MULTI-DIMENSIONAL LIMITING PROCESS FOR HYPERBOLIC CONSERVATION LAWS ON UNSTRUCTURED GRIDS

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ABSTRACT

The present paper deals with a robust, accurate and efficient limiting strategy on unstructured grids within the framework of finite volume method. The basic idea of the present limiting strategy is to control the distribution of both cell-centered and cell-vertex physical properties to mimic a multi-dimensional nature of flow physics, which can be formulated as so called the MLP condition. The design principle of the proposed method is based on the multi-dimensional limiting condition and the maximum principle, which can ensure the multi-dimensional monotonicity through the global/local L_∞ stability. Consequently, it can be shown that the MLP limiting does satisfy the local extremum diminishing (LED) condition in a truly multi-dimensional way. Various numerical analyses and extensive computations validate superior characteristics, such as efficient controlling multi-dimensional oscillations and accurate capturing of both discontinuous and continuous multi-dimensional flow features.

INTRODUCTION

Robust and efficient high resolution schemes are one of the key ingredients for computing large scale hyperbolic conservation laws. High-order accurate reconstruction schemes make it quite feasible to accurately capture complex flow structure with acceptable computational cost. At the same time, non-linearity and hyperbolic feature of these equations require sophisticated numerical treatments across physical discontinuities, particularly in multi-dimensional flows. However, most oscillation-free schemes including TVD and ENO-type schemes are mainly based on the mathematical analysis of one-dimensional convection equation, and applied to systems of equations with the help of some linearization step. Though this approach may work successfully in many cases, it is often insufficient or almost impossible to control oscillations near shock discontinuity in multi-dimensional flows. These schemes can be readily applicable to arbitrary dimension by dimensionally splitting manner, but it is not at all straightforward to directly adapt these schemes on unstructured grids. To truly maintain the merits of unstructured grid techniques, which can be summarized as flexible tessellation and mesh adaptation, the development of a robust and accurate oscillation control strategy on unstructured grids is essential.

In order to find out a suitable criterion for oscillation control in multiple dimensions, the one-dimensional monotonic condition was extended to multi-dimensional flow situations and the multi-dimensional limiting process (MLP) was successfully developed. From a series of researches, it has been clearly demonstrated that the MLP limiting strategy possesses favorable characteristics, such as enhanced accuracy and convergence behaviors in inviscid and viscous computations on structured grids [1, 2]. Recently, the MLP limiting strategy has been successfully extended on unstructured grids [3-5]. It was observed that the MLP limiting on unstructured grids is quite effective to control multi-dimensional oscillations as well as accurate in capturing multi-dimensional flow features, both in continuous and discontinuous regions. In particular, MLP on unstructured grids yields the designed accuracy with a MUSCL-type second-order reconstruction, and thus it can efficiently and accurately capture local flow structures, which are usually smeared out by conventional slope limiters. The present work is going to show enhanced features of MLP limiting on unstructured grids, and explore multi-dimensional monotonicity of this strategy from the maximum principle and LED condition [6].

MULTI-DIMENSIONAL LITMITING CONDITION

Summary of the MLP Condition

In order to enforce the multi-dimensional monotonicity, the present limiting strategy exploits the MLP condition, which can be regarded as the extension of the one-dimensional monotonic condition. The basic idea of the MLP condition is to control the distribution of both cell-centered and cell-vertex physical properties to mimic the multi-dimensional nature of flow physics. Especially, we pay attention to the observation that well-controlled vertex values at interpolation stage make it possible to produce multi-dimensional monotonic distribution of cell-centered values. Based on the observation, the vertex values are required to satisfy the following MLP condition (Eq. (1)).

$$\overline{q}^{\min}_{v_i,neighbor} \le q_{v_i} \le \overline{q}^{\max}_{v_i,neighbor}, \tag{1}$$

where q is the state variable and q_{v_i} is the reconstructed (or interpolated) value at vertex. $\left(\overline{q}^{\min}_{v_i,neighbor}, \overline{q}^{\max}_{v_i,neighbor}\right)$ are the minimum and maximum of the cell-averaged values among all neighboring cells sharing the same vertex v_i. In principle, the MLP condition can be implemented regardless of grid topology.

On structured grids, a physical property at vertex is readily estimated by summing one-dimensional monotonic variation along each coordinate direction. Thus, the MLP limiting can be naturally implemented by combining the TVD-MUSCL framework and detailed derivation and analyses results can be found in Refs. [1, 2]. On unstructured grids, there is no explicit reference direction and thus it is not feasible to obtain conservative directional variations. Interpolation within a cell may start from the unstructured version of MUSCL-type reconstruction as follows.

$$q_j(\mathbf{x}) = \overline{q}_j + \phi \nabla \overline{q}_j \cdot \mathbf{r}, \tag{2}$$

where $\overline{q}_j$ and $\nabla \overline{q}_j$ are the cell-averaged value and the gradient on the tetrahedron T_j. ϕ is a slope limiter to be determined and $\mathbf{r}$ is the vector from the centroid of the cell T_j. From the MLP formulation on structured grids, it is observed that the maximum and minimum values along the cell boundary should be checked to prevent spurious oscillations [3, 4]. On unstructured grids, local extrema appear at cell-vertex, and each vertex value should be monitored by the MLP condition. Considering all of the physical distributions around the vertex itself, the MLP range for the multi-dimensional slope limiter can be obtained as follows.

$$0 \le \phi \le \max\left(\frac{\overline{q}_{neighbor}^{\min} - \overline{q}}{\nabla \overline{q} \cdot \mathbf{r}_{vertex}}, \frac{\overline{q}_{neighbor}^{\max} - \overline{q}}{\nabla \overline{q} \cdot \mathbf{r}_{vertex}} \right). \tag{3}$$

Monotonicity of the MLP Condition

One of the attractive points of the MLP condition is to satisfy the maximum principle, which is the crucial condition in ensuring the multi-dimensional monotonicity.

Theorem *(MLP condition and Maximum principle)* Consider the multi-dimensional scalar hyperbolic conservation law of

$$q_t + \nabla f(q) = 0\,, \tag{4}$$

with a Lipschitz continuous monotone numerical flux function. If the linear reconstruction satisfies the MLP condition, then the corresponding fully discrete finite volume scheme satisfies the maximum principle under a proper CFL restriction, *i.e.*,

$$\text{If } \overline{q}_{j,neighbor}^{\min,n} \le \overline{q}_j^n \le \overline{q}_{j,neighbor}^{\max,n},\ \text{then } \overline{q}_{j,neighbor}^{\min,n} \le \overline{q}_j^{n+1} \le \overline{q}_{j,neighbor}^{\max,n}. \tag{5}$$

Complete proof will be provided in the presentation. There are other limiting strategies which also satisfy the maximum principle, but the major difference is the stencil involved in limiting process and the maximum principle. The MLP condition exploits all of the cell-averaged values sharing not only the cell vertex but also the cell face. We will call the resulting stencil the MLP stencil. The MLP condition on the MLP stencil makes it possible to capture multi-dimensional flow structures accurately while maintaining the desired accuracy.

From Eq. (5), it is clear to see that the MLP limiting provides the global/local L_∞ stability of computed solutions. In addition, it is also satisfied the LED condition [6] and it can be summarized as following corollary.

Corollary If the above-mentioned numerical discretization of hyperbolic conservation laws (Eq. (4)) satisfies the MLP condition, it also satisfies the LED condition in the MLP stencil. It indicates that the MLP limiting does satisfy the LED condition in a truly multi-dimensional way, *not in a dimensional-splitting way*.

MLP SLOPE LIMITERS ON UNSTRUCTURED GRIDS

With the range of the MLP limiting on unstructured grids (Eq. (3)), the MLP slope limiter can be generally formulated as follows.

$$\phi_{MLP} = \min_{\forall v_i \in T_j} \begin{cases} \Phi\left(r_{v_i,j}\right) & \text{if } \nabla \overline{q} \cdot \mathbf{r}_{v_i,j} \ne 0 \\ 1 & \text{otherwise} \end{cases}, \tag{6}$$

where $r_{v_i,j}$ is the ratio of the minimum or maximum allowable variation to the estimated variation at the vertex v_i,

$$r_{v_i,j} = \max\left(\frac{\overline{q}_{v_i,j}^{\min} - \overline{q}_j}{\nabla \overline{q} \cdot \mathbf{r}_{v_i,j}}, \frac{\overline{q}_{v_i,j}^{\max} - \overline{q}_j}{\nabla \overline{q} \cdot \mathbf{r}_{v_i,j}} \right). \tag{7}$$

For monotonicity, Φ should be in the range of $0 \le \Phi(r) \le \min(1, r_{v_i,j})$. The immediate form of the characteristic function Φ is to choose the upper bound of the limiting region. Equation (6) with this Φ denotes as MLP-u1, which can be written as follows.

$$\Phi(r) = \min(1, r_{v_i,j}). \tag{8}$$

MLP-u1 is non-differentiable, which might have a potential to hamper the convergence of steady state problems. Adapting the modification by Venkatakrishnan of Barth's limiter, we also propose MLP-u2 limiter for steady state problem as follows.

$$\Phi\left(\frac{\Delta_+}{\Delta_-}\right) = \frac{1}{\Delta_-}\left[\frac{(\Delta_+^2 + \varepsilon^2)\Delta_- + 2\Delta_-^2\Delta_+}{\Delta_+^2 + 2\Delta_-^2 + \Delta_+\Delta_- + \varepsilon^2}\right], \tag{9}$$

where $\varepsilon^2 = (K\Delta x)^3$. The role of value ε is to distinguish a nearly smooth region from a fluctuating one. Like TVB or ELED limiters, it also plays a role of preventing the clipping phenomenon.

NUMERICAL RESULTS

Isentropic Vortex Advection

Since vortex flow is a purely multi-dimensional continuous flow, it is a good test case to examine the accuracy of a numerical scheme in multi-dimensional flows without shock waves and turbulence. Since the flowfield is inviscid, the exact solution is just the passive advection of the initial vortex with mean flow. The mean flow, which is considered as a free stream, is $\rho_\infty = 1$, $p_\infty = 1$ and $(u_\infty, v_\infty) = (0,0)$. The perturbation values for the isentropic vortex are given by

$$(\delta u, \delta v) = \frac{\varepsilon}{2\pi} e^{0.5(1-r^2)}(-\bar{y}, \bar{x}),\ \delta T = -\frac{(\gamma-1)\varepsilon^2}{8\gamma\pi^2} e^{1-r^2},\ \ \delta S = 0. \tag{10}$$

The strength of vortex is $\varepsilon = 5$. Here, $(\bar{x}, \bar{y}) = (x - x_0, y - y_0)$, where (x_0, y_0) is the coordinate of the center of the initial vortex, and $r^2 = \bar{x}^2 + \bar{y}^2$. The computational domain is $-5 \le x \le 5$ and $-5 \le y \le 5$ and the periodic boundary condition is applied. Triangular mesh is created by dividing uniform square elements along the diagonal. RoeM scheme is applied as a numerical flux.

Figure 1 shows the comparison of density distributions across the vortex center, which clearly shows the low-dissipative characteristic of the proposed limiting strategy. Table 1 presents the analysis of the order of accuracy, and it also presents the higher order accuracy of MLP-u limiters over Barth's limiter.

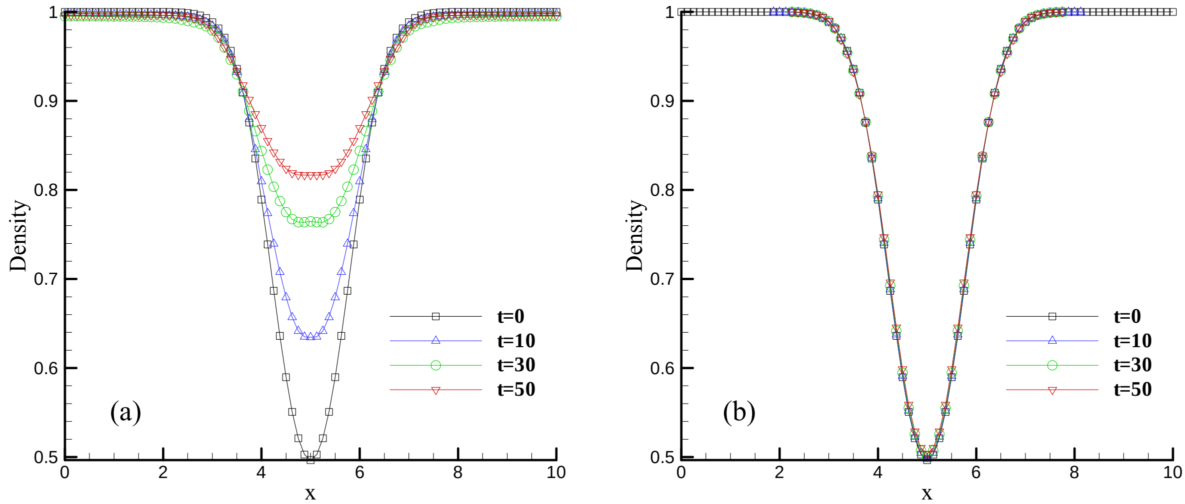

Figure 1. Density distributions along the vortex center: (a) Barth's limiter and (b) MLP-u1

Table 1. Comparison of numerical accuracy

	Barth's Limiter				MLP-u1 Limiter			
	L_∞	Order	L_1	Order	L_∞	Order	L_1	Order
10x10x2	2.08E-01	-	1.13E-2	-	1.72E-01	-	9.59E-03	-
20x20x2	1.19E-01	0.81	4.73E-3	1.26	4.16E-02	2.05	2.05E-03	1.94
40x40x2	5.98E-02	1.00	2.19E-3	1.13	8.62E-03	2.27	5.87E-04	2.09
80x80x2	3.20E-02	0.90	1.07E-3	1.01	1.58E-03	2.44	1.29E-04	2.19
160x160x2	1.68E-02	0.93	5.40E-4	0.99	4.63E-04	1.78	3.00E-5	2.11

Double Mach Reflection

This is a very popular test case for high-resolution schemes. The whole computational domain is [0, 4] x [0, 1]. The reflective wall is located at the bottom of computational domain beginning at $x = 1/6$. At first, a right-moving shock with $M_s = 10.0$ is located at $(x = 1/6, y = 0)$ inclined $60°$ angle with respect to the x-axis. Lax-Friedrich scheme is used as a numerical flux and the computation was carried out till $t = 0.2$.

Figure 2 shows the comparison of density contours with the vertical height h of triangular meshes $1/480$. Both limiters give monotone solutions, but MLP-u1 limiter shows a much more enhanced resolution for shock discontinuity and the complicated flow structure below the Mach stem.

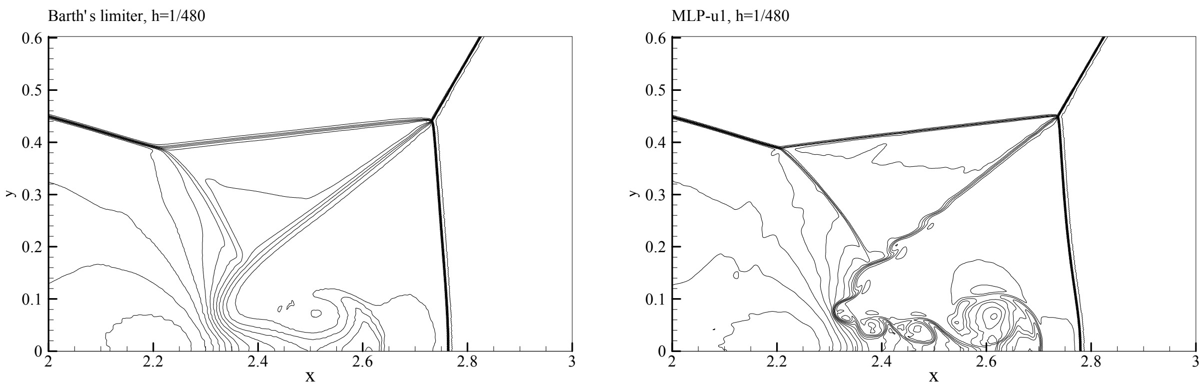

Figure 2. Density contours around the region of the double Mach stem

CONCLUSION

Based on the observation on multi-dimensional flow physics, a new multi-dimensional limiting process (MLP) on unstructured grids has been developed within the framework of finite volume discretization. The basic idea is to control physical values at vertex point as well as cell-centered point. The most distinguishable property of MLP is to provide non-oscillatory problems in multi-dimensional flows. The satisfaction of the maximum principle and LED condition guarantees the global/local L_∞ stability to ensure the multi-dimensional monotonicity. Thanks to the property, MLP can significantly increase accuracy, convergence/robustness and efficiency in multi-dimensional flows containing physical discontinuities. This progress can also be served as an important stepping stone to achieve a higher-order multi-dimensional limiting in conjunction with some higher-order discretizations, such as discontinuous Galerkin method. Various numerical results confirm the desirable characteristics of the proposed scheme, such as multi-dimensional monotonicity, improved accuracy and efficiency.

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REFERENCES

1. Kim, K. H. and Kim, C., "Accurate, efficient and monotonic numerical methods for multi-dimensional compressible flows Part II: Multi-dimensional limiting process," *Journal of Computational Physics*, Vol. 208, 2005, pp. 570-615.
2. Yoon, S.H., Kim, C. and Kim, K.H., "Multi-dimensional limiting process for three-dimensional flow physics," *Journal of Computational Physics*, Vol. 227, 2008, pp. 6001-6043.
3. Park, J. S., Yoon, S.H. and Kim, C., "Multi-dimensional limiting process for hyperbolic conservation laws on unstructured grids," *Journal of Computational Physics,* Vol. 229, 2010, pp. 788-812.
4. Park, J. S. and Kim, C., "Multi-dimensional limiting process for compressible flow analyses on tetrahedral grids," In preparation, 2011.
5. Park, J. S. and Kim, C., "Multi-dimensional Limiting Process for Discontinuous Galerkin Methods on Unstructured Grids," *Proc. of the ICCFD6*, St. Petersburg Russia, July 2010.
6. Jameson, A., "Analysis and Design of Numerical Schemes for Gas Dynamics 1: Artificial Diffusion, Upwind Biasing, Limiters and their Effect on Accuracy and Multigrid Convergence," *International Journal of Computational Fluid Dynamics*, Vol. 4, No. 3, 1995, pp. 171–218.