

# A DUAL ITERATIVE SUBSTRUCTURING METHOD WITH A PENALTY TERM

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## ABSTRACT

An iterative substructuring method with Lagrange multipliers is considered for the second order elliptic problem, which is a variant of the FETI-DP method. The standard FETI-DP formulation is associated with the saddle-point problem which is induced from the minimization problem with a constraint for imposing the continuity across the interface. Starting from the slightly changed saddle-point problem by addition of a penalty term with a positive penalization parameter  $\eta$ , we propose a dual substructuring method which is implemented iteratively by the conjugate gradient method. Performance of such a dual iterative substructuring method is directly connected with the condition number of a relevant dual system. For  $\eta = 0$ , the proposed method is reduced to the FETI-DP method. For the preconditioned FETI-DP with the optimal Dirichlet preconditioner, it is well-known that the condition number is bounded by a polylogarithmic factor:  $(1 + \log(H/h))^2$  in two dimensions and  $(H/h)(1 + \log(H/h))^2$  in three dimensions. To the contrary, in spite of the absence of any preconditioners, it is shown that the proposed method is numerically scalable in the sense that for a large value of  $\eta$ , the condition number of the resultant dual problem is bounded by a constant independent of both the subdomain size  $H$  and the mesh size  $h$ . We deal with a computational issue and present numerical results. Furthermore, we extend the proposed method to the three dimensional problem.

## INTRODUCTION

We consider the following Poisson model problem with the homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^d$  with  $d = 2, 3$  and  $f$  is a given function in  $L^2(\Omega)$ . For simplicity, we assume that  $\Omega$  is partitioned into two nonoverlapping subdomains  $\{\Omega_i\}_{i=1}^2$  such that  $\bar{\Omega} = \bigcup_{i=1}^2 \bar{\Omega}_i$ . It is well-known that the problem (1) is equivalent to the constrained minimization as

$$\min_{\substack{v_i \in H^1(\Omega_i) \\ v_i = 0 \text{ on } \partial\Omega \cap \partial\Omega_i \\ v_1 = v_2 \text{ on } \partial\Omega_1 \cap \partial\Omega_2}} \sum_{i=1}^2 \left( \frac{1}{2} \int_{\Omega_i} |\nabla v_i|^2 dx - \int_{\Omega_i} f v_i dx \right). \tag{2}$$

In the domain-decomposition approach, a key point is how to convert the constrained minimization problem into an unconstrained one. Most studies for treatment of constrained minimizations

started in the field of optimal control problem. There are three most popular methods developed for different purposes; the Lagrangian method, the method of penalty function, the augmented Lagrangian method. Such various ideas have been introduced for handling constraints as the continuity across the interface in (2) (see [4–6]). The FETI-DP method is one of the most advanced dual substructuring methods, which introduces Lagrange multipliers to enforce the continuity constraint by following the Lagrangian method and solves the resultant dual problem from the process of seeking a saddle-point of the relevant Lagrangian functional.

In this paper, we propose a dual iterative substructuring algorithm which deals with the continuity constraint across the interface in view of the augmented Lagrangian method. To the Lagrangian functional, we add a penalty term which measures the jump across the interface and includes a positive penalization parameter  $\eta$ . In the same way as in most dual substructuring approaches, the saddle-point problem related to the augmented Lagrangian functional is reduced to the dual problem with Lagrange multipliers as unknowns. Then we solve it by the conjugate gradient method (CGM). Many studies for the augmented Lagrangian method have been done in the frame of domain-decomposition techniques which belong to families of nonoverlapping Schwarz alternating methods, variants of FETI method, etc. (cf. [1,3,6,7])

When focusing on the iterative routine of CGM, a key indicator of the efficiency of a domain decomposition method is the bound on the growth of the condition number of the relevant system when the mesh is refined and the number of subdomains increases. We prove that the proposed method is scalable in the sense that for a large  $\eta$ , the condition number of the relevant dual system  $F_\eta$  has a constant bound which is independent of the subdomain size  $H$  and the mesh size  $h$ , unlike most efficient iterative substructuring methods including FETI-DP where the polylogarithmic growth  $(1 + \log(H/h))^2$  in the condition number is observed. To the best of our knowledge, the algorithm with such a constant bound of condition number is unprecedented according to the previous studies in the field of domain decomposition. Furthermore, it is noticeable that in spite of the absence of any preconditioner, the proposed method has a strong scalability while the condition number of the FETI-DP method with the optimal preconditioner is bounded by a polylogarithmic factor.

In addition, we treat computational issues in practical sense. In fact, an extremely large value of  $\eta$  is not necessary for either improvement of accuracy or speed-up of iterative solver. But, in order to avoid trouble in choosing a properly large  $\eta$ , we check how the proposed method behaves in practice as  $\eta$  increases. Unfortunately, in the process of CGM implementation, we solve a system  $K_{rr}^\eta$  which becomes more ill-conditioned in proportion to increment of  $\eta$ . To increase the ease of use and the practical efficiency of the presented method, an optimal preconditioner  $M$  with respect to  $\eta$  is developed. Based on the fact that the proposed method is a variant of the FETI-DP method, we compare two methods from many perspectives such as the conditioning of the dual system, the CG iteration number for convergence, and the virtual wall clock time. According to the numerical results, the presented method is superior to the preconditioned FETI-DP by the optimal Dirichlet preconditioner.

Finally, we suggest an extension of the proposed method for the three dimensional problem with a modified penalty term. The well-designed penalty term in three dimensions makes the extended method keep the strong scalability, a constant bound on the condition number of the dual system.

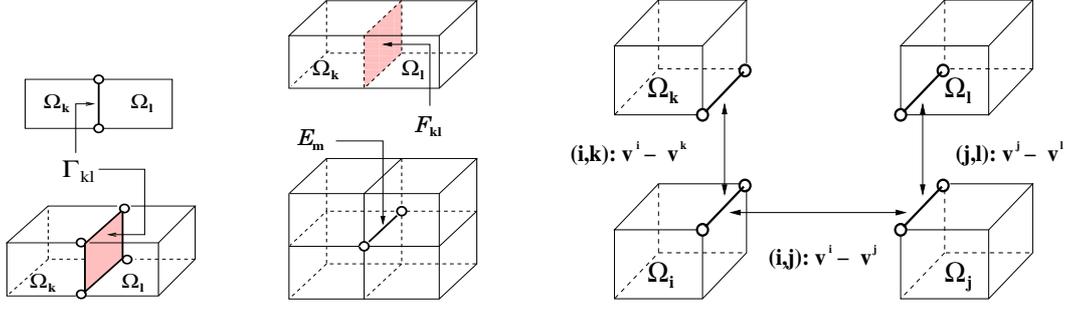


Figure 1. Left: Interface  $\Gamma_{kl}$  in 2D and 3D. Center: 3D objects: a face  $F_{kl}$  and an edge  $E_m$ . Right: Choice of three pairs of adjacent subdomains which share an edge.

## UNDERSTANDING OF MAIN IDEA

The proposed method is developed through the process of adding a penalty term to the standard FETI-DP formulation where the continuity constraint is imposed in terms of the pointwise matching. Since the augmented term penalizes the jump across the interface, it is obvious that one of them is redundant in highlighting only the aim of treatment of continuity constraint. The combination of dualization and penalization is purposed to have the best of both methods. Unlike most penalty methods, our method guarantees the convergence of solution without making the penalization parameter  $\eta$  large since it is in company with dualization process. Moreover, the penalty term accelerates the convergence of the dual iterative solver. The strongest point of the proposed method is the constant bound on the condition number of the dual system, which is independent of  $H$  and  $h$ . It mainly results from the addition of the penalty terms which measure the jump across the interface  $\Gamma$  in appropriate ways. In this view, we shall take a closer look at the augmented penalty terms both in two dimensions and in three dimensions.

Let  $\Gamma_{kl}$  be the interface shared by adjacent subdomains  $\Omega_k$  and  $\Omega_l$  (see Fig. 1 for detailed geometrical description). The penalty term  $J_\eta$  in 2D is defined as

$$J_\eta(u, v) = \sum_{k < l} \frac{\eta}{h} \int_{\Gamma_{kl}} (u^k - u^l)(v^k - v^l) ds, \quad \eta > 0. \quad (3)$$

Since the degrees of freedom at subdomain corners are shared by neighboring subdomains, the penalty term in (3) plays a role in decreasing the jump on each edge node except vertices. We can easily extend the presented method to the three dimensional problem by adopting the same penalty term as in (3), that is, by penalizing the jump on  $\Gamma_{kl}$  where each  $\Gamma_{kl}$  is regarded as a common face including edge nodes except vertices. But, due to the coupling between face nodes and edge nodes in  $\Gamma_{kl}$ , the same penalty term makes an algebraic structure of the 3D algorithm somewhat complicated so that the algorithm becomes less efficient in practical sense. In this content, by considering the interface as a union of two separate objects: faces and edges, we introduce a modified penalty term

$$J_\eta(u, v) = \eta(J_{\mathcal{F}}(u, v) + J_{\mathcal{E}}(u, v)), \quad \eta > 0, \quad (4)$$

where

$$J_{\mathcal{F}}(u, v) = \frac{1}{h} \sum_{k < l} \int_{F_{kl}} (u_{F_{kl}}^k - u_{F_{kl}}^l)(v_{F_{kl}}^k - v_{F_{kl}}^l) dx$$

and

$$J_{\mathcal{E}}(u, v) = \sum_{E_m} \sum_{(i,j) \in I_{E_m}} \int_{E_m} (u^i - u^j)(v^i - v^j) ds.$$

Here,  $u_{F_{kl}}^k$  is a part of  $u$ , which is related to the contribution to  $u^k$  on a face  $F_{kl}$  only from the face nodal basis functions except the edge nodal basis functions.  $I_{E_m}$  is the set of indices of subdomain pairs which have an edge  $E_m$  in common.

In addition, we should pay attention to the weights  $h^\alpha$ . In general view of finite element analysis, it might be essential to maintain a balance between the penalty term and the energy norm, which guarantees the convergence of a finite element approximate solution. But, the proposed method ensures convergence even if we use different weights in (3) and (4). In fact, the weights in (3) and (4) play a major role in making the condition number of the relevant dual system bounded by a constant independent of the subdomain size  $H$  and the mesh size  $h$ .

## DUAL ITERATIVE SUBSTRUCTURING METHOD

In this section, we propose a dual iterative substructuring algorithm which deals with the continuity constraint across the interface in view of the augmented Lagrangian method. For simplicity, we present the main theoretical results only for the two dimensional problem.

To the Lagrangian functional in the FETI-DP method, we add the penalty term  $J_\eta$  suggested in the previous section. Finding a finite element solution for (1) is equivalent to seeking the saddle-point for the augmented Lagrangian. The saddle-point problem for the augmented Lagrangian functional is written in an algebraic form

$$\begin{bmatrix} K_{rr}^\eta & K_{rc} & B_r^T \\ K_{rc}^T & K_{cc} & 0 \\ B_r & 0 & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_c \\ \lambda \end{bmatrix} = \begin{bmatrix} f_r \\ f_c \\ 0 \end{bmatrix}, \quad (5)$$

where  $u_c$  denotes the degrees of freedom (dof) at the subdomain corners,  $u_r$  the remaining of dof, and  $\lambda$  the Lagrange multipliers introduced for imposing the continuity constraint across the interface. Eliminating  $u_r$  and  $u_c$  in (5) yields the system for the Lagrange multipliers

$$F_\eta \lambda = d_\eta \quad (6)$$

where

$$F_\eta = F_{rr} + F_{rc} F_{cc}^{-1} F_{rc}^T, \quad d_\eta = d_r - F_{rc} F_{cc}^{-1} d_c$$

with

$$F_{rr} = B_r (K_{rr}^\eta)^{-1} B_r^T, \quad F_{rc} = B_r (K_{rr}^\eta)^{-1} K_{rc}, \quad F_{cc} = K_{cc} - K_{rc}^T (K_{rr}^\eta)^{-1} K_{rc}.$$

Noting that  $F_\eta$  is symmetric positive definite, we solve the resultant dual system (6) iteratively by the conjugate gradient method. Hence, the key issue is to provide a sharp estimate for the condition number of  $F_\eta$ .

**Theorem 1** *Assume that a triangulation  $\mathcal{T}_{h_k}$  on each subdomain  $\Omega_k$  is quasi-uniform. Then, for a sufficiently large  $\eta$ , there exists a constant  $C$  such that*

$$\kappa(F_\eta) \leq C,$$

where the constant  $C$  is independent of both of the mesh size  $h$  and the subdomain size  $H$ . In particular, assuming that each triangulation  $\mathcal{T}_{h_k}$  on  $\Omega_k$  is uniform,  $C = 3$ .

In the process of solving the dual system by CGM, we solve  $K_{rr}^\eta$  iteratively. To observe the practical efficiency of the proposed method, we shall estimate the condition number of  $K_{rr}^\eta$ . Note that

$$K_{rr}^\eta = K_{rr} + \eta \tilde{J} = \begin{bmatrix} A_{ii} & A_{ie} \\ A_{ie}^T & A_{ee} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix},$$

where  $\eta J$  is induced from the augmented penalty term  $J_\eta$ . Thanks to the specific type of discrete Sobolev inequality in Lemma 3.4 of [2], we get the following estimate.

**Theorem 2** *For each  $\eta > 0$ , we have that*

$$\kappa(K_{rr}^\eta) \lesssim \left(\frac{H}{h}\right)^2 \left(1 + \log \frac{H}{h}\right) (1 + \eta).$$

Theorem 2 shows how severely  $\eta$  deteriorates the property of  $K_{rr}^\eta$  as  $\eta$  is increased. Hence, it is expected that the large condition number of  $K_{rr}^\eta$  shown above may cause the computational cost relevant to  $K_{rr}^\eta$  to be more expensive. We shall establish a good preconditioner for  $K_{rr}^\eta$  in order to remove a bad effect of  $\eta$ . We introduce the preconditioner  $M$  as follows

$$M = \bar{K}_{rr} + \eta \tilde{J} = \begin{bmatrix} A_{ii} & 0 \\ 0 & A_{ee} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix}.$$

**Theorem 3** *The condition number of the preconditioned problem grows asymptotically as*

$$\kappa(M^{-1}K_{rr}^\eta) := \frac{\lambda_{\max}(M^{-1}K_{rr}^\eta)}{\lambda_{\min}(M^{-1}K_{rr}^\eta)} \lesssim \frac{H}{h} \left(1 + \log \frac{H}{h}\right).$$

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