 + T/EA}\mathbf{x}'ds = \int_{l_0} (\delta\mathbf{x})^{\mathrm{T}}(\mathbf{q} + \mathbf{f})ds \tag{8}$$

As it mentioned above, the cable tension is the function of the shape of cable. Thus, the equation of motion is the non-linear equation on the position variable of cable. We employ Newton-Raphson iteration scheme to calculate the solution of the weak form (8) at time $t + \Delta t$. To derive the linearized incremental form of weak form, we define the incremental form of the position variable and tension.

$$\begin{aligned} \mathbf{x}_k^{t+\Delta t} &= \mathbf{x}_{k-1}^{t+\Delta t} + \Delta\mathbf{x} = \underline{\mathbf{x}} + \Delta\mathbf{x} \\ T_k^{t+\Delta t} &= T_{k-1}^{t+\Delta t} + \Delta T = \underline{T} + \Delta T \end{aligned} \tag{9}$$

where k denotes k th iteration. According to 1st order Taylor expansion, the tension increment ΔT can be expressed in terms of position variables

$$\Delta T = T_k^{t+\Delta t} - T_{k-1}^{t+\Delta t} \approx \frac{EA}{1+\underline{T}/EA}(\underline{x}'\Delta x' + \underline{y}'\Delta y') \tag{10}$$

Using the above expression, we may obtain the linearized incremental form of equation (8)

$$\int_{l_0} \rho(\delta\mathbf{x})\mathbf{I}\Delta\ddot{\mathbf{x}}ds + \int_{l_0} \frac{d(\delta\mathbf{x})}{ds}\mathbf{D}_c\frac{d\Delta\mathbf{x}}{ds}ds = \int_{l_0}(\delta\mathbf{x})(\mathbf{f}^{t+\Delta t} - \rho\underline{\ddot{\mathbf{x}}})ds - \int_{l_0}(\delta\mathbf{x}')\frac{\underline{T}}{1+\underline{T}/EA}\underline{\mathbf{x}}'ds \tag{11}$$

where $\mathbf{I}$ and $\mathbf{D}_c$ are 2nd order unit matrix and tangential stiffness matrix of cable, respectively.

$$\mathbf{D}_c = \frac{\underline{T}}{1+\underline{T}/EA}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{EA}{(1+\underline{T}/EA)^3}\begin{bmatrix}\frac{d\underline{x}}{ds}\frac{d\underline{x}}{ds} & \frac{d\underline{x}}{ds}\frac{d\underline{y}}{ds}\\ \frac{d\underline{x}}{ds}\frac{d\underline{y}}{ds} & \frac{d\underline{y}}{ds}\frac{d\underline{y}}{ds}\end{bmatrix} \tag{12}$$

The position variable $\mathbf{x}$ is then interpolated in the shape function $\mathbf{N}^e(s)$ according to

$$\mathbf{x} \cong \sum_e \mathbf{N}^e\mathbf{X}^e \tag{13}$$

Upon introducing the spatial discretization (13) of $\mathbf{x}$ into incremental form (11), we obtain the discrete equation of motion in matrix form

$$\mathbf{M}_c\Delta\ddot{\mathbf{X}} + \mathbf{K}_c\Delta\mathbf{X} = \Delta\underline{\mathbf{f}} \tag{14}$$

where $\Delta\mathbf{X}$ and $\Delta\ddot{\mathbf{X}}$ are the incremental change of nodal position and increment of nodal acceleration, respectively, $\mathbf{M}_c$ is the constant mass matrix, $\mathbf{K}_c$ is discretized tangential stiffness matrix, and $\Delta\underline{\mathbf{f}}$ is the unbalanced force.

STABILIZTTION SCHEME

The cable model derived previous section has serious numerical problem when we face with the distinct possibility of zero or negative tension. If we rewrite equation (12), we may obtain the case that negative tension make the stiffness matrix lose ellipticity.

$$\varepsilon < -\cos^2\theta \text{ or } \varepsilon < -\sin^2\theta \tag{15}$$

where θ denotes the angle of tangent at the particle on the cable. The situation in equation (15) occurs more frequently for the case of horizontal cable or inclined cable with small angle. This situation induced by the ellipticity loss makes computational instability. To recover ellipticity, as a stabilizing term, bending stiffness and rotational inertia of cable should be considered which we used to ignore for the perfectly flexible cable.

To consider bending stiffness and rotational inertia, rotational degree of freedom has to be considered but it is not defined in the cable model. If we combine cable element and beam element (figure 1), it is easy to define rotational degree of freedom. As a beam element, Bernouli beam is applied for the beam element. The equilibrium equation of motion of beam element is given by

$$\left\{\rho_b \frac{\partial^2 w}{\partial t^2} - I_o \frac{\partial^2}{\partial \xi^2}\left(\frac{\partial^2 w}{\partial t^2}\right)\right\} + \frac{\partial^2}{\partial \xi^2}\left(EI \frac{\partial^2 w}{\partial \xi^2}\right) = 0 \tag{16}$$

where ρ_b =mass per unit length which is equivalent to the cable; t =time; w =vertical displacement along the local co-ordinate of beam; I_o =mass moment of inertia of the beam element for rotation; ξ =horizontal distance from one end along the local co-ordinate; and EI =bending stiffness (Figure 2). The weak form of beam element can be obtained by multiplying virtual displacement δw by equation (16) and integrating over the length of element. Final expression is obtained by integration by parts twice.

$$\begin{aligned} &\int_l (\delta w)\rho_b \ddot{w}\, d\xi + \int_l \frac{\partial(\delta w)}{\partial \xi} I_o \frac{\partial \ddot{w}}{\partial \xi} d\xi + \int_l \frac{\partial^2(\delta w)}{\partial \xi^2} EI \frac{\partial^2 w}{\partial \xi^2} d\xi = \\ &\delta w(0)V(0) + \delta w(l)V(l) + \delta\theta(0)M(0) + \delta\theta(l)M(l) = \delta \mathbf{d}^e \cdot \mathbf{f}_b^e \end{aligned} \tag{17}$$

where θ =rotation; $\delta \mathbf{d}^e$ =end displacement; $\mathbf{f}_b^e$ =end force; V and M =shear force and moment. Displacement w is then interpolated in the shape function $\mathbf{N}_b$ according to

$$w = \mathbf{N}_b \cdot \mathbf{d}^e \tag{18}$$

Satisfying all virtual displacement in Equation (17), we can rewrite equation (17) as follows

$$\begin{aligned} \mathbf{f}_b^e &= \int_l \mathbf{N}_b \rho_b \mathbf{N}_b d\xi \ddot{\mathbf{d}}^e + \int_l \frac{\partial \mathbf{N}_b}{\partial \xi} I_o \frac{\partial \mathbf{N}_b}{\partial \xi} d\xi \ddot{\mathbf{d}}^e + \int_l \frac{\partial^2 \mathbf{N}_b}{\partial \xi^2} EI \frac{\partial^2 \mathbf{N}_b}{\partial \xi^2} d\xi \mathbf{d}^e \\ &= \overline{\mathbf{m}}_b^e \ddot{\mathbf{d}}^e + \mathbf{m}_I^e \ddot{\mathbf{d}}^e + \mathbf{k}_b^e \mathbf{d}^e \end{aligned} \tag{19}$$

Since beam element is introduced only for rotational behavior, the shape function of the rigid body motion has to be excluded from the mass matrix $\overline{\mathbf{m}}_b^e$. New mass matrix of beam becomes

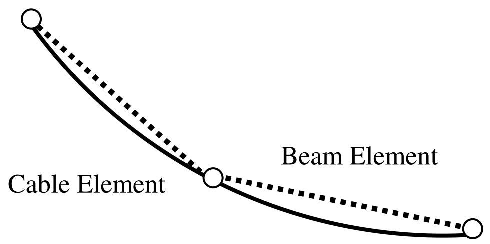

Figure 1. combination of cable and beam element

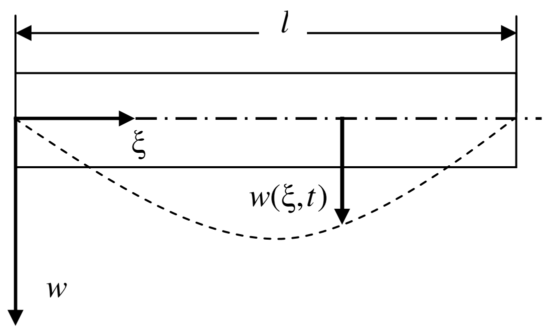

Figure 2. Beam element

$$\mathbf{m}_b^e = \int_l (\mathbf{N}_b - \mathbf{N}_r)\rho_b(\mathbf{N}_b - \mathbf{N}_r)d\xi \tag{20}$$

where $N_r = ((1-\xi/l_b^e),\ 0,\ \xi/l_b^e,\ 0)$. After substituting $\mathbf{m}_b^e$ into $\overline{\mathbf{m}}_b^e$ in equation (19), we may obtain new cable model including bending behavior by combining equation (14) and (19).

$$(\mathbf{M}_c + \mathbf{M}_b)\Delta\ddot{\mathbf{U}} + (\mathbf{K}_c + \mathbf{K}_b)\Delta\mathbf{U} = \Delta\underline{\mathbf{F}} \tag{21}$$

where $\mathbf{X}_k^{t+\Delta t} = \mathbf{X}^0 + \mathbf{U}_k^{t+\Delta t} = \mathbf{X}^0 + \mathbf{U}_{k-1}^{t+\Delta t} + \Delta\mathbf{U} = \mathbf{X}_{k-1}^{t+\Delta t} + \Delta\mathbf{U}$ or $\Delta\mathbf{X} = \Delta\mathbf{U}$.

Newmark's β method is used for time integration. We note that β =0.25 and τ =0.5 correspond to the trapezoidal rule; this choice of the parameter β and τ renders the algorithm unconditionally stable in the linear case and second order accurate [1].

NUMERICAL SIMULATION

Numerical simulation is concerned with the free vibration of horizontal cable with 100 m span length and 2m sag. The finite element mesh consists of 20 elements with linear isoparametric interpolation function for cable element and 40 elements with Hermitian interpolation function for beam element. Damping effect is neglected and 1/500 sec is selected for time interval. Initial displacement is around 0.5m at the center. Convergence criterion of the k -th iteration for the example is selected as follows

$$\varepsilon = \frac{\left\|\mathbf{U}_k^{t+\Delta t}\right\|}{\left\|\mathbf{U}^{t+\Delta t}\right\|} \le 10^{-9} \tag{22}$$

where $\mathbf{U}_k^{t+\Delta t}$ and $\mathbf{U}^{t+\Delta t}$ are the displacement of k -th iteration at time $t+\Delta t$ and total displacement at time $t+\Delta t$, respectively.

Figure 3 illustrates changing of the tension with 'unstable' cable model. As far as the tension maintain in the positive tension region, result shows stable condition but, after it goes to negative region, the tension becomes very unstable. On the other hand, when we consider bending stiffness, the stable result is achieved as shown in Figure 4. Subtle trembling during vibration is presumed that tension is influenced by high frequency transverse oscillation.

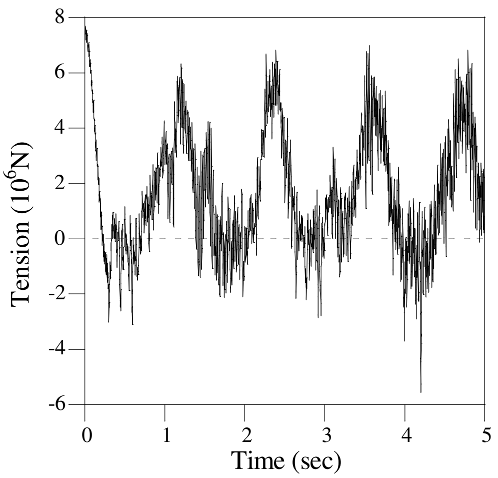

Figure 3. Tension with 'unstable' cable model.

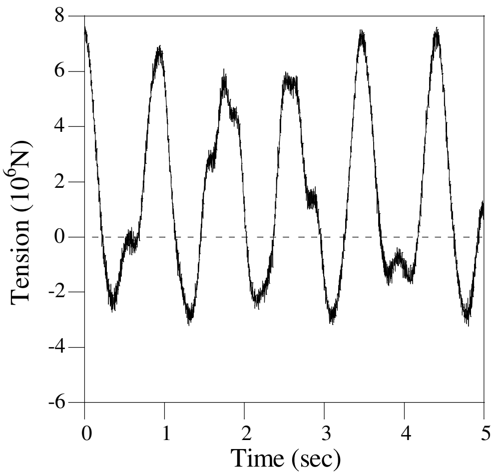

Figure 4. Tension with 'stable' cable model.

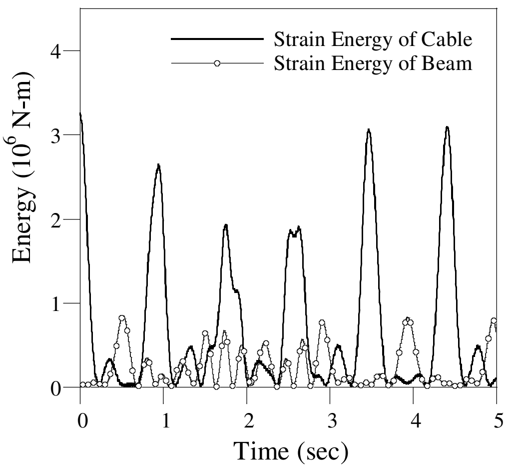

Figure 5. Strain energy.

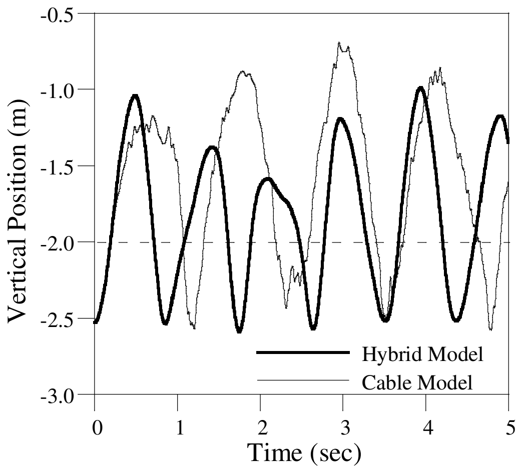

Figure 6. Vertical vibration at the center of cable.

As verifying the validity of basic concept of the stabilizing scheme, the strain energy of cable element and beam element is presented in Figure 5. In the Figure 5, the strain energy of the beam element contributes dominantly in the negative tension region. The strain energy of cable element, however, does not disappear in the negative tension region because negative tension absorbs the strain energy as a compression. Figure 6 illustrates the comparison between vertical vibration of 'stable' and 'unstable' cable model at the center of cable. 'stable' cable model presents periodicity. Due to ellipticity loss, 'unstable' cable model shows strong local oscillation and periodicity loss.

CONCLUSION

This paper deals with the modeling of slack cables. The equation of motion for perfectly flexible cable is derived in a global co-ordinate system, accounting for axial deformation. The tangent stiffness of the equation consists of cable tension and loses ellipticity when negative tension introduced. This situation induces numerical instability. Bending stiffness is considered as a stabilizing term by combining beam element and cable element. To verify proposed method, we demonstrated numerical example for slack cable with sag-span ratio 1/50. The result shows that bending stiffness successfully recovers ellipticity loss and stabilizes well.

REFERENCES

(1) Book

1. Humar, J. L., 1990, *Dynamics of Structures,* Prentice-Hall, Englewood Cliffs, NJ.

(2) Paper in a journal

2. Irvine, H.M., 1978, "Free Vibrations of Inclined Cables," *Proceedings of the American Society of Civil Engineers*, Vol. 104, No. ST2, February.
3. Ahn S. S. and Jang, S. P., 1998, "Free Vibration Characteristics of an Inclined Cable," *J KSCE*, Vol. 18, No. I-5, pp. 615-623.
4. Dastous, J.-B, and Pierre, J.-B. 1996, "Experimental Investigation on the Dynamic Behavior of Flexible Conductors between Substation Equipment during an Earthquake," *IEEE Transactions on Power Delivery*, Vol. 11, No. 2, April, pp.801-807.
5. Triantafyllou, M.S., 1984, "The Dynamics of Taut Inclined Cables," *Quarterly Journal of Applied Mathematics*, 37, pp. 421-440.